The exact solution of quasi one-dimensional Hecke algebra models

von

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Abstract

We study a class of two dimensional discrete models whose local transfer matrices form a representation of the Hecke algebra and can therefore be parametrized in order to satisfy the Yang-Baxter equation. They are generalizations of the well known six-vertex model and of Pasquier's ADE-models and they are closely related to the quantum groups $su(k)_q$. Generically, they are represented as Interaction-Round-a-Face (IRF) models with the height variables living in the vertex set of a graph, but for a subclass of them a dual description involving $su(k)_q$ spins is known.

The transfer matrix we consider has arbitrary boundary conditions and hence no diagonalization technique is known. Our main result is the computation of the specific free energy of the quasi one-dimensional system obtained by taking the thermodynamic limit in space direction but keeping the system size in time direction fixed to one lattice spacing. We solve the equations for the su(2) models completely and compute the free energy.

In the last part of the work we generalize the above duality to a duality between gauge and path algebra theories and we discuss some of its possible applications.

Zusammenfassung

Wir untersuchen eine Klasse zweidimensionaler Modelle deren lokale Transfer-Matrizen eine Darstellung der Hecke-Algebra bilden, und die somit — mit einer geeigneten Parametrisiserung versehen — die Yang-Baxter-Gleichung erfüllen. Sie sind eine Verallgemeinerungen des wohlbekannten Sechs-Vertex-Modells und von Pasquiers ADE-Modellen, und es besteht eine enge Verbindung zu $su(k)_q$ Quantengruppen. In ihrer allgemeinen Form sind sie als IRF-Modelle beschrieben, deren 'height' Variablen in der Vertexmenge eines Graphen leben, aber für eine Unterklasse von ihnen existiert eine äquivalente Beschreibung als $su(k)_q$ Spin-Modelle.

Unsere Transfer-Matrix beschreibt beliebige Randbedingungen und ist mit den bekannten Techniken nicht diagonalisierbar. Unser Hauptresultat ist die Berechnung der spezifischen freien Energie des quasi-eindimensionalen Systemes, das man erhält, wenn man den thermodynamischen Limes nur in Raumrichtung nimmt, die Systemausdehnung in Zeitrichtung aber auf einen Gitterabstand fixiert. Für die su(2)-Modelle lösen wir die Gleichungen vollständig und berechnen die freie Energie.

Im letzten Teil der Arbeit verallgemeinern wir die obige Dualität zu einer Dualität zwischen Eich- und Pfadalgebratheorien und wir diskutieren einige mögliche Anwendungen.

Chapter I. After a brief presentation of the models we introduce the physical and mathematical concepts that we use. We explain the dimensional reduction of the configuration space with the transfer matrix formalism and discuss the possibility to iterate it. Then we briefly describe quantum groups, fusion algebras, star- and von Neumann algebras, graphs and their associated path algebras.

Chapter II. We describe the Hecke algebra models in detail, introducing at first the more general class of IRF models. We define the face transfer matrices as a function of the Hecke- (or the generalized Temperley-Lieb-) algebra generators g, represented on graphs related to the su(k)$_q$ fusion algebra. For the subclass of models whose height variables are the unitary representations on the su(k)$_q$ fusion graph, we introduce the dual description as a su(k)$_q$ spin model, the generators g being the quantum R-matrix.

Chapter III. We study the sequence of partition functions Z^N of a tube of variable length N but fixed width 1. We discuss the relation between partition operators, traces and boundary conditions. We compute the thermodynamic limit of the partition operators and show that it is related to an energy normalization at infinite temperature, and that the Jones index, which appears there in a natural manner, determines the system's entropy.

Finally, we find a generalized transfer matrix formalism based on a level k recurrence formula

$$\sum_{i=0}^{k} \rho_i \mathcal{G}_i Z^{N-i} = 0$$

We show that the coefficients ρ_i are universal in the sense that they depend only on the deformation and the spectral parameters. All dependence on the models is absorbed into the graphs $\mathcal{G}_i$ who generate a representation of the fusion algebra. Furthermore we show that this relation determines the Z^N uniquely because we can use the su$(k-1)$ recurrence in order to determine the su(k) initial conditions. We compute the ρ_i up to su(3) and give a general algorithm for su(k). We determine explicitly the free energy for all su(2) models as a function of deformation- and spectral parameter.

Chapter IV. We generalize the duality between Spin and IRF models to a duality between gauge theories based on a quasitriangular Hopf algebra and path algebra theories whose underlying graph is determined by the local representation of the gauge algebra.

We show that this duality is related to the Fourier transformation when we take as Hopf algebra the group algebra $C[U(1)]$. We discuss the possibility to use the duality in order to describe discrete processes.

Contents

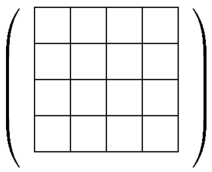
Chapter 1

Introduction

1.1 The Hecke algebra models

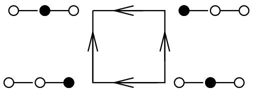
The Hecke algebra models, a generalization of Pasquier's ADE-models, are a class of discrete integrable models of statistical mechanics describing a critical behaviour, and hence their continuum limit is described by a conformal field theory, the so called *coset theories* [Pas87, Pas88a, Pas88b, DJMO87, JMO88].

In the standard description, the Hecke algebra models are given as interaction-round-a-face (IRF) models: this means that the local variables (we refer to them as *heights*[1], and reserve the name *spins* for the variables appearing in the Spin-description) live on the sites of a square lattice, and only the heights situated at the four corners of a lattice face interact. The weight of a configuration is given as the product over its face weights

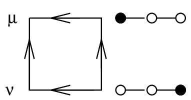
W 

$$= \prod_{\text{faces} f} W\left(\boxed{f} \right) \qquad (1.1)$$

More precisely, the square lattice is taken to be oriented, the height variables are in the vertex set of an oriented graph, and the face weight vanishes whenever the configuration is not continuous in the sense of graphs. So the face weights can be only non-vanishing if the vertices adjacent to two adjacent sites on the lattice are adjacent on the target graph as shown in the following example.



$$\neq 0,$$



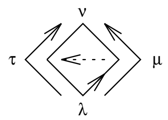
$$= 0 \qquad (1.2)$$

It is common to change to a diagonal-to-diagonal transfer matrix formalism [Fra92] (see sect. 1.2), interpreting the face weights as a *face* or *local transfer matrix* X which acts on a set of three

[1] because these are the height variables appearing in the SOS or RSOS models

adjacent heights, changing only the middle one

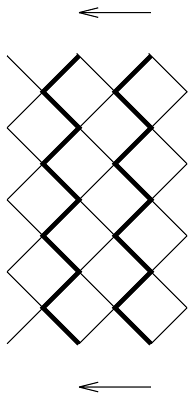
$$\mathrm{X}^{\lambda'\tau\nu'}{}_{\lambda\mu\nu} = \delta_{\lambda,\lambda'}\delta_{\nu,\nu'}\, W(\lambda\mu\tau\nu) = \tag{1.3}$$



Because of the continuity condition the face transfer matrix X can be interpreted as a linear mapping on the space spanned by the paths of length 3, leaving the boundaries fixed.

The complete diagonal to diagonal transfer matrix T acting on a zig-zag cut is then given as a product over the face transfer matrices X_i acting on the different lattice sites i.

$$T = \prod_{i \text{ even}} \mathrm{X}_i \prod_{i \text{ odd}} \mathrm{X}_i = \tag{1.4}$$



Here X_i acts on the sites $(i-1, i, i+1)$ of the zig-zag. Like the local transfer matrices, T does not change the heights at the boundaries of the lattice cut and is hence an element of the path algebra of the underlying graph (see sect. 1.4.3).

In this work we will use local face weights that form a representation of the Hecke algebra (2.6). They depend on a spectral parameter α and satisfy the Yang-Baxter equation (2.3). This induces that the standard transfer matrices, the ones propagating parallel to lattice links, commute for different values of the spectral parameter when we impose periodic boundary conditions and can hence be diagonalized with Bethe ansatz or related techniques [Bax82]. However, the objects that we consider in sect. 3 are products of the local generators with open boundaries and hence these operators do not commute and cannot be diagonalized with the standard techniques. In a recent work [BPO95] Pearce et al. modified a formalism due to Sklyanin [Skl88] and used the duality between spin and IRF models of sect. 2.3 in order to treat fixed boundary conditions in related models. However, these results have not yet been generalized to the models that we studied and our approach is slightly different due to the diagonal transfer matrix that we use and that still can not be diagonalized.

These models are strongly related to the quantum group $\mathrm{su(k)}_q$: in the Spin-description (sect. 2.2) the X_i are in the algebra generated by the quantum R-matrix (see sect. 1.3.1) and hence they commute with the global action of the quantum group induced by the coproduct Δ. Furthermore, the graphs that appear in the general Hecke algebra models (sect. 2.1.2) seem all to carry a representation of the fusion algebra of $\mathrm{su(k)}_q$ (see sect. 1.3.2).

Before we explain these models in more detail in sect. 2, we recall briefly the concepts needed and introduce the notations we use.

1.2 Transfer matrix formalism

In the transfer matrix formalism, the Boltzmann weights are interpreted as matrix elements of some operator (the *transfer matrix*), and hence the summation over the intermediate spaces becomes an operator composition.

Reduction 2d to 1d with the transfer matrix formalism. We explain the reduction with the example of a spin model with the spins situated at the sites of a square lattice. We denote $i = 1 \ldots N$ and $j = 1 \ldots M$ the coordinates on the lattice and $(\sigma_{i,j}^{N,M})$ the configuration of a system of size $N \times M$. The partition function is given as

$$Z^{N,M} = \sum_{(\sigma_{i,j})} W^{N,M}[(\sigma_{i,j}^{N,M})] \tag{1.5}$$

where $W^{N,M}$ is the functional associating the Boltzmann weight to the configurations on systems of size (N, M). We note that this defines in fact a double sequence of systems, indexed by the lattice size (N, M) which are all locally equivalent. We suppose, that W contains only nearest-neighbor interactions[2]. In this case, the sum over the inner sites of any area can be executed independently as we want to explain with the following example.

Suppose that we are interested in the partition function of a rectangular plate P which is composed out of two smaller plates P_1, P_2

$$\boxed{\quad P \quad} = \boxed{P1 \,\big|\, P2} \tag{1.6}$$

We denote by $Z_P(\sigma_{\mathrm{N}}, \sigma_{\mathrm{S}}, \sigma_{\mathrm{W}}, \sigma_{\mathrm{E}})$ the partition function with the values of the spins at the boundary fixed to $(\sigma_{\mathrm{N}}, \sigma_{\mathrm{S}}, \sigma_{\mathrm{W}}, \sigma_{\mathrm{E}})$, so that we just sum over the spins in the interior of P and we have

$$Z_P(\sigma_{\mathrm{N}}, \sigma_{\mathrm{S}}, \sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) = \sum_{\sigma_{i,j} \in \mathrm{int} P} W[\sigma_{\mathrm{N}} \ldots \sigma_{\mathrm{E}}, \sigma_{i,j}] \tag{1.7}$$

The partition function Z_P can then be obtained using the partial partition functions Z_{P_1} and Z_{P_2} associated with the plates P_1, P_2 as

$$Z_P(\sigma_{\mathrm{N}}, \sigma_{\mathrm{S}}, \sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) = \sum_{\sigma} Z_{P_1}(\sigma_{\mathrm{N}1}, \sigma_{\mathrm{S}1}, \sigma_{\mathrm{W}}, \sigma) Z_{P_2}(\sigma_{\mathrm{N}2}, \sigma_{\mathrm{S}2}, \sigma, \sigma_{\mathrm{E}}) \tag{1.8}$$

The transfer matrix method is founded on the same principle, using the above decomposition in order to find a recurrence relation for the $Z^{N,M}$. We choose at first one direction to be the

[2] It is always possible to find such a description if the number of spins coupled by the interaction does not depend on the system size (N, M), which means that there are no long range interactions.

time direction, say the one corresponding to N. Then we note, that the plate $P^{N+1,M}$ can be decomposed into $P^{N+1,M} = P_N^{2,M} \cup P^{N,M}$ with the common boundary at time N. Using the above formula and defining the *transfer matrix* $(T_N^{(M)})^{\sigma_W}{}_{\sigma_E} := Z_N^{2,M}(\sigma_W, \sigma_E)$ (north and south are evidently fixed by the boundaries of east and west), we obtain the recurrence relation

$$Z^{N+1,M}(\sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) = \sum_\sigma (T_N^{(M)})^{\sigma_{\mathrm{W}}}{}_\sigma \, Z^{N,M}(\sigma, \sigma_{\mathrm{E}}) \equiv (T_N^{(M)} \cdot Z^{N,M})(\sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) \tag{1.9}$$

with the initial condition $Z^2 = T$. The solution of this equation is

$$Z^{N,M}(\sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) = \left(\prod_{n=1}^{N} T_n^{(M)} \right) (\sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) = |_{T=\mathrm{const}}((T^{(M)})^N)(\sigma_{\mathrm{W}}, \sigma_{\mathrm{E}}) \tag{1.10}$$

where the last equality holds if the transfer matrix is independent of n. In a common terminology, we refer to the variable which we absorbed into the matrix composition as the *time variables* and to the transverse one describing the common boundary of the two plates as the *space variable.*

Implementing boundary conditions. The physical partition function depends not only on the local dynamics but also on the boundary conditions. We were able to encode the local dynamics into the transfer matrix, thereby reducing the two dimensional configuration space to the one dimensional space of boundary conditions on which the partition function can be seen as a N step transfer matrix.

The boundary chosen conditions are now described by the matrix B and the associated partition function – now a scalar – is

$$Z_{\mathrm{phys}}^{N,M} = \mathrm{tr}(B \cdot Z^{N,M}) \tag{1.11}$$

Any type of boundary conditions can be encoded this way: *open* boundaries are given by $B^\sigma{}_{\sigma'} = 1$, boundaries *fixed* to vectors $|a\rangle$, $|b\rangle$ by $B^\sigma{}_{\sigma'} = a^\sigma \, b_{\sigma'}$ and *periodic* ones by $B^\sigma{}_{\sigma'} = \delta_{\sigma,\sigma'}$. If $B^2 = \mathrm{id}$ they are called *antiperiodic* and all other types of boundary conditions are called *nonperiodic*.

The thermodynamic limit. Taking the limit in time direction – the system hence being on a infinite ribbon of width M – the partition function is determined by the infinite product of transfer matrices T^∞. When we suppose that T is diagonalizable then T^∞ is (without accident due to the boundary conditions) dominated by the eigenvalue of $T^{(M)}$ of greatest modulus $\gamma_0^{(M)}$

$$Z^{N,M} \sim_{N\to\infty} \left(\gamma_0^{(M)}\right)^N \tag{1.12}$$

which determines the free energy par lattice site $f = -\log|\gamma_0^{(M)}|$.

The thermodynamical limit $M \to \infty$ in the space direction is conceptually different. The transfer matrices $T^{(M)}$ and $T^{(M+1)}$ act on different spaces and hence live in different algebras $\mathcal{A}^M$ and $\mathcal{A}^{M+1}$. In the most intuitive approach, we try to diagonalize the transfer matrices for each M separately. If we know how to do this we can easily take the thermodynamic limit $M \to \infty$ in the space direction as well.

Reduction 1d to 0d. If we do not know diagonalizing $T^{(M)}$ with M considered as an arbitrary parameter we can try to repeat the transfer matrix formalism and to find a recurrence for the $T^{(M)}$ or the $Z^{N,M}$ with N fixed (maybe to ∞). In general there is an inclusion $\mathcal{A}^M \subset \mathcal{A}^{M+1}$ such that $\mathrm{id}_M \mapsto \mathrm{id}_{M+1}$[3] due to the fact that a system of size $M+1$ contains a system of size M. However we have to be careful, because the natural trace operation, which extracts the physically measurable quantities out of the formalism, differs in these two algebras. In the simple case of spin models described above (which is not the case of the general Hecke models), $\mathcal{A}^M = \bigotimes_{i=1}^M \mathcal{A}_{loc}$ is just the tensor product of M local algebras and the inclusion is $T^{(M)} \mapsto T^{(M)} \otimes \mathrm{id} \in \mathcal{A}^{M+1}$. Due to the nearest neighbor interaction the transfer matrix is a product of local transfer matrices $T^{(M)} = X_M \ldots X_1$, where X_i is acting on the neighboring sites $(i, i+1)$. If the system is space translationally invariant, all the X_i are locally the same.

We note that the product of the X_i is not the same as the product of the global transfer matrices $T_n^{(M)}$ because of the shift giving this product a tensor character. It is only after inclusion into the inductive limit of the algebras (see sect. 1.4 as well) that we can interpret this operation as an algebra multiplication, and hence the X_i as transfer matrices over a zero dimensional space. We note however that X_i and X_{i+1} in general do not commute because of the interaction.

Summary. We summarize briefly the path that we followed in order to simplify the problem. Starting with a two-dimensional classical system with Boltzmann weights $W^{N,M}$, we first reduced it to the one dimensional system of transversal or space coordinates, the left and right boundaries of time indexed paths in the 2d configuration space. We then reduced the transversal configuration space once again, to an oriented pair of points. The price we pay for this dimensional reduction is that the geometry of the configuration space becomes more and more complicated: in the two-dimensional case, the weight operator $\sum_\sigma W^{N,M}[\sigma]\ |\sigma\rangle\langle\sigma|$ is diagonal. In the one dimensional space this weight operator becomes a product of transfer matrices $T_N \ldots T_1$, but, because $T_i = T_j \equiv T$, the T_i's commute for different times i, j and are hence simultaneously diagonalizable. The transfer matrices over the zero dimensional space X_i finally do not necessarily commute for adjacent times, leading to serious technical problems when we compute the resulting product.

1.3 Quantum groups

In this section we recall the subset of definitions and relations concerning *quantum groups* (we mean by this term *quasitriangular Hopf algebras*) that we will use in this work; for a more detailed review and further references see [Fuc92].

[3] This characterizes a tower inclusion.

1.3.1 Quantum groups and R matrix

Let $\mathcal{A}$ be a Hopf algebra, in our case $\mathrm{su(k)}_q$. We denote by $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ the coproduct, which is also an algebra homomorphism. We use the following common notation

$$\Delta(x) \equiv \Delta^{(1)}(x) \otimes \Delta^{(2)}(x) \equiv x^{(1)} \otimes x^{(2)} := \sum_{\alpha} \Delta^{(1,\alpha)}(x) \otimes \Delta^{(2,\alpha)}(x) \tag{1.13}$$

where we sometimes suppress the summations appearing in a tensor product. This coproduct is coassociative, which means that the two possibilities to build the triple coproduct lead to the same result

$$\Delta_3 \equiv \Delta^{(1)} \otimes \Delta^{(2)} \otimes \Delta^{(3)} := \Delta^{(11)} \otimes \Delta^{(21)} \otimes \Delta^{(2)} = \Delta^{(1)} \otimes \Delta^{(12)} \otimes \Delta^{(22)} \tag{1.14}$$

One shows by induction that this remains true for higher coproducts as well. We denote the N-fold coproduct by $\Delta_N : \mathcal{A} \to \mathcal{A}^{\otimes N}$. Especially, $\Delta_1(x) = x$ and $\Delta_2 = \Delta$. When we insist on how to set the (co-) parentheses, we may use the mixed notation

$$\Delta_{N+M}(x) = \Delta_N(x^{(1)}) \otimes \Delta_M(x^{(2)}) \tag{1.15}$$

When we are given a tensor product of representations we use a simplified notation where we do not note explicitly the representation mapping. Let Γ_i be the *local representations* – this means representations on some standard representation space carrying the local degrees of freedom – we write

$$\Delta_{ijk\ldots}(x) := (\Gamma_i \otimes \Gamma_j \otimes \Gamma_k \otimes \cdots)\,(x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes \cdots) \tag{1.16}$$

for the *global representation* induced by the coproduct on the tensor product of the local spaces.

If $\mathcal{A}$ is a Lie algebra, then the coproduct is on the Lie algebra generators

$$\Delta_N(X) = X \otimes \cdots \otimes \mathrm{id} + \mathrm{id} \otimes X \otimes \mathrm{id} \cdots + \cdots + \mathrm{id} \otimes \cdots \otimes X \tag{1.17}$$

and on any general element it can be calculated with the homomorphism property. The Casimir operator of su(2) for example becomes

$$\Delta(\vec{J}^2) = \vec{J}^2 \otimes \mathrm{id} + \mathrm{id} \otimes \vec{J}^2 + 2\vec{J} \dot{\otimes} \vec{J} \equiv \vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2 \tag{1.18}$$

If Δ is a coproduct, and Π is the permutation operator which acts on $\mathcal{A} \otimes \mathcal{A}$ as $\Pi\,(x \otimes y)\,\Pi = y \otimes x$ then $\Delta' := \Pi \Delta \Pi^{-1}$ is a coproduct as well. If $\Delta = \Delta'$, which is the case for Lie algebras, then the algebra is called *cocommutative*. In this case the coproduct does not depend on the ordering of the local representation spaces. A Hopf algebra is called *quasitriangular* if the action of Π on $\Delta(\mathcal{A})$ is interior in $\mathcal{A} \otimes \mathcal{A}$. This means there exists an invertible element $\mathrm{R} \in \mathcal{A} \otimes \mathcal{A}$, the quantum R-matrix

$$\mathrm{R} \equiv \mathrm{R}^{(1)} \otimes \mathrm{R}^{(2)} := \sum_{\alpha} \mathrm{R}^{(1,\alpha)} \otimes \mathrm{R}^{(2,\alpha)}, \tag{1.19}$$

such that

$$\Delta'(x) = \Pi \Delta(x) \Pi = \mathrm{R} \Delta(x) \mathrm{R}^{-1}. \tag{1.20}$$

Evidently a cocommutative algebra is quasitriangular taking $\mathrm{R} = \mathrm{id}$.

1.3.2 Quantum groups and Fusion

Let $\mathcal{A}$ be a quantum group, for us always $\mathrm{su(k)}_q$, and $\{\lambda\}$ the set of its irreducible, unitary representations and $\mathcal{A}_\lambda$ the image of $\mathcal{A}$ under the representation λ. Let $\mathcal{A}_t := \bigoplus \mathcal{A}_\lambda$. By definition, this is the semi-simple part of $\mathcal{A}$. We will refer to it as to the *truncated algebra* as well. We have $\mathcal{A} = \mathcal{A}_t \oplus \mathcal{A}_{ns}$, and $\mathcal{A}$ is semi-simple iff $\mathcal{A}_t = \mathcal{A}$. For $\mathrm{su(k)}_q$ at generic q one has $\mathcal{A}_{ns} = 0$. For q a root of unity $\mathcal{A}_t$ is the set of representations in the truncated Weyl chamber. The set $\mathcal{A}_t \subset \mathcal{A}$ is a subalgebra and by construction each unitary representation of $\mathcal{A}$ induces a unitary representation of $\mathcal{A}_t$.

Because $\mathcal{A}_t$ is semi-simple its unitary representations are fully reducible. In particular, the coproduct representation $\Delta : \mathcal{A}_t \to \mathcal{A} \otimes \mathcal{A}$ is fully reducible, hence $\mathcal{A} \otimes \mathcal{A}$ blocks into irreducible representations of $\mathcal{A}_t$ and a part of non-semi-simple representations which has to be truncated. Let $\mathbf{1}_\lambda \in \mathcal{A}_t$ denote the central projection onto λ, hence $\Delta(\mathbf{1}_\lambda) \in \mathcal{A} \otimes \mathcal{A}$ is the projection onto the central representations of type λ contained in $\mathcal{A} \otimes \mathcal{A}$. A second set of central projections is given by the local projections $\mathbf{1}_\mu \otimes \mathbf{1}_\nu \in \mathcal{A}_t \otimes \mathcal{A}_t$.

The *fusion coefficients* $N_{\mu\nu}{}^\lambda$ count the number of representations of type λ contained in $\mu \otimes \nu$

$$(\mu) \otimes (\nu) = \sum_\lambda N_{\mu\nu}{}^\lambda \cdot (\lambda) \tag{1.21}$$

Because $\mathbf{1}_\mu \otimes \mathbf{1}_\nu \cdot \Delta(\mathbf{1}_\lambda)$ is the identity operator of the representations of type λ on the tensor product of the μ, ν-representation spaces and its trace factorizes as a trace over the λ-representation space and its commutant, the above multiplicities can be computed as

$$N_{\mu\nu}{}^\lambda = \frac{1}{\mathrm{tr}\mathbf{1}_\lambda} \mathrm{tr}(\mathbf{1}_\mu \otimes \mathbf{1}_\nu \cdot \Delta(\mathbf{1}_\lambda)) \tag{1.22}$$

Note that $\mathrm{tr}(\mathbf{1}_\lambda) = d_\lambda$ is the dimension of the representation space of type λ.

These relations define the — associative and commutative — *fusion algebra* which is spanned by the (μ). The coefficients $N_{\mu\nu}{}^\lambda$ define its fundamental representation[4]. The image of the abstract algebra element (μ) under this representation is the matrix N_μ with

$$(N_\mu)_\nu{}^\lambda = N_{\mu\nu}{}^\lambda \tag{1.23}$$

These matrices satisfy the representation property

$$\sum_\tau (N_\mu)_\rho{}^\tau (N_\nu)_\tau{}^\sigma = \sum_\lambda N_{\mu\nu}{}^\lambda (N_\lambda)_\rho{}^\sigma \tag{1.24}$$

The fusion algebra is generated by the subset of representations (i) corresponding to the diagrams containing only a single column. We note the matrices corresponding to these *fundamental representations* N_i.

[4]The representation on the algebra itself via left multiplication.

Because all the fusion matrices N_μ commute, they are diagonalizable in a common basis $|a>$. Because the transposed of a matrix N_μ is the matrix associated with its contragradient representation $N_{\bar\mu}$ all these matrices are *normal*[5] and hence the basis $|a>$ can be taken orthonormal. The matrix elements are nonnegative integers, and the matrix is irreducible in the sense that it is not a direct sum. The Perron-Frobenius theorem [MM92] states in that case that one of the eigenvalues with greatest modulus is real, and that the associated *Perron-Frobenius vector* $|0>$ has only positive entries. We will note the eigenvalues

$$N_\mu\,|a> = \gamma(a,\mu)\,|a> \tag{1.25}$$

Often we will interpret the matrices N_μ as adjacency matrices of some graph $\mathcal{G}_\mu$ (see sect. 1.4.1). We call the graph associated with the fundamental representation $N_1 \equiv \mathcal{G}_1$ the *fundamental fusion graph.* Its eigenvalues are noted as $\gamma_a \equiv \gamma(a,(1))$.

1.4 Path algebras

1.4.1 Graphs and Paths

A convenient tool for describing a graph is its *adjacency matrix* $\mathcal{G}$: it is a matrix whose indices run through the vertex set of the graph and the matrix element $\mathcal{G}^\mu{}_\nu$ is the number of oriented links from ν to μ. The adjacency matrix can be seen as an operator on the vertex Hilbert space. This is the space spanned by the orthonormalized basis of the vertices $|\mu>$. One writes

$$<\mu|\,\mathcal{G}\,|\nu> = \#\{\text{links from } \nu \text{ to } \mu\} \tag{1.26}$$

In the following we will always identify a graph with its adjacency matrix.

The graph with the orientation inversed is the transposed, or the hermitian conjugate $\mathcal{G}^+$ if the vertex space is taken to be complex. A graph is *non-oriented* iff $\mathcal{G}^+ = \mathcal{G}$. One can interpret $\mathcal{G}$ as describing the paths of length 1 on the graph. Matrix multiplication is path concatenation, hence $<\mu|\,\mathcal{G}^N\,|\nu>$ is the number of paths from ν to μ with lengh N.

A graph is *normal* iff it commutes with its conjugate $\mathcal{G}^+$. In this case, it can be diagonalized in an orthonormalized basis $|a>$

$$\mathcal{G}\,|a> = \gamma_a\,|a> \quad <a|b> = \delta_{a,b} \tag{1.27}$$

Under the hypothesis that the Perron-Frobenius eigenvalue is not degenerate we can compute the propagation on the path for large n to be

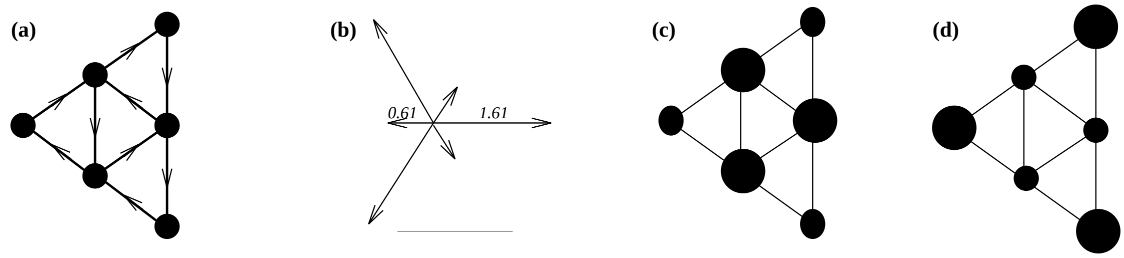
$$<\mu|\,\mathcal{G}^n\,|\nu> \;\sim_{n\to\infty}\; \gamma_0^n <\mu|0><0|\nu> \tag{1.28}$$

[5] They commute with their transpose.

In problems where we are interested in the propagation of an initial probability distribution $|\rho>$ by the graph this propagation is for large n always dominated by the Perron-Frobenius sector: Both the $<\mu|\rho>$ and $<0|\mu>$ are non-negative for all and strictly positive for at least one μ, so $<0|\rho>$ is strictly positive.

However, when the graph has certain symmetry properties, the Perron-Frobenius eigenvalue is degenerate in the sense that there are several eigenvalues with the same modulus. Consider for example that the graph is *C-colorable*, that means we can associate to each vertex a color c being a C-cyclic integer such that the sites adjacent to a site of color c carry always the color $c+1$[6]. In this case there are selection rules arising from the conserved charge 'color at $n=0$'. Starting at a type c site at time $n=0$ one will always be at a type $c+n$ site at the time n. The path space decomposes into C subspaces of paths that meet never.

The spectrum of the graph in this case is degenerate in modulus. The phase of the eigenvalue of a C colored graph is a power of $e^{2\pi i/C}$ and it induces a representation of the color symmetry group Z_C. In particular, the Perron-Frobenius eigenvector is the one with the trivial color representation. See figure 1.1 for a tricolorable graph and its eigensystem.



(a) the graph (b) the eigenvalues in the complex plane (number = length) (c) the coefficients in the Perron Frobenius sector (surface = $|\text{coeff}|^2$) (d) in the subdominant sector

Figure 1.1: A tricolorable graph and its eigensystem

1.4.2 Star- and von Neumann algebras

Without going too much into the details, we give here a brief introduction to the ideas of C^*- and W^*- (von Neumann-) algebra theory and to Jones subfactor theory. For details see for example the textbooks [Con94], [Sak71] and [BR79] for the general and [GdlHJ89] for the Jones theory and Bratteli diagrams.

For our purposes a *C^* algebra* is an operator algebra which is closed with respect to the usual operator norm topology and with a complex conjugation $x \mapsto x^*$. A *von Neumann (W^*-algebra)* is a C^* algebra of operators on a Hilbert space which is weakly closed[7]. This property is quite useful.

[6] More mathematically that means there is a morphism onto the cyclic graph $(Z_C, +1)$ (see footnote 10 on page 13 for the notation).

[7] In the sense of matrix elements: when all matrix elements of a sequence of operators converge then the operator

We can for example define an integration over von Neumann algebras on the matrix elements as

$$<m|\int \mathrm{d}z\, A(z)\,|n> := \int \mathrm{d}z\, <m|\,A(z)\,|n> \tag{1.29}$$

As a consequence for an operator A with discrete spectrum the projection onto the spectral domain D is

$$\mathbf{1}_D = \oint_{\partial D} \mathrm{d}z\,[z-A]^{-1} \tag{1.30}$$

and hence it is in particular contained in the algebra.

A *factor* N is a von Neumann algebra with a trivial center, this means its commutant[8] $N' \cap N \sim$ id. A n-dimensional factor is always the matrix algebra $M_n(C)$. There is a decomposition theorem which states roughly that any von Neumann algebra is the direct sum (or integral) of factors. A factor is *simple* because it contains no proper ideals.

A *trace* is a cycliclinear form on an algebra with the possible value ∞. On a factor a trace is unique up to scalar multiplication. As a consequence the space of traces is isomorphic as a vector space to the space spanned by the central projections $\mathbf{1}_\lambda$. A *weight vector* $\alpha = \sum \alpha^\lambda \mathbf{1}_\lambda$ determines a trace via $\mathrm{tr}_\alpha(x) := \mathrm{tr}(\alpha x)$ with tr any reference trace. The phase of a trace on the positive elements of a factor x^*x is unique, this means $\mathrm{tr}(x^*x) = q(x^*x)e^{i\alpha}$ with $q(x^*x)$ real. A trace is *positive* iff $e^{i\alpha} = 1$.

A *unitary or star representation* is an algebra homomorphism into the operator algebra of a Hilbert space which maps the star operation onto the hermitian conjugation. Unitary factor representations are essentially unique of the form $x \mapsto x \otimes \mathrm{id}$ with id the identity of the commutant algebra. A representation Γ is called *irreducible* if it contains no proper subspaces, which means that for each nonzero $v \in \mathcal{H}$ the set $\Gamma(\mathcal{A})v$ is dense in $\mathcal{H}$.

A *state* φ is a positive linear form on $\mathcal{A}$. It induces a scalar product on the algebra via $(x, y) := \varphi(x^*y)$. The kernel of φ is a both sided ideal $\mathcal{I} \subset \mathcal{A}$. On the quotiented algebra $\mathcal{A}_\varphi := \mathcal{A}/\mathcal{I}$, seen as a vector space, φ induces a positive definite scalar product. The representation of $\mathcal{A}$ on (the completion of) $\mathcal{A}_\varphi$ via left multiplication is a star representation. It can be shown that all possible star representations can be obtained by this so called Gel'fand-Naimark-Segal *GNS construction*.

1.4.3 Path algebras

The concept of path algebras has been introduced as a very intuitive picture in order to study the inductive limits of algebra towers[9], in particular in C^* and W^* algebra theory, see [GdlHJ89]. The transfer matrix of the IRF-models is naturally an element of the path algebra of the underlying

itself converges to the same limit.

[8] The commutant of a set $S \subset \mathcal{A}$ is the subalgebra of operators $S' \subset \mathcal{A}$ which commute with each $x \in S$.

[9] An Algebra tower is a sequence of algebras $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ with $\mathrm{id}_i \mapsto \mathrm{id}_{i+1}$.

graph, and these algebras build the bridge between the IRF models — in particular Pasquier's ADE models [Pas87] — and the construction of subfactors by Jones [Jon83, Wen88, Jon89, Jon90].

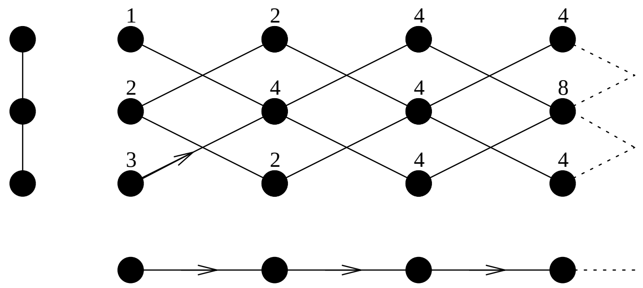
Given a graph $\mathcal{G}$ one defines its *path space* to be the Hilbert space $\mathcal{H}_{N,M}$ spanned by the orthonormal basis of the paths starting at the coordinate $n = N$ and ending at $n = M$. The path algebra $\Gamma_{N,M}$ of $\mathcal{H}_{N,M}$ is the subalgebra of operators in $\mathrm{GL}(\mathcal{H}_{N,M})$ that do not change the heights at the boundaries $n = N, M$. If we assume the graph to be simply laced, then the paths can be indexed by sequences of adjacent vertices $(\lambda_N \ldots \lambda_M)$. The condition of fixed boundaries at N, M induces for the matrix elements that

$$<\lambda_N \ldots \lambda_M|\, z \,|\mu_N \ldots \mu_M> \sim \delta_{\lambda_N \mu_N} \delta_{\mu_M \mu_M} \tag{1.31}$$

Often one treats path algebras on a *Bratteli graph.* For our purposes it is defined as the graph whose adjacency matrix is the tensor product of the graph and the 'time graph' $(Z, +1)$[10]. More explicitly, when we note λ the vertices of a graph $\mathcal{G}$, then the vertices of the Bratteli graph are given as the pair (λ, i) with i an integer, and (λ_2, i_2) adjacent (λ_1, i_1) iff λ_2 adjacent to λ_1 on $\mathcal{G}$ and $i_2 = i_1 + 1$. A *Bratteli diagram* is a Bratteli graph whose vertices (λ, i) carry integers $n_{\lambda,i}$ which satisfy $n_{\lambda,i+1} = \sum \mathcal{G}^{\lambda}{}_{\mu} n_{\mu,i}$ and hence

$$n_{\lambda,N} = \sum_{\mu} n_{\mu,M} \times \#\{ \text{ paths from } (\mu, M) \text{ to } (\lambda, N) \ \} \tag{1.32}$$

see figure 1.2 for an example. Often Bratteli diagrams are defined more generally, allowing different graphs at the different levels i, see [GdlHJ89]. The advantage using the associated Bratteli graph instead of the graph itself in order to represent a path algebra is that it contains an explicit time (or path length) coordinate.



The graph (left), the time graph (bottom) and the Bratteli graph constructed out of these two (center) together with the numbers $n_{\lambda,i}$ making a Bratteli diagram out of the graph. The numbers at the left hand side are chosen arbitrarily, the others are determined by (1.32)

Figure 1.2: A graph and its Bratteli diagram

The path algebra is a *multi-matrix-algebra*, that means the direct sum of finite dimensional matrix algebras and hence a von Neumann algebra. The center is the tensor product of the spaces

[10] Given a discrete set $\{a\}$ and a family of actions on it (f_i), we define the graph $\mathcal{G} = (\{a\}, (f_i))$ to be the graph whose vertices are the points a and whose adjacency matrix is $\mathcal{G}^b{}_a = \#\{f_i | f_i(a) = b\}$.

spanned by the central projections at the two boundaries. The space of traces can hence be identified with the space of physical boundary conditions. The factors contained in the path algebra $\Gamma_{M,N}$ are the restrictions ${\Gamma^{\nu,N}}_{\mu,M}$ onto the paths from vertex μ at M to ν at N .

In Jones subfactor theory [GdlHJ89] one considers the towers of path algebras ordered by increasing path length together with a given trace Tr and its (thermodynamic) limit under the GNS construction. The tower (inclusion-) structure of the path algebras is encoded in the adjacency matrix of the underlying graph $\mathcal{G}$: Like stated above, the algebras ${\Gamma^{\lambda,N+1}}_{\mu,M}$ and ${\Gamma^{\nu,N}}_{\mu,M}$ are factors, and ${\Gamma^{\nu,N}}_{\mu,M} \subset {\Gamma^{\lambda,N+1}}_{\mu,M}$ is naturally included with $\mathrm{id} \hookrightarrow \mathrm{id}$. More explicitly, the inclusion maps $x \hookrightarrow x \otimes \mathrm{id}_{{\mathcal{G}^\lambda}_\nu}$ with $\mathrm{id}_{{\mathcal{G}^\lambda}_\nu}$ the identity matrix of the ${\mathcal{G}^\lambda}_\nu$ dimensional multiplicity space. Hence this factor inclusion has the *Jones index*

$$[{\Gamma^{\lambda,N+1}}_{\mu,M} : {\Gamma^{\nu,N}}_{\mu,M}] = {\mathcal{G}^\lambda}_\nu \quad \text{and} \quad [{\Gamma^{\nu,N}}_{\lambda,M-1} : {\Gamma^{\nu,N}}_{\mu,M}] = {\mathcal{G}^\mu}_\lambda = {(\mathcal{G}^+)^\lambda}_\mu \tag{1.33}$$

Consequently, the natural traces ${\mathrm{tr}^{\nu,N}}_{\mu,M}$ on ${\Gamma^{\nu,N}}_{\mu,M}$ behave as

$$\begin{aligned} {\mathrm{tr}^{\nu,N+1}}_{\mu,M}(x) &= \sum_\alpha {\mathcal{G}^\nu}_\alpha \, {\mathrm{tr}^{\alpha,N}}_{\mu,M}(x) \\ {\mathrm{tr}^{\nu,N}}_{\mu,M-1}(x) &= \sum_\alpha {\mathrm{tr}^{\nu,N}}_{\alpha,M}(x) \, {\mathcal{G}^\alpha}_\mu \end{aligned} \tag{1.34}$$

For a special choice of the trace, explicitly the trace whose weight vector is the Perron-Frobenius vector of the graph, the inductive limit is a II_1-factor[11].

There is an interesting structure appearing here: in order to obtain a factor as the limit of an algebra tower, the trace of a given element z has to be independent of the algebra in the tower in which we compute it. This is only possible when we take as weight vector an eigenvector of the graph and when we normalize the trace with the corresponding eigenvalue γ. But this choice of the trace leads for any finite level of this construction to an algebra which is not a factor.

[11] A factor allowing arbitrary non-discrete trace values on the central projections. This can be interpreted as a continuous dimension for the associated subspaces.

Chapter 2

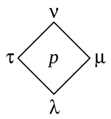
The Hecke algebra models

For the basic Hecke algebra models that we describe in sect. 2.1.1 we know two equivalent descriptions, the standard IRF-description where the transfer matrix is a matrix acting on a path space, and the Spin-description, where the face weights are determined by a representation of the quantum R-matrix. However, the general Hecke algebra models are a real generalization for which no Spin-description is known.

2.1 The IRF-description

In the IRF picture, the space of the local variables is spanned by the vertices of some given graph. The face weights induce the face transfer matrix

$$\mathrm{X} : V^{\otimes 3} \to V^{\otimes 3}, \quad \mathrm{X}^{\lambda'\tau\nu'}{}_{\lambda\mu\nu} = \delta_{\lambda,\lambda'}\delta_{\nu,\nu'}\, W(\lambda\mu\tau\nu) =: \delta_{\lambda,\lambda'}\delta_{\nu,\nu'} \tag{2.1}$$



with V being the vector space spanned by the local variables at each site. Note that, contrary to what is stated sometimes in the literature, X does not have the form $\mathrm{X} = \mathrm{id} \otimes X \otimes \mathrm{id}$, but $\mathrm{X} = \sum \mathbf{1}_\mu \otimes X_{\mu\nu} \otimes \mathbf{1}_\nu$ with $\mathbf{1}_\mu,\ \mathbf{1}_\nu$ central projections.

Because the weights vanish if (λ, μ, ν) or (λ', τ, ν) are not paths on the graph, it is consistent to interpret these face transfer matrices as being in the path algebra acting on paths of length 2, and hence the complete transfer matrix to be an element of the path algebra whose length is determined by the number of lattice sites in the space direction.

It is assumed that the face transfer matrix depending on the spectral parameter α

$$\mathrm{X}_i(q, \alpha) = F(q, \alpha)\mathrm{id} + G(q, \alpha)\, U_i \tag{2.2}$$

satisfies the *Yang-Baxter equation*

$$\mathrm{X}_i(\alpha_1)\mathrm{X}_{i+1}(\alpha_1 + \alpha_2)\mathrm{X}_i(\alpha_2) = \mathrm{X}_{i+1}(\alpha_2)\mathrm{X}_i(\alpha_1 + \alpha_2)\mathrm{X}_{i+1}(\alpha_1) \tag{2.3}$$

There

$$F(q,p) = \frac{qp^{-1} - q^{-1}p}{q - q^{-1}}, \quad G(q,p) = \frac{p - p^{-1}}{q - q^{-1}}, \quad p = e^{i\alpha} \text{ or } p = e^{\alpha} \tag{2.4}$$

We define η via $q = e^{i\eta}$. The solution with $p = e^{i\alpha}$

$$F(\eta,\alpha) = \frac{\sin(\eta - \alpha)}{\sin(\eta)}, \quad G(\eta,\alpha) = \frac{\sin(\alpha)}{\sin(\eta)} \tag{2.5}$$

is usually called the *trigonometric solution*, the other one the *hyperbolic solution*.

The U_i satisfy the Hecke algebra relations with parameter q

$$\begin{aligned} U_i\,U_j &= U_j\,U_i & |i-j| \geq 2 \\ U_i\,U_{i+1}\,U_i - U_i &= U_{i+1}\,U_i\,U_{i+1} - U_{i+1} \\ {U_i}^2 &= (q + q^{-1})\,U_i =: \beta\,U_i \end{aligned} \tag{2.6}$$

with $\beta := q + q^{-1} = 2\cos(\eta)$. These relations (2.4) to (2.6) guarantee the Yang-Baxter equation (2.3). After a changement in the generators[1] we obtain an equivalent set of relations for the $g_i = q - U_i$:

$$\begin{aligned} g_i\,g_j &= g_j\,g_i & |i-j| \geq 2 \\ g_i\,g_{i+1}\,g_i &= g_{i+1}\,g_i\,g_{i+1} \\ {g_i}^2 &= 1 + (q - q^{-1})\,g_i \end{aligned} \tag{2.7}$$

Due to the quadratic relation the only possible eigenvalues of g_i are q and $-q^{-1}$ and hence

$$g = -q^{-1}\,P_0 + q\,P_1, \quad P_iP_j = \delta_{i,j}P_i \text{ spectral projections} \tag{2.8}$$

The generator U is then a multiple of the projection onto the eigenvalue $-q^{-1}$.

The models are based on the quantum group $\mathrm{su(k)}_q$. The generators satisfy a supplementary condition, depending on k. In order to give this condition, we have to introduce yet another set of generators $x_i = (g_i)^{-1}$. Now we choose for every element σ of the symmetric group S_k a minimal decomposition $I(\sigma)$ of σ in terms of transpositions exchanging two neighboring elements τ_i. So $\sigma = \prod_{I(\sigma)} \tau_i$ and we pose $\hat{\sigma}(x_N \ldots x_{N+k-1}) := \prod_I x_{i+N}$. This operation is independent of the choice of the minimal decomposition[2]. The supplementary condition reads (see [Fra92])

$$\sum_{\sigma \in S_{k+1}} (-q)^{|\sigma|} \hat{\sigma}(x_N \ldots x_{N+k-1}) = 0 \quad \forall N \tag{2.9}$$

where $|\sigma|$ is the number of transpositions of $I(\sigma)$. For $k = 2$, the quotiented algebra is usually called the Temperley-Lieb or Temperley-Lieb-Jones algebra. We refer to the algebra with arbitrary k as *Temperley-Lieb (k)* algebra or as TL_k in that which follows.

For $q = 1$, when $\mathrm{su(k)}_q$ is the Lie algebra and the Hecke algebra becomes the algebra of the symmetric group, then the left hand side of (2.9) is the k-row antisymmetrizer acting on a tensor

[1] For an overview over the different sets of generator see Appendix C.

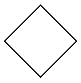
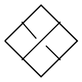
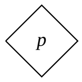
[2] The word 'minimal' here is crucial: taking for example $\tau^2 = \mathrm{id}$ we find of course that $X_{\tau^2} = X_\tau^2 \neq X_{\mathrm{id}}$.

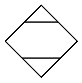
product of isomorphic vector spaces. The quotiented algebra $\mathrm{TL}_k(q=1)$ is hence the commutant of su(k) due to the Brauer-Weyl duality, the algebra of all representations of the symmetric group which are antisymmetric in not more than k indices [FH91, Fuc92]. This relation survives the deformation, so $\mathrm{TL}_k(N,q)$ is the commutant of $\mathrm{su(k)}_q$ in the N-fold tensor product of fundamental representations [Res87], a fact that will be crucial in that which follows.

For higher k, these antisymmetrizers operators become quite long, so in the literature explicit expressions are normally only given for $k=2,3$ [Fra92]

$$\begin{aligned} U_N\, U_{N+1}\, U_N - U_N &= 0 & k=2 \\ (U_N\, U_{N+1}\, U_N - U_N)(U_N + U_{N+1} - U_{N+2}\, U_{N+1}\, U_N) &= 0 & k=3 \end{aligned} \tag{2.10}$$

Finally we introduce the graphical notation for the different generators

$U =$  , $g =$  , $X_p =$ 

, id =  , (2.11)

2.1.1 The basic models

These models are strongly related to the underlying quantum group $\mathrm{su(k)}_q$: The algebra of the face transfer matrices satisfies the $\mathrm{su(k)}_q$ relation (2.9) and the graph on which this algebra is represented is the fundamental fusion graph of $\mathrm{su(k)}_q$. For q not a root of unity this is the fundamental fusion graph of the Lie algebra su(k), and hence the projection of the infinite oriented hypercube with boundary $(N^k, \pm 1)$ along the main diagonal $N(1,\ldots,1)$. It can e.g. be defined as a quotiented graph

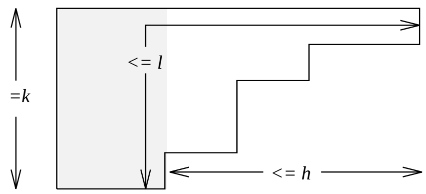
$$\mathcal{G}_{\mathrm{su}(k)} = (N^k, \pm 1)\, /\, N(1,\ldots,1) \sim 0 \tag{2.12}$$

The vertices of the graph are indexed with the Young diagrams with at most $k-1$ rows or equivalently with the classes of diagrams with k rows under the equivalence relation induced by adding columns with exactly k rows, which shows explicitly the quotienting above.

For q a root of unity, the graphs are truncated at a certain height h with[3]

$$q = e^{\pi i/l} \Longleftrightarrow h = l - k \tag{2.13}$$

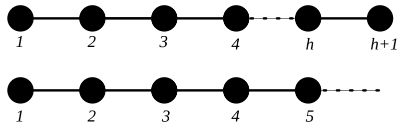
Truncation means that the longest row of the diagram contains at most h boxes or equivalently that the maximal hook length is at most l:



(2.14)

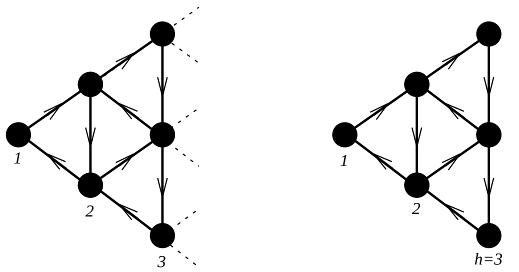
[3] See Appendix C for the different conventions concerning q.

We show the graphs for $\mathrm{su}(2)_q$ for generic q and $q = e^{\pi i/(h+2)}$



(2.15)

and for $\mathrm{su}(3)_q$ for generics q and with $q = e^{2\pi i/5}$



(2.16)

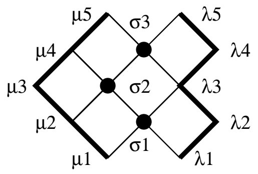
The transfer matrix T is completely determined when we know all its matrix elements $<\gamma|\, T\, |\gamma'>$. Because the fundamental fusion graph does not contain multiple links, the paths can be indexed with the heights they pass:

$$<\gamma|\, T\, |\gamma'> = <\lambda_1'\lambda_2'\ldots|\, T\, |\lambda_1\lambda_2\ldots> \tag{2.17}$$

Knowing the matrix elements of the face transfer matrices between the paths of length 2 – the face weights – one can compute the matrix elements of a product of them, T for example. For the bookkeeping of how to contract the indices, the graphical algorithm shown in the following example is quite convenient: Take $T = X_2X_1X_3$. Then

$$\begin{aligned} <\mu_1,\ldots,\mu_5|\, T^2\, |\lambda_1,\ldots,\lambda_5> = \sum_{\sigma_1\sigma_2\sigma_3} & W(\lambda_1\lambda_2\sigma_1\lambda_3)W(\lambda_3\lambda_4\sigma_3\lambda_5)W(\sigma_1\lambda_3\sigma_2\sigma_3) \\ & W(\mu_1\sigma_1\mu_2\sigma_2)W(\sigma_2\sigma_3\mu_4\mu_5)W(\mu_2\sigma_2\mu_3\mu_4) \end{aligned}$$

or graphically

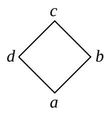
$$<\mu_1,\ldots,\mu_5|\, T^2\, |\lambda_1,\ldots,\lambda_5> = \sum_{\sigma_1,\sigma_2,\sigma_3}$$ 

(2.18)

2.1.2 The general Hecke algebra models

We call a *general Hecke algebra model* on a graph $\mathcal{G}$ an IRF-model whose face transfer matrix $X(q,u)$ is given by (2.2) with U being the image of the Hecke generator U under a representation of the Hecke algebra in the length 2 path algebra of $\mathcal{G}$. The U_i satisfy per definition the Hecke algebra relations (2.6) and hence the $X(u)$ satisfy the Yang-Baxter equation (2.3). In dependence of the supplementary condition (2.9) that the generators fulfill, we will call these models *Temperley-Lieb* (k) or TL_k models.

For $k = 2$ these models have been completely classified: these are Pasquier's ADE-models [Pas87, Pas88a] whose target graph is an $A-$ (for the basic models) or an $D-, E-$ (for the general models) Coxeter graph. The face weights are explicitly given as



$$= \delta_{a,c} \cdot \sqrt{\frac{\Psi_d \Psi_b}{\Psi_a \Psi_c}} \tag{2.19}$$

where Ψ_i is the i-th component of the Perron-Frobenius vector of $\mathcal{G}$.

For $k \geq 3$ however, neither a complete classification of the possible target graphs nor an explicit formula giving weights in terms of objects related to the graphs is known. The representations have been constructed case by case [Soc91, Fra92]. It seems that all the target graphs allow a representation of a su(k)$_q$ fusion algebra [Fra92]. More explicitly: let s, t, u indices of a Hilbert basis of the vertex vector space of $\mathcal{G}$, then there is a set of graphs $(\tilde{N}_\mu)^s{}_t$ indexed by the su(k)$_q$ unitary representations (μ) which satisfy the representation condition (1.24)

$$(\tilde{N}_\mu)^s{}_t (\tilde{N}_\nu)^t{}_u = N_{\mu\nu}{}^\lambda (\tilde{N}_\lambda)^s{}_u \quad \text{or} \quad \tilde{N}_\mu \cdot \tilde{N}_\nu = \sum_\lambda N_{\mu\nu}{}^\lambda \tilde{N}_\lambda \tag{2.20}$$

and such that the image of the fundamental representation is $\tilde{N}_1 = \mathcal{G}$

2.2 The Spin-description of the basic models

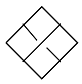
As announced above there is another equivalent description of the basic models: The local variables in this case are *spins* which are assigned to the links of the square lattice. More precisely, the local configuration space associated with each edge is a representation space of the quantum group su(k)$_q$ in the fundamental representation, and the Hecke generators appear as images of the quantum R-Matrices acting on these spaces.

The important consequence of the defining property of the R-matrix (1.20) for us is that

$$(\Pi \mathrm{R})\Delta(x) = \Delta(x)(\Pi \mathrm{R}) \tag{2.21}$$

and hence, when we define [SZ90]

$$g = $$



$$:= q^{\frac{1}{2}} \Pi \mathrm{R} \tag{2.22}$$

then g commutes with the image of the coproduct in $\mathcal{A} \otimes \mathcal{A}$. Because an element that is in the commutant of Δ_N is in the commutant of Δ_{N+M} as well – this can be shown using that $\Delta_{N+M}(x) = \Delta_N(x^{(1)}) \otimes \Delta_M(x^{(2)})$ – g is in the commutant of $\Delta_N, N \geq 2$.

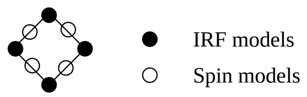
For the case that all the local algebras are in the fundamental representation one can even show completeness [Res87], which means that the commutant of su(k)$_q$ in the $N+1$ fold tensor product of fundamental representation spaces is generated (as an algebra) by the $g_i = \Pi_{ii+1} \mathrm{R}_{ii+1}$.

Furthermore, the g_i fulfill the Hecke algebra relations (2.7) as well as the supplementary condition (2.9) defining the Temperley-Lieb algebras TL_k. The face transfer matrix X_i is hence given as a linear combination (2.2) of g_i and id_i acting a priori on the local spaces i and $i+1$.

2.3 The relation between the two descriptions

Up to now we have seen that there are two types of models whose transfer matrices are generated by the U_i which satisfy the Temperley-Lieb relations (2.6) and (2.9). One can show that the spin models are equivalent to the basic models, the models whose associated graph is the fusion graph with the fundamental representation of $\mathrm{su(k)}_q$. We will outline this equivalence in that which follows; for a more detailed description see [Pas88b].

The concrete representation space of these two descriptions is different: In the first case, it is the path space of the fundamental fusion graph. For simply laced graphs, these paths can be indexed with the heights[4]. In the second case, the space is the N fold tensor product of fundamental representation spaces living on the links of the lattice.



Note that this direct correspondence that we establish now exists only between the basic models and the spin models. For the general Hecke models, whose graph is not the fusion graph, but rather one of its non-trivial representations (2.20), such a construction relating the two pictures is impossible.

A natural commutant basis. In that which follows we denote $V_i, i \geq 1$ the local space at site i in the spin picture, carrying a fundamental $\mathrm{su(k)}_q$ representation and V_0 an auxiliary space which carries every representation of the (truncated, see sect. 1.3.2) algebra exactly once. Then the coproduct Δ induces a representation $\Delta_{0\ldots N}$ of $\mathrm{su(k)}_q$ on $V_0 \otimes \cdots \otimes V_N$.

Let $\mathbf{1}_\lambda \in \mathrm{su(k)}_q$ denote the projection onto the irreducible representation associated with the Young tableau λ. The $\{\mathbf{1}_\lambda\}$ form a complete set of orthogonal projections. Let ${\mathbf{1}_\lambda}^N := \Delta_{0\ldots N}(\mathbf{1}_\lambda)$ denote the projection induced on $V_0 \otimes \cdots \otimes V_N$.

The projections $\mathbf{1}_\lambda$ are central in $\mathrm{su(k)}_q$, and with the same argument as after (2.22) one shows that all these projections commute

$$ {\mathbf{1}_\lambda}^N {\mathbf{1}_{\lambda'}}^{N'} = {\mathbf{1}_{\lambda'}}^{N'} {\mathbf{1}_\lambda}^N \quad \forall N, N', \lambda, \lambda' \tag{2.23} $$

and even more that they are central on $V_0 \otimes \cdots \otimes V_N$ in the sense that

$$ [\Delta_{0\ldots N}(\mathrm{su(k)}_q), {\mathbf{1}_\lambda}^M] = 0 \quad \forall M \leq N \tag{2.24} $$

[4] Recall that for the basic models the vertices of the graph are the unitary irreducible representations of the algebra.

Furthermore they are complete and orthogonal for a fixed N

$$\sum_\lambda \mathbf{1}_\lambda{}^N = \mathbf{1} \qquad \mathbf{1}_\lambda{}^N \mathbf{1}_{\lambda'}{}^N = \delta_{\lambda,\lambda'} \mathbf{1}_\lambda{}^N \quad \forall \lambda, \lambda' \tag{2.25}$$

We define the path projection onto the path $\gamma = (\lambda_0 \dots \lambda_N)$ to be

$$\mathbf{1}_\gamma \equiv \mathbf{1}_{(\lambda_0\lambda_1\dots\lambda_N)} := \mathbf{1}_{\lambda_0}{}^0 \mathbf{1}_{\lambda_1}{}^1 \dots \mathbf{1}_{\lambda_N}{}^N \tag{2.26}$$

With the construction above, the space $V_0 \otimes \cdots \otimes V_N$ carries a representation of su(k)$_q$ and $\mathbf{1}_\lambda{}^N$ is the projection onto the subspaces carrying representations of type λ. Hence the path projection projects the representation space of $\Delta_{0\dots i}$ down to a pure type λ_i space, or equivalently $\mathbf{1}_{(\lambda_0\lambda_1\dots\lambda_N)}\Delta_{0\dots i}$ is of pure type λ_i for $i = 0\dots N$.

In order to justify the name 'path projection' we note that only the projections $\mathbf{1}_{\lambda_0\dots\lambda_N}$ with λ_i adjacent λ_{i+1} on the fundamental fusion graph are non-vanishing: The projection $\mathbf{1}_\lambda{}^N$ makes a pure λ representation space out of $V_0 \otimes \cdots \otimes V_N$, and hence the only representations that can be carried by $(V_0 \otimes \cdots \otimes V_N) \otimes V_{N+1}$ are those contained in the tensor product of λ and the fundamental representation.

The equation (2.24) states exactly that all the $\mathbf{1}_\lambda{}^M$ for $M \leq N$ are in the commutant of $\Delta_{0\dots N}$. Furthermore, because the fundamental fusion graph is simply laced – equivalently this means that the fusion with the fundamental representation does not produce multiplicities – each of these projections $\mathbf{1}_\gamma$ is one dimensional (times the identity of a λ_N representation space) and hence the set of the $\{\mathbf{1}_\gamma\}$ projects onto a natural basis of the commutant of $\Delta_{0\dots N}$ on $V_0 \otimes \cdots \otimes V_N$.

Construction of the matrix elements. The Hecke algebra representation on $V_1 \otimes \cdots \otimes V_N$ induced via the R-matrix representations $R_{i,i+1}, i = 1\dots N-1$ (and hence on $V_0 \otimes \cdots \otimes V_N$ as well) lives in the commutant of $\Delta_{1\dots N}$ and so of $\Delta_{0\dots N}$. It follows from the representation theory of semi-simple algebras[5] that $V_0 \otimes \cdots \otimes V_N$ decomposes into standard su(k)$_q$ representation spaces V_λ and their respective commutants denoted $V_\lambda^{c,N}$, see [Fuc92]. In this basis, each $x \in$ su(k)$_q$ is represented as

$$\Delta_{0\dots N}(x) = \oplus_\lambda \mathrm{id}_\lambda^c \otimes x_\lambda \quad \text{and} \quad V_0 \otimes \cdots \otimes V_N = \bigoplus_\lambda V_\lambda^{c,N} \otimes V_\lambda \tag{2.27}$$

There x_λ is the image of x under a the standard representation of type λ. After Schur's lemma, each element z in the commutant of $\Delta_{0\dots N}$ can be decomposed in a similar manner as

$$[z, \Delta_{0\dots N}] = 0 \quad \Longrightarrow \quad z = \oplus_\lambda z_\lambda^c \otimes \mathrm{id}_\lambda \tag{2.28}$$

With the aid of the projections $\mathbf{1}_\gamma = \mathbf{1}_{\lambda_0\lambda_1\dots\lambda_N}$ above we constructed explicitly a basis of the commutant space: The vectors associated with the operators $\mathbf{1}_\gamma$ with λ_N fixed span $V_\lambda^{c,N}$ and

[5] See sect. 1.3.2 for semi-simplicity and truncation at q a root of unity

each element of the commutant z can be expressed in terms of its matrix elements in this basis $<\gamma|\, z\,|\gamma'>$. They can be computed as

$$<\gamma|\, z\,|\gamma'> = \frac{1}{\mathrm{tr}(\mathbf{1}_{\lambda_N})} \cdot \mathrm{tr}(\hat{\mathbf{1}} \cdot \mathbf{1}_\gamma z \mathbf{1}_{\gamma'}) \tag{2.29}$$

with the operator $\hat{\mathbf{1}}$ being defined as

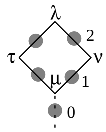
$$\hat{\mathbf{1}} := \sum_{\gamma\gamma'} |\gamma><\gamma'|\,, \quad <\gamma|\, \hat{\mathbf{1}}\,|\gamma'> = 1 \tag{2.30}$$

Note that this definition is basis dependent. We see from this equation that z can be interpreted to be in the path algebra of the graph, which means it keeps the boundaries fixed: z commutes with ${\mathbf{1}_{\lambda_N}}^N$ because of (2.24) and with ${\mathbf{1}_{\lambda_0}}^0$ because it acts non-trivially only on $V_1 \otimes \cdots \otimes V_N$ and hence $\lambda_N = \lambda'_N$ and $\lambda_0 = \lambda'_0$.

Matrix elements and tower inclusion. Up to now, we studied the case of a fixed path length N. Often we want to compose operators acting on different pieces of the path, for example U_1 and U_2 that live naturally in the algebras (012) and (123), but that are locally the same. In order to be able to compose them we have to interpret them as living in an algebra that is large enough to contain them both, here (0123). Evidently they do not change the paths outside their original domain, so the inclusions $(012) \subset (0123)$ and $(123) \subset (0123)$ are tower inclusions[6].

The matrix elements related to one operator z under this inclusion are related. This relation is the reason why we introduced the auxiliary space V_0 at the beginning of this section, which represents all the contributions from the representations below. So the matrix elements of any Hecke algebra element can be computed using the algorithm of (2.18) once we know the ones of U on $V_0 \otimes V_1 \otimes V_2$

$$<\mu\tau\lambda|\, U\,|\mu\nu\lambda> =$$



$$= \frac{1}{\mathrm{tr}(\mathbf{1}_\lambda)} \cdot \mathrm{tr}(\hat{\mathbf{1}} \cdot \mathbf{1}_{\mu\tau\lambda} U \mathbf{1}_{\mu\nu\lambda}) \tag{2.31}$$

This can be proven using the fact that an operator z which acts non-trivially only on $V_{m+1} \otimes \cdots \otimes V_n$ and that commutes there with the global action induced by $\Delta_{m+1\ldots n}$ commutes with ${\mathbf{1}_\lambda}^i$ for $i = 0 \ldots m$ and $i = n \ldots N$ and hence its matrix elements are

$$<\lambda_0 \ldots \lambda_N|\, z\,|\mu_0 \ldots \mu_N> = \delta_{\lambda_0\mu_0} \ldots \delta_{\lambda_m\mu_m} \delta_{\lambda_n\mu_n} \ldots \delta_{\lambda_N\mu_N} <\lambda_m \ldots \lambda_n|\, z\,|\mu_m \ldots \mu_n> \tag{2.32}$$

An example. The explicit calculation relating the matrix elements of spin and the vertex models is too lengthy to be given in detail here. It can be found in [Pas88b]. We just give an example for the calculus using a toy partition operator and restricting ourselves to non-deformed su(2).

[6] See footnote 9 on page 12.

The operator that we are interested in for this example is

$$z_2 := e^{-(\vec{J}_{(1)}+\vec{J}_{(2)})^2} \tag{2.33}$$

where the $\vec{J}$ are su(2) spins in the fundamental spin 1/2 representation. We index the su(2) representations with $\lambda = 2s+1$. Hence the λ are positive integers, $\lambda = 1$ is the trivial and $\lambda = 2$ is the fundamental representation. The fusion rules give the su(2) structure constants

$$N_{\lambda\mu}{}^{\nu} = \sum_{\alpha=(|\lambda-\mu|+1)/2}^{(\lambda+\mu-1)/2} \delta_{\nu,2\alpha} \tag{2.34}$$

in particular $N_{\mu 1}{}^{\nu} = \delta_{\nu,\mu-1} + \delta_{\nu,\mu+1}$ whenever $\mu \geq 2$ and hence the paths of a model with the local spins being in the fundamental representation satisfy $\lambda_{i+1} = \lambda_i \pm 1$. The operator z_2 is in the commutant of Δ_2, so the minimum path length is 3. The only possible paths are $|\lambda, \lambda+1, \lambda+2>$, $|\lambda, \lambda+1, \lambda>$, $|\lambda, \lambda-1, \lambda>$ and $|\lambda, \lambda-1, \lambda-2>$. The matrix element of $\vec{J}^2$ then becomes

$$\mathbf{1}_{\lambda_1,\lambda_2,\lambda_3} \vec{J}^2 \mathbf{1}_{\lambda_1',\lambda_2',\lambda_3'} = |\lambda_1, \lambda_2, \lambda_3><\lambda_1, \lambda_2, \lambda_3| \, \vec{J}^2 \, |\lambda_1', \lambda_2', \lambda_3'><\lambda_1', \lambda_2', \lambda_3'| \tag{2.35}$$

The operator z_2 acts trivially on the λ_1 space. Furthermore, $\mathbf{1}_{\lambda_1\lambda_2\lambda_3} = P^1_{\lambda_1} P^2_{\lambda_2} P^3_{\lambda_3}$. Both exterior projections P^1 and P^3 commute and hence the matrix element vanishes if $\lambda_1 \neq \lambda_1'$ or $\lambda_3 \neq \lambda_3'$.

The operator $\vec{J}^2$ is diagonal in the fusion basis (32)1, but the matrix elements are computed in the basis 3(21), hence we have to use the $6j$-symbols

$$|\lambda_1, \lambda_2, \lambda_3> = \sum_{\lambda} \left\{ \begin{array}{ccc} \lambda_1 & 1/2 & \lambda_2 \\ 1/2 & \lambda_3 & \lambda \end{array} \right\} |\lambda_1(\lambda)\lambda_3> \tag{2.36}$$

Introducing a complete set of projections we get

$$<\lambda_1, \lambda_2, \lambda_3| \, \vec{J}^2 \, |\lambda_1, \lambda_2', \lambda_3> = \sum_{\lambda} \frac{1}{4}(\lambda^2-1)<\lambda_1, \lambda_2, \lambda_3|\lambda_1(\lambda)\lambda_3><\lambda_1(\lambda)\lambda_3|\lambda_1, \lambda_2', \lambda_3> \tag{2.37}$$

and hence

$$<\lambda_1, \lambda_2, \lambda_3| \, z_2 \, |\lambda_1, \lambda_2', \lambda_3> = \sum_{\lambda} e^{-\frac{1}{4}(\lambda^2-1)} \overline{\left\{ \begin{array}{ccc} \lambda_1 & 1/2 & \lambda_2 \\ 1/2 & \lambda_3 & \lambda \end{array} \right\}} \left\{ \begin{array}{ccc} \lambda_1 & 1/2 & \lambda_2' \\ 1/2 & \lambda_3 & \lambda \end{array} \right\} \tag{2.38}$$

Equation (2.32) gives then for longer paths

$$<\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots| \, z_2 \, |\ldots, \lambda_{-1}', \lambda_0', \lambda_1', \ldots> = <\lambda_{-1}, \lambda_0, \lambda_1| \, z_2 \, |\lambda_{-1}, \lambda_0', \lambda_1> \cdot \prod_{i\neq 0} \delta_{\lambda_i \lambda_i'} \tag{2.39}$$

We recall in this context that the Hecke generator U for a su(2) model is the projection onto the representation $\lambda = 1$ on the two fold tensor product of fundamental representation spaces, and hence

$$U_i = \mathrm{id} - \frac{4}{3} J^2_{ii+1} \tag{2.40}$$

where $J_i^2 = \Delta_{ii+1}(J^2)$ is the su(2) Casimir operator acting on the spaces i and $i+1$.

Chapter 3

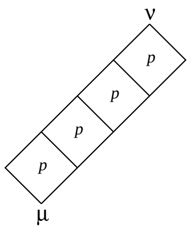
Partition functions

In this chapter, we study the partition function Z and the associated free energy $f = -\log Z$. Recall from sect. 1.2 that the space-size N partition function Z^N of a tube of width 1 and length N can be expressed as the trace of a partition operator $z^N = T^{(N)}$ with $T^{(N)}$ being the transfer matrix acting naturally on paths of length $N+1$.

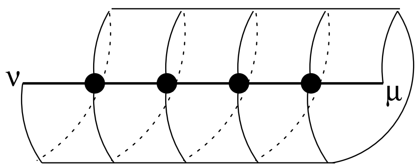
The argumentation used here is valid in the two pictures – except the translation invariance decomposition (3.6) which has to be conjectured for some IRF models – and hence even if most formulas are given rather in a spin picture notation, the translation into IRF picture notations should be obvious.

Recall from sect. 2.1 that the transfer matrix itself is given as a product over the face transfer matrices and so $z^N = T^{(N)} = X_N \ldots X_1$. Recall that the product there has to be understood as being evaluated in an algebra which is large enough to contain all the interesting X_i, in the thermodynamical limit hence the inductive algebra limit (see sect. 1.4), and that the partition function Z is a matrix in the space of boundary conditions in space direction. Note in this context that the fact that the transfer matrix keeps the two boundaries fixed can be interpreted – even for the partition operator z being an arbitrary power of T – as an identification of all the upper and lower boundary spins. These lead to a closure of the tube at the two ends and hence to a spherical or *sausage geometry* with the two poles fixed.

$Z^{\nu}{}_{\mu} =$



$\equiv$ tr



(3.1)

We note that we introduced a length scale – the radius of the tube – and that we hence expect the conformal symmetry to break down even in the thermodynamic limit.

In section 3.1 we derive a symmetry property of the partition functions in the different path algebra sectors $Z^{\mu}{}_{\nu}$. called *translation invariance.* It states that the $Z^{\mu}{}_{\nu}$ are determined by the $Z^{\lambda} := Z^{\lambda}{}_{0}$.

In sect. 3.2 we consider the thermodynamic limit process. In a first step, we take the algebra limit, embedding all the X_i in a huge algebra where the product is internal, and we study the conditions under which there is a trace operation which allows us to extract the measurable quantities out of the operator formalism.

In the remaining part of this chapter, we use the generalized Temperley-Lieb condition in order to derive a recurrence formula for the Z^N, which turns out to be universal, and we compute the free energy of an arbitrary su(2) system explicitly, finding a non-analytical behaviour in dependence of the spectral parameter.

3.1 Reducing the partition function via fusion

At first, we treat the basic models. We saw in sect. 2.3 that a central element z is determined by its matrix elements $z^{\gamma}{}_{\gamma'}$ between paths starting at 0 and that the other matrix elements are related, even if an explicit formula describing this dependence is not obvious. Taking the trace, the dependencies simplify and we are able to give the explicit formulas.

Recall that we computed the matrix elements between paths starting at λ_0 introducing an auxiliary space V_0 which is reduced by $\mathbf{1}_{\lambda_0}{}^0$ to a multiplicity one λ_0 representation space. The partition function in the sector (λ_0, λ_N) is then given by the expression

$$Z^{\lambda_N}{}_{\lambda_0} = \frac{1}{\mathrm{tr}\mathbf{1}_{\lambda_N}} \mathrm{tr}_{V_0\otimes\cdots\otimes V_N}(\mathbf{1}_{\lambda_0}{}^0\, \mathbf{1}_{\lambda_N}{}^{0\ldots N} \cdot z^N) \tag{3.2}$$

which means, the partition function between the boundaries (λ_N, λ_0) is given as a trace over the product of the – spectral parameter α dependent – face transfer operators $X_N(\alpha)\ldots X_1(\alpha)$. In the IRF-picture this trace is computed as a sum over paths with the boundaries fixed to (λ_N, λ_0), in the spin picture – this is the expression (3.2) – it is a trace over a tensor product of representation spaces, with the weight vector of the trace being the central projection $(\mathbf{1}_{\lambda_0}{}^0\, \mathbf{1}_{\lambda_N}{}^{0\ldots N}$.

Like in sect. 2.3 we now use the fact that each element in the commutant of a unitary representation of $\mathrm{su(k)}_q$ can be written as $z = \sum z^{\lambda} \otimes \mathbf{1}_{\lambda}$ where $\mathbf{1}_{\lambda}$ is the identity of the irreducible representation of type λ in $\mathrm{su(k)}_q$ — and hence the central projection onto this representation — and z^{λ} is an element of the commutant of the type λ representations, and we define the projections $\mathbf{1}_{\lambda}{}^{1\ldots N} := \Delta_{1\ldots N}(\mathbf{1}_{\lambda})$ which are central in $\Delta_{1\ldots N}(\mathrm{su(k)}_q)$ acting on $V_1\otimes\cdots\otimes V_N$. Then (3.2) becomes

$$Z^{\lambda_N}{}_{\lambda_0} = \frac{1}{\mathrm{tr}\mathbf{1}_{\lambda_N}} \cdot \sum_{\lambda} \mathrm{tr}_{V_0\otimes\cdots\otimes V_N}(\mathbf{1}_{\lambda_0}{}^0 \cdot \mathrm{id}^0 \otimes (z^{\lambda} \otimes \mathbf{1}_{\lambda}{}^{1\ldots N}) \cdot \mathbf{1}_{\lambda_N}{}^{0\ldots N}) \tag{3.3}$$

Using that the trace over a factorizing element of a tensor product space factorizes $\mathrm{tr}_{A\otimes B}(a\otimes b) \sim \mathrm{tr}_A(a)\cdot \mathrm{tr}_B(b)$ and the coassociativity of Δ we get

$$Z^{\lambda_N}{}_{\lambda_0} = \frac{1}{\mathrm{tr}\mathbf{1}_{\lambda_N}} \cdot \sum_{\lambda} \mathrm{tr}_{(V_1\otimes\cdots\otimes V_N)^{c,\lambda}}(z^{\lambda}) \cdot \mathrm{tr}_{V_0\otimes V^r}(\mathbf{1}_{\lambda_0}{}^0 \mathbf{1}_{\lambda}{}^r \mathbf{1}_{\lambda_N}{}^{0\otimes r}) \tag{3.4}$$

where $(V^r, \mathbf{1}_\lambda{}^r)$ represent the (multiplicity one) irreducible representation spaces of the representation of su(k)$_q$ on $V_1 \otimes \cdots \otimes V_N$ and $(V_1 \otimes \cdots \otimes V_N)^{c,\lambda}$ is the multiplicity space of the representations of type λ on $V_1 \otimes \cdots \otimes V_N$. Comparing this expression with (1.22) we see that the second term gives the fusion coefficients $N_{\lambda_0 \lambda}{}^{\lambda_N}$.

We call the first term

$$Z^\lambda := Z^\lambda{}_0 = \mathrm{tr}_{(V_1 \otimes \cdots \otimes V_N)^{c,\lambda}}(z^\lambda) = \frac{1}{\mathrm{tr}\mathbf{1}_\lambda} \mathrm{tr}_{V_1 \otimes \cdots \otimes V_N}(\mathbf{1}_\lambda{}^{1\ldots N} \cdot z) \tag{3.5}$$

the *reduced partition function* and we rewrite (3.4) in order to define the condition which we call *translation invariance*[1]

$$Z^\nu{}_\mu = \sum_\lambda Z^\lambda \cdot N_{\lambda\mu}{}^\nu \tag{3.6}$$

As a consequence of this equation and because of the commutativity of the fusion algebra (1.24), the common eigenvectors of the fusion matrices N_λ also diagonalize the partition function $Z^\lambda{}_\nu$. As in (1.25) we call $\gamma(a,\lambda)$ the eigenvalues of the N_λ in the common eigenbasis $|a\rangle$. We call the reduced partition function expressed in this basis

$$Z^a := \sum_\lambda \gamma(a,\lambda)\, Z^\lambda \tag{3.7}$$

the generalized *Fourier transformation* of Z^λ.

3.1.1 Reduced partition function in the general Hecke algebra models

We conjecture here the formula generalizing (3.6) for the general Hecke algebra models introduced in sect. 2.1.2. We recall that beside the graph $\mathcal{G}$ whose path algebra carries the Hecke algebra representation there exists a family of graphs $\tilde{N}_\mu$ with $(\tilde{N}_\mu)^t{}_s = \tilde{N}_{\mu s}{}^t$ satisfying the representation condition (2.20) and with $\tilde{N}_1 = \mathcal{G}$.

Then the partition function in the (s,t) path sector $\tilde{Z}^t{}_s = \mathrm{tr}^t{}_s(\ldots \tilde{X}_2 \tilde{X}_1)$ – here the $\tilde{X}_i$ are defined by (2.2) with U_i being the representation of the Hecke generator in the path algebra of $\mathcal{G}$ – of this model is given with the coefficients Z^λ above as

$$\tilde{Z}^t{}_s = \sum_\lambda Z^\lambda \cdot \tilde{N}_{\lambda s}{}^t \tag{3.8}$$

There is a variety of arguments which make this conjecture plausible. For the case of su(2) it can be rigorously proven using the explicit formula for the weights (2.19) [Pas88b]. Some others rely on the connection between modular invariance and partition functions of systems with boundaries [Car89, Zub].

[1] This is a translation invariance in the target space of the $\{\mu\}$. The name is due to the fact that for the Lie algebra $u(1)$, whose unitary representations are given by $(\lambda): U(1) \ni e^{i\alpha} \mapsto e^{i\lambda a} \in \mathrm{End}V$ with λ an arbitrary integer, the fusion coefficients are $N_{\mu\nu}{}^\lambda = \delta_{\lambda,\mu+\nu}$ and hence a translation invariant function satisfies $Z^\nu{}_\mu = Z^{\nu-\mu}$, see sect. 4.1.2.

Yet another argument is given by an interpretation of the above formula in terms of characters of the Hecke algebra [Fra92]: the unitary representations of the Temperley-Lieb (k) algebras at q a root of unity can be indexed with the Young diagrams appearing in the $\mathrm{su(k)}_q$ fusion graph [Wen88], and Z^λ is due to the definition the character of z under the representation of type λ. So the character of z under an arbitrary Hecke algebra representation can be decomposed into the fundamental characters Z^λ. Formula (3.8) describes this decomposition, allowing hence to interpret $\tilde{N}_{\lambda s}{}^t$ as the multiplicity of λ of the representation in the (s,t) sector of the path algebra.

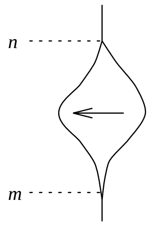
3.2 Thermodynamic limit I: the algebras

We study the thermodynamic limit process in two steps: at first, we take the algebra limit keeping the partition operator z^N fixed but interpreting it as living in a tower of algebras via the identification $z^N \cong z^N \otimes \mathrm{id}$. Physically this corresponds to a system which is at finite temperature only in the area associated with N. In a second step, we look at the behaviour when we pass to an infinite system of finite temperature taking the limit $N \to \infty$.

The transfer matrices $T^{(N)}$ and hence the partition operator of the tube $z^N = T^{(N)}$ live in different algebras, the path algebras $\Gamma_{m,n}$ acting on paths running on the fundamental fusion graph or its Bratteli graph (see sect. 1.4.3) on the sites labelled from m to $n = m + N + 1$. They build a (two dimensional) tower of algebras. The *tower inclusions* are injective algebra homomorphisms which map the identity onto the identity (see the paragraph before (2.32) as well) leading to the inclusion structure

$$\begin{array}{ccccccccccc}
 & & & & & & \cdots & & & & \\
 & & & & \Gamma_{1,1} & \subset & \Gamma_{1,2} & \subset & \Gamma_{1,3} & \subset & \cdots \\
 & & & & \cap & & \cap & & \cap & & \\
 & & \Gamma_{0,0} & \subset & \Gamma_{0,1} & \subset & \Gamma_{0,2} & \subset & \Gamma_{0,3} & \subset & \cdots \\
 & & \cap & & \cap & & \cap & & \cap & & \\
\Gamma_{-1,-1} & \subset & \Gamma_{-1,0} & \subset & \Gamma_{-1,1} & \subset & \Gamma_{-1,2} & \subset & \Gamma_{-1,3} & \subset & \cdots \\
\cdots & & & & & & & & & &
\end{array} \tag{3.9}$$

This tower structure leads to an interpretation of the $\Gamma_{m,n}$ to be an algebra acting on paths of an arbitrary length, keeping the paths in the area outside of (m,n) fixed

$\Gamma_{m,n} \sim$ 

(3.10)

We recall however, that $\Gamma_{m,n}$ defined via this equivalence relation does not posses a natural trace operation.

All these algebras are included in the inductive limit $\Gamma_{-\infty,\infty} \supset \Gamma_{m,n}$. The algebra $\Gamma_{-\infty,\infty}$ is an infinite dimensional algebra and hence the trace operation is non-trivial. What we are looking for is a trace on $\Gamma_{-\infty,\infty}$, we call it 'Tr', which is defined by its restrictions to all the $\Gamma_{m,n}$ such that $\mathrm{Tr}^{b,n'}{}_{a,m'}(z_{n,m})$ does not depend on n', m' once it is defined.

Construction of an invariant trace. Starting the other way around, we study how the natural traces 'tr' on the finite dimensional algebras $\Gamma_{m,n}$ behave. Recall from sect. 1.4 that $\Gamma_{m,n}$ is a multi-matrix algebra whose center is spanned by the projections onto the paths running from μ to λ and that a trace is uniquely determined by its weight vector, i.e. its values on the central projections of the algebra.

At first we study the case of boundary conditions fixed to some vectors a and b, hence the trace is computed with the trace vectors $|a\rangle = (a^\mu)$ and $\langle b| = (b_\nu)$[2]. For boundary conditions fixed to a certain vertex μ_0 we had for example $a^\mu = \delta_{\mu,\mu_0}$. Then the trace of a fixed operator z^N is given as

$$\mathrm{tr}^{b,n}{}_{a,m}(z^N) = b_\nu\, Z^{\nu,n}{}_{\mu,m}\, a^\mu \equiv \langle b|\, Z_{m,n}\, |a\rangle, \qquad \text{with } Z^{\nu,n}{}_{\mu,m} = \mathrm{tr}^{\nu,n}{}_{\mu,m}(z^N) \tag{3.11}$$

where the *natural partition function* Z is defined with the natural trace $\mathrm{tr}^{\nu,n}{}_{\mu,m}$ on the natural factor decomposition of $\Gamma_{m,n}$ into finite dimensional matrix algebras, see sect. 1.4.3.

Let now the underlying graph be $\mathcal{G}$. We know from (1.34) that

$$Z^{\nu,n+1}{}_{\mu,m} = \sum_\alpha \mathcal{G}^\nu{}_\alpha\, Z^{\alpha,n}{}_{\mu,m} \quad \text{and} \quad Z^{\nu,n}{}_{\mu,m-1} = \sum_\alpha Z^{\nu,n}{}_{\alpha,m}\, \mathcal{G}^\alpha{}_\mu \tag{3.12}$$

Taking as weight vectors eigenvectors of $\mathcal{G}$ (we suppose the graph to be normal)

$$\mathcal{G}\,|a\rangle = \gamma_a\,|a\rangle \quad \langle b|\,\mathcal{G} = \langle b|\,\gamma_b^* \quad \langle a|b\rangle = \delta_{a,b} \tag{3.13}$$

then we obtain for the natural partition function defined with the trace in $\Gamma_{m-m',n+n'}$ as a function of the one in $\Gamma_{m,n}$

$$Z^{b,n+n'}{}_{a,m-m'} = \gamma_a^{m'}\gamma_b^{n'} \cdot Z^{b,n}{}_{a,m} \tag{3.14}$$

$$Z^{\nu,n+n'}{}_{\mu,m-m'} = \sum_{a,b} \gamma_a^{m'}\gamma_b^{n'} \cdot Z^{n,b}{}_{m,a} \cdot \langle\nu|b\rangle\langle a|\mu\rangle \tag{3.15}$$

whenever $Z_{m,n}[z^N]$ is defined.

We define a trace $\mathrm{Tr}^b{}_a$ on $\Gamma_{-\infty,\infty}$ which is invariant under the tower inclusions with the following construction: Suppose that we are interested in the trace of an element $z \in \Gamma_{-\infty,\infty}$. We choose a $z_{m,n} \in \Gamma_{m,n}$ such that its image under the inclusion $\Gamma_{m,n} \subset \Gamma_{-\infty,\infty}$ is z and we define

$$\mathrm{Tr}^b{}_a(z) := \gamma_a^{-n}\gamma_b^m \mathrm{tr}^{b,n}{}_{a,m}(z_{m,n}) \tag{3.16}$$

[2]Raising and lowering the vector indices is defined with respect to the standard complex scalar product, so $a^\mu = (a_\mu)^*$ is the complex conjugated c-number.

This definition is independent of the choice of (m, n). We call the analogue of Z which is computed with the invariant trace the *renormalized partition function* $\bar{Z}$

$$\bar{Z}^b{}_a[z] := <b|\,\mathrm{Tr}(z)\,|a> \tag{3.17}$$

Because $Z^\nu{}_\mu$ is translation invariant (3.6) and the matrices N_λ commute, the eigenvectors of $\mathcal{G} = N_2$ diagonalize all the N_λ and hence Z. Like in (1.25) we denote these eigenvectors $|a>$ and its eigenvalues $\gamma(a, \lambda)$, so in particular $\gamma_a = \gamma(a, 2)$. Inserting the reduced partition function (3.5)

$$Z^{N,\lambda} := \mathrm{tr}_{(V_1\otimes\cdots\otimes V_N)^{c,\lambda}}(z^{N,\lambda}) = \frac{1}{\mathrm{tr}\mathbf{1}_\lambda}\mathrm{tr}_{V_1\otimes\cdots\otimes V_N}(\mathbf{1}_\lambda{}^{1\ldots N}\cdot z^N) \tag{3.18}$$

and using the translation invariance decomposition (3.6) we can write the invariant trace in terms of the Fourier transformed partition function (3.7) as

$$\mathrm{Tr}^b{}_a(z) = \delta_{a,b}\,\gamma_a^{-N}\left(\sum_\lambda \gamma(a,\lambda)\,Z^{N,\lambda}\right) = \delta_{a,b}\cdot\gamma_a^{-N} Z^{N,a} \equiv \delta_{a,b}\cdot\bar{Z}^a \tag{3.19}$$

The natural partition function finally becomes in the Fourier basis

$$Z^{b,n}{}_{a,m} = \gamma_a^{n-m}\,\bar{Z}^b{}_a = \delta_{a,b}\,\gamma_a^{n-m}\,\bar{Z}^a \tag{3.20}$$

and in the canonical vertex space basis

$$<\nu|\,Z_{m,n}\,|\mu> = \sum_a <\nu|a><a|\mu>\,\gamma_a^{n-m}\cdot\bar{Z}^a \tag{3.21}$$

General boundary conditions. We derived the formulas above for the special case of fixed boundary conditions. Now we consider the general case, where the physical partition function is given as $Z_B = \mathrm{tr}(B\cdot Z)$ like in (1.11). We look at a closed chain of length $N + L + 1$ and a $N + 1$ sites partition operator z^N. Applying formula (3.12) gives for the partition function with boundary condition matrix $B^\lambda{}_\mu$

$$Z_B^{N+L} = \sum_{\mu\nu\lambda}(\mathcal{G}^L)^\mu{}_\nu\,Z^\nu{}_\lambda\,B^\lambda{}_\mu \equiv \mathrm{tr}(G^L\cdot Z\cdot B) \tag{3.22}$$

Evaluating the trace as the sum over the common eigenbasis $|a>$ of the N_λ and hence $\mathcal{G}$ and Z, using the translation invariance decomposition (3.6) and the commutativity of the fusion algebra, and introducing a complete system $\sum|\lambda><\lambda|$ between Z and B we obtain that

$$Z_B^{N+L} = \sum_{a,\lambda}<a|\,B\,|a>\,\gamma_a^L\,\gamma(a,\lambda)Z^{N,\lambda} = \sum_a <a|\,B\,|a>\,\gamma_a^L\,Z^{N,a} \tag{3.23}$$

We see that the only possibility to obtain an invariant trace is to take it in each sector separately as

$$\mathrm{Tr}^a(z) = \gamma_a^{-N} Z^{N,a} \tag{3.24}$$

3.3 Thermodynamic limit II: the operators

Now we examine the physical thermodynamic limit. Mathematically speaking, we have a sequence of partition operators z^N all seen as elements of a limit-algebra Γ_∞ and what we are interested in is the dependence of $Z^N = \mathrm{tr}(z^N)$ on the system size in space direction N. Recall that $Z = Z^\mu{}_\nu$ (or $\tilde{Z}^t{}_s$) is a matrix in the two fold reduced (and hence zero-dimensional) configuration space of path boundary conditions and that this matrix is determined by the reduced partition function $Z^{N,\lambda}$ via translation invariance (3.6) and (3.8).

3.3.1 The recurrence formulas for the basic models

In order to illustrate the formalism on a non-physical example, we take at first z^N to be the deformation of the cyclic permutation $g_N \ldots g_1$. The object that we compute here is the reduced partition function Z^λ of (3.5).

$$Z^{N,\lambda} = \mathrm{tr}^{N,\lambda}{}_{0,0}(z^N) = \frac{1}{\mathrm{tr}(\mathbf{1}_\lambda)} \mathrm{tr}_N(g_N \ldots g_1 \cdot \mathbf{1}_\lambda{}^N) \tag{3.25}$$

For $q = 1$ we can evaluate this expression with the aid of the Brauer-Weyl duality between representations of the symmetric- and the linear groups. Then Z^N is the S_{N+1} character of the cyclic permutation of length $N+1$ which is known to be an alternating sum over the hook diagrams. We show in appendix A that the $Z^{N,\lambda}$ satisfy the matrix recurrence relation

$$\sum_{i=0}^{k} (-1)^i N_i \cdot Z^{N-i} = 0 \tag{3.26}$$

with the fundamental fusion matrices N_i.

In the following theorem we give the generalization of this formula for arbitrary q.

Theorem & Conjecture 3.3.1 (Recurrence I) *Let $k = 2$, $N \geq k+1$ and let the non-physical partition function $Z^{N,\lambda} = \mathrm{tr}_N^\lambda(g_N \ldots g_1)$ be the trace in the sector of paths running from 0 to λ, and let N_i denote the representation of the fusion matrices corresponding to the fundamental representations (i) of $\mathrm{su}(k)_q$ on the vertex space of the graph if $0 \leq i \leq k$ and to be identically zero if not. Then*

$$\sum_{i=0}^{\infty} (-q)^i N_i \cdot Z^{N-i} = 0$$

The sum is in fact a sum up to k and the multiplication is the matrix multiplication $(N_i \cdot Z^N)^\mu \equiv (N_i)^\mu{}_\nu \, Z^{N,\nu}$.

The same recurrence formula holds for $N = k$ when we set $Z^0 := N_1$.

We conjecture that this theorem holds for any $k \geq 3$.

Proof. We give here the proof for $k = 2$. It uses basically the Temperley-Lieb relations including the supplementary condition (2.9), the cyclicity of the trace and the knowledge of Z_2.

(1) We write the Temperley-Lieb condition (2.9) in the generators g_i as (see [SZ90])

$$1 - q^{-1}(g_1 + g_2) + q^{-2}(g_1 g_2 + g_2 g_1) - q^{-3}(g_1 g_2 g_1) = 0$$

(2) Multiplying this with $g_N \dots g_3 =: x g_3$ and taking the trace over the paths of length N we get

$$\begin{aligned} &\mathrm{tr}_N(x g_3) - q^{-1}\mathrm{tr}_N(x g_3 g_1) - q^{-1}\mathrm{tr}_N(x g_3 g_2) \\ &\quad + q^{-2}\mathrm{tr}_N(x g_3 g_1 g_2) + q^{-2}\mathrm{tr}_N(x g_3 g_2 g_1) - q^{-3}\mathrm{tr}_N(x g_3 g_1 g_2 g_1) = 0 \end{aligned}$$

(3) Evidently $\mathrm{tr}_N(g_N \dots g_{k+1}) = N_1^k \cdot Z^{N-k}$ in the sense of matrices. We simplify each term, using the Hecke relations (2.7) and the cyclicity of the trace in order to obtain

$$N_1^2 \cdot Z^{N-2} - q^{-1}(Z^{N-2} \cdot Z^1) - q^{-1} N_1 \cdot Z^{N-1} + q^{-2} Z^N + q^{-2} Z^N - q^{-3}(N_1 \cdot Z^{N-1} + (q - q^{-1}) Z^N) = 0$$

The second term is meant as $(Z^{N-2} \cdot Z^1)^\mu = (Z^{N-2})^\mu{}_\lambda Z^{1,\lambda} = Z^{N-2,\tau} Z^{1,\lambda} N_{\tau\lambda}{}^\mu$ which follows from the factorization of the trace over $x g_3 g_1$ and translation invariance in the index i of the g_i.

(4) One shows by inspection that $\mathrm{tr}_2(U) = \beta N_2$ in the sense of matrices. Given that $g = q - U$ (see app. C) it follows that $Z^1 = \mathrm{tr}(g) = q N_1^2 - \beta N_2$. Using these relations we get

$$(q^{-2} + q^{-4}) Z^N - (q^{-1} + q^{-3}) N_1 Z^{N-1} + (1 + q^{-2}) N_2 Z^{N-2} = 0$$

Dividing through $(q^{-2} + q^{-4})$ we obtain the desired result for $\mathrm{su}(2)_q$. ∘

In order to prove the recurrence for $N = k$ we take the trace of the supplementary condition itself and verify that the connected part with the highest indices produces the expressions in $Z^{N-1} \dots Z^0$ when we set $Z^0 := N_1$.

(2a) Taking the trace of the supplementary condition we get

$$\begin{aligned} &\mathrm{tr}_2(1) - q^{-1}\mathrm{tr}_2(g_1) - q^{-1}\mathrm{tr}_2(g_2) \\ &\quad + q^{-2}\mathrm{tr}_2(g_1 g_2) + q^{-2}\mathrm{tr}_2(g_2 g_1) - q^{-3}\mathrm{tr}_2(g_1 g_2 g_1) = 0 \end{aligned}$$

(3a,4a) Evaluating the above traces, we interpret the connected part with the highest indices as $Z^N, N = 0 \dots 2$, such that

$$\mathrm{tr}_2(1) = N_1^2 \cdot Z^0, \quad \mathrm{tr}_2(g_1) = \mathrm{tr}(g) \cdot Z^0, \quad \mathrm{tr}_2(g_2) = N_1 \cdot Z^1, \quad \mathrm{tr}_2(g_2 g_1) = Z^2$$

Inserting this we obtain

$$N_1^2 \cdot Z^0 - q^{-1}(q N_1^2 - (q + q^{-1}) N_2) \cdot Z^0 - q^{-1} N_1 \cdot Z^1 + 2 q^{-2} Z^2 - q^{-3}(N_1 \cdot Z^1 + (q - q^{-1}) Z^2) = 0$$

which gives the desired result when collecting the terms and dividing through $q^{-2} + q^{-4}$ ∘

The conjectured formula is known to be true for the case $q = 1$ for any k. Because of the increasing complexity of the antisymmetrizer – it contains about $k!$ terms – the practical computation with the above algorithm becomes however impossible for k significantly greater than three or four. •

Recurrence for the physical generators. Now we prove a similar formula for the face transfer matrices X_p defined in (2.2). So $Z^N = \mathrm{tr}(X_N \dots X_1)$ is the partition function of the quasi-one-dimensional tube shown in (3.1). The reduced partition function $Z^{N,\lambda}(p,q)$ depends on the system size N, on a vertex coordinate λ that lives on the fundamental fusion graph, and on the spectral and the deformation parameters p and q. The only dependence on k comes from the fusion matrices N_i.

Theorem 3.3.2 (Recurrence II) *Let $k = 2, 3$, $N \geq k+1$, let $Z^{N,\lambda} = \mathrm{tr}_N^{\lambda}(X_N \dots X_1)$ be the partition function in the sector of paths running from 0 to λ, and let N_i denote the representation of the fundamental fusion matrices of $\mathrm{su}(k)_q$ on the vertex space of the graph if $0 \leq i \leq k$ and to be identically zero if not. Then*

$$\begin{aligned} Z^N &= \frac{qp^{-1} - q^{-1}p}{q - q^{-1}} N_1 \cdot Z^{N-1} + \frac{(p-p^{-1})(q^2p^{-1} - q^{-2}p)}{(q-q^{-1})^2} N_2 \cdot Z^{N-2} \\ &\quad + \frac{(p-p^{-1})^2(q^3 - q^{-3})(pq^{-1} + p^{-1}q)}{(q-q^{-1})^3(q+q^{-1})} N_3 \cdot Z^{N-3} \end{aligned}$$

or, with $p = e^{i\alpha}$ the spectral parameter and $q = e^{i\eta}$ the deformation parameter

$$\begin{aligned} Z^N &= \frac{\sin(\eta - \alpha)}{\sin(\eta)} N_1 \cdot Z^{N-1} + \frac{\sin(\alpha)\ \sin(2\eta - \alpha)}{\sin^2(\eta)} N_2 \cdot Z^{N-2} \\ &\quad + \frac{\sin^2(\alpha)\ \sin(3\eta)\ \cos(\eta - \alpha)}{\sin^3(\eta)\ \cos(\eta)} N_3 \cdot Z^{N-3} \end{aligned}$$

The multiplication is the matrix multiplication $(N_i \cdot Z_N)^\mu \equiv (N_i)^\mu{}_\nu\, Z^{N,\nu}$, and for $\mathrm{su}(2)_q$ the coefficient of the last term vanishes because there $N_3 = 0$.

The same recurrence formula holds for $N = k$ when we set $Z^0 := N_1$.

Proof.

(1) In the first step we invert the relation (2.2) $X = F + GU$ with $F = \frac{qp^{-1}-q^{-1}p}{q-q^{-1}}$, $G = \frac{p-p^{-1}}{q-q^{-1}}$ to $U = A + BX$ with $A = \frac{pq^{-1}-p^{-1}q}{p-p^{-1}}$ and $B = \frac{q-q^{-1}}{p-p^{-1}}$. Then we express the supplementary condition for TL_3 (2.10) in terms of the X_i as

$$\begin{aligned} &[(A+BX_1)(A+BX_2)(A+BX_1) - (A+BX_1)] \times \\ &\quad [(A+BX_1) + (A+BX_2) - (A+BX_3)(A+BX_2)(A+BX_1)] = 0 \end{aligned}$$

(2) We multiply the above formula out and collect with respect to the X, leading to 39 terms $\mathrm{id}, X_1, \dots, X_1X_2X_1X_3X_2X_1$ that are multiplied with polynominals in A, B.

(3) We multiply from the left with $X_N \dots X_4$ and take the trace. Using the Hecke algebra relations $X^2 = sX + t$ with $s = p + p^{-1}$, $t = \left(\frac{p-p^{-1}}{q-q^{-1}}\right)^2 - 1$, and $X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1} - X_{i+1} + X_i$, and the cyclicity of the trace we can reduce the expression to terms of the form $\mathrm{tr}(X_N \dots X_4 X_{i_1} \dots X_{i_l})$ with $4 > i_1 > \dots > i_l \geq 1$ that multiply a rational function of p, q.

(3a) As an example we treat $\mathrm{tr}(X_N \dots X_4 X_2 X_3 X_2 X_1)$. We introduce the notation $[i_1 \dots i_l] := \mathrm{tr}(X_{i_1} \dots X_{i_l})$ with $4 \sim X_N \dots X_4$. We compute

$$\begin{aligned} [42321] &= [24321] = [43212] = [43121] - [431] + [432] \\ [43121] &= [14321] = [43211] = s\,[4321] + t\,[432] \end{aligned}$$

leading in total to

$$\mathrm{tr}(X_N \dots X_4 X_2 X_3 X_2 X_1) = s\,\mathrm{tr}(X_N \dots X_1) + (t+1)\,\mathrm{tr}(X_N \dots X_2) - \mathrm{tr}(X_N \dots X_3 X_1)$$

(4) Now we break up the terms under the traces, giving

$$\begin{aligned} \mathrm{tr}(X_N \dots X_{l+1}) &= N_1^l \cdot Z^{N-l} & \mathrm{tr}(X_N \dots X_3 X_1) &= \mathrm{tr}(X) \cdot Z^{N-2} \\ \mathrm{tr}(X_N \dots X_4 X_1) &= \mathrm{tr}(X) \cdot N_1 \cdot Z^{N-3} & \mathrm{tr}(X_N \dots X_4 X_2) &= \mathrm{tr}(X) \cdot N_1 \cdot Z^{N-3} \\ \mathrm{tr}(X_N \dots X_4 X_2 X_1) &= \mathrm{tr}(XX) \cdot Z^{N-3} \end{aligned}$$

where $\mathrm{tr}(X) = \mathrm{tr}(X_1) = Z^1$ and $\mathrm{tr}(XX) = \mathrm{tr}(X_2 X_1) = Z^2$ are the traces in the minimal algebras (length 2 and 3 respectively), and the multiplication is meant in the sense of matrices.

(5) We now use the knowledge of $\mathrm{tr}(X)$ and $\mathrm{tr}(XX)$ (see theorem 3.3.3)

$$\begin{aligned} \mathrm{tr}(X) &= F\,\mathrm{tr}(\mathrm{id}) + G\,\mathrm{tr}(U) = \tfrac{qp^{-1} - q^{-1}p}{q - q^{-1}}\, N_1^2 + \tfrac{p - p^{-1}}{q - q^{-1}}\,(q + q^{-1})\, N_2 \\ \mathrm{tr}(XX) &= F^2\, N_1^3 + (2\beta F G + G^2)\, N_1 N_2 + G^2(\beta^2 - 1)\, N_3 \end{aligned} \tag{3.27}$$

and we rewrite the condition now containing the terms $Z^N, N_1 Z^{N-1}, N_1^2 Z^{N-2}, N_2 Z^{N-2}, N_1^3 Z^{N-3}$, $N_1 N_2 Z^{N-3}, N_3 Z^{N-3}$ that are multiplied by some rational functions of p, q.

(6) As an example, we give in figure 3.1 a part of the rational function that we obtained by the algorithm above for the coefficient of N_1^3. The other remaining five coefficients are of comparable size, making it necessary to treat the problem with some computer-algebra system, in our case Mathematica. Simplifying these functions leads to the announced recurrence relation. •

Note here, that the coefficients of N_1^2, $N_1 N_2$ and N_1^3 vanish due to non-trivial cancellations. This implies that the only matrices that appear in the recurrence relation are the fundamental fusion matrices N_i, which means the ones associated with the diagrams with one single row. We did not find any general arguments for why the generic matrices N_λ do not intervene here.

3.3.2 The recurrence bootstrap

In the previous section we derived a recurrence relation for the sequence of partition functions of TL_2 and TL_3 theories. Introducing the notations $X_{i,k}$ for the operator X_i in the algebra TL_k of the basic models, $Z_k^N = \mathrm{tr}(X_{N,k} \dots X_{1,k})$ for the size N partition function, and $Z_k^0 := N_1(q,k)$, we can write the recurrence as

$$Z_k^N = \sum_{i=1}^{k} \rho_i(p,q)\, N_i(q,k) Z_k^{N-i} \tag{3.28}$$

$$
\begin{aligned}
&\frac{2\,p\left(\frac{1}{p}+p\right)^2\left(-\frac{1}{q}+q\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^4-q^2\right)\left(-1+q^2\right)^2}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}+\frac{4\,p^3\left(\frac{1}{p}+p\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^2-q^2\right)\left(p^4-q^2\right)\left(-1+q^2\right)^4}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)^2}\\
&-\frac{p^2\left(\frac{1}{p}+p\right)^2\left(-\frac{1}{q}+q\right)^2\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+q^2\right)^2\left(-p^2+q^2\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}+\frac{p^2\left(\frac{1}{p}+p\right)\left(-\frac{1}{q}+q\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(-1+q^2\right)^2\left(-p^2+q^2\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}\\
&+\frac{p^2\left(\frac{1}{p}+p\right)^3\left(-\frac{1}{q}+q\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(-1+q^2\right)^2\left(-p^2+q^2\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}-\frac{p^4\left(\frac{1}{p}+p\right)\left(-\frac{p}{q}+\frac{q}{p}\right)\left(p^2-q^2\right)\left(-1+q^2\right)^4\left(-p^2+q^2\right)}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)}\\
&+\frac{p^4\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^2-q^2\right)\left(-1+q^2\right)^4\left(-p^2+q^2\right)}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)^2}+\frac{2\,p^4\left(\frac{1}{p}+p\right)^2\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^2-q^2\right)\left(-1+q^2\right)^4\left(-p^2+q^2\right)}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)^2}\\
&+\frac{p^2\left(\frac{1}{p}+p\right)\left(-\frac{p}{q}+\frac{q}{p}\right)\left(p^4-q^2\right)\left(-1+q^2\right)^3\left(-p^2+q^2\right)^2}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)}+\frac{p^2\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^4-q^2\right)\left(-1+q^2\right)^3\left(-p^2+q^2\right)^2}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)^2}\\
&+\frac{p^3\left(\frac{1}{p}+p\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(-1+q^2\right)^3\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)^2}+\frac{2\,p\left(-\frac{p}{q}+\frac{q}{p}\right)\left(p^2-q^2\right)\left(p^4-q^2\right)\left(-1+q^2\right)^2\left(p^2-q^4\right)}{\left(-1+p^2\right)^5 q^6\left(-\frac{1}{q}+q\right)}\\
&-\frac{p\left(-\frac{1}{q}+q\right)^3\left(p^4-q^2\right)\left(-1+q^2\right)^2\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}+\frac{2\,p\left(\frac{1}{p}+p\right)\left(-\frac{1}{q}+q\right)^2\left(-\frac{p}{q}+\frac{q}{p}\right)\left(p^4-q^2\right)\left(-1+q^2\right)^2\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}\\
&+\frac{2\,p\left(-\frac{1}{q}+q\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^4-q^2\right)\left(-1+q^2\right)^2\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}+\frac{4\,p^3\left(-\frac{p}{q}+\frac{q}{p}\right)\left(p^2-q^2\right)\left(p^4-q^2\right)\left(-1+q^2\right)^4\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)}\\
&-\frac{2\,p^2\left(\frac{1}{p}+p\right)\left(-\frac{1}{q}+q\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(-1+q^2\right)^2\left(-p^2+q^2\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}-\frac{p^4\left(p^2-q^2\right)\left(-1+q^2\right)^4\left(-p^2+q^2\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-1+p^2\right)^6 q^6}\\
&+\frac{p^2\left(p^4-q^2\right)\left(-1+q^2\right)^3\left(-p^2+q^2\right)^2\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-1+p^2\right)^6 q^6}+\frac{p^3\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+q^2\right)^3\left(-p^2+q^2\right)^3\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-1+p^2\right)^6 q^6\left(-\frac{1}{q}+q\right)}\\
&+\frac{p^2\left(p^4-q^2\right)\left(-1+q^2\right)^3\left(p^2-q^4\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-1+p^2\right)^5 q^6}+\frac{p^3\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+q^2\right)^3\left(-p^2+q^2\right)\left(p^2-q^4\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)}{\left(-1+p^2\right)^5 q^6\left(-\frac{1}{q}+q\right)}\\
&+\frac{p^2\left(-\frac{1}{q}+q\right)^2\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+q^2\right)^2\left(-p^2+q^2\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)^2}{\left(-\frac{1}{p}+p\right)^3\left(-1+p^2\right)^3 q^3}+\frac{p^4\left(p^2-q^2\right)\left(-1+q^2\right)^4\left(-p^2+q^2\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)^2}{\left(-1+p^2\right)^6 q^6}\\
&+\frac{\left(\frac{1}{p}+p\right)\left(-\frac{1}{q}+q\right)\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(\frac{2\left(p^2-q^2\right)}{-q+p^2\,q}+\frac{\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^3 q^3}\right)}{\left(-\frac{1}{p}+p\right)^3}+\frac{2\,p^2\left(-\frac{p}{q}+\frac{q}{p}\right)^2\left(p^2-q^2\right)\left(-1+q^2\right)^2\left(\frac{2\left(p^2-q^2\right)}{-q+p^2\,q}+\frac{\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^3 q^3}\right)}{\left(-1+p^2\right)^3 q^3\left(-\frac{1}{q}+q\right)^2}\\
&+\frac{p\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+q^2\right)\left(-p^2+q^2\right)^2\left(\frac{2\left(p^2-q^2\right)}{-q+p^2\,q}+\frac{\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^3 q^3}\right)}{\left(-1+p^2\right)^3 q^3\left(-\frac{1}{q}+q\right)}+\frac{\left(p^2-q^2\right)\left(p^2-q^4\right)\left(\frac{2\left(p^2-q^2\right)}{-q+p^2\,q}+\frac{\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^3 q^3}\right)}{\left(-1+p^2\right)^2 q^3}\\
&+\frac{p\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+q^2\right)\left(p^2-q^4\right)\left(\frac{2\left(p^2-q^2\right)}{-q+p^2\,q}+\frac{\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^3 q^3}\right)}{\left(-1+p^2\right)^2 q^3\left(-\frac{1}{q}+q\right)}+\frac{\left(-\frac{1}{q}+q\right)^2\left(-\frac{p}{q}+\frac{q}{p}\right)\left(-1+\frac{\left(-\frac{1}{p}+p\right)^2}{\left(-\frac{1}{q}+q\right)^2}\right)\left(\frac{2\left(p^2-q^2\right)}{-q+p^2\,q}+\frac{\left(-p^2+q^2\right)^3}{\left(-1+p^2\right)^3 q^3}\right)}{\left(-\frac{1}{p}+p\right)^3}\\
&+\cdots=0
\end{aligned}
$$

The rational function that gives the recursion coefficient multiplying N_1^3 obtained by the algorithm of theorem 3.3.2. The ellipses replaces eight more lines. The remaining five rational functions are similar.

Figure 3.1: An example for the apparent rational functions

When we set $N_i(q,k)=0$ whenever $i>k$ then we can formally sum up to infinity and the only dependence on the k comes from the N_i.

In order to obtain this relation we have to know the input of the k initial values $Z_k^0 \dots Z_k^{k-1}$ and the recurrence then determines $Z_k^k, Z_k^{k+1}, \dots$ We show in theorem 3.3.3 that the input $Z_k^0 \dots Z_k^{k-1}$ can in fact be determined when we know $Z_{k-1}^0 \dots Z_{k-1}^{k-1}$ leading to a finite algorithm to compute the recurrence for the Z_k^N for any fixed k.

Notations. Throughout this section, N_i represents the *universal fusion matrices* defined by the multiplication laws of Young diagrams [Fuc92], that act on the non-quotiented space of Young diagrams, that means $(N_i)^\nu{}_\mu$ can be only non-zero if ν contains exactly i boxes more then μ. The specialization to a fusion graph for su(k)$_q$ consists in erasing all links to diagrams with more than k rows, identifying the diagrams differing only in the number of columns with k boxes and – for q a root of unity – truncation. Coherently we use the Bratteli-graph-description of the path algebra where a path of length N adds always N boxes to the diagram of its starting-point. The projection onto pathes on the su(k)$_q$ fusion graph is the same as above. Furthermore, let $\iota=(\iota_1\dots\iota_l)$ be a partition, so $\iota_1\geq\dots\geq\iota_l\geq 1$. We note $|\iota|:=\iota_1+\dots+\iota_l$ and we use N_ι as a shorthand for $N_{\iota_1}\cdots N_{\iota_l}$ when we develop the partition functions into the fusion matrices as

$$Z_k^N=\sum_\iota \alpha_\iota^{N,k}\,N_\iota := \sum_\iota \alpha_\iota^{N,k}\,N_{\iota_1}\cdots N_{\iota_l} \tag{3.29}$$

We will see that the coefficients $\alpha_\iota^{N,k}$ satisfy the selection rules $\alpha_\iota^{N,k}=0$ if $|\iota|\neq N+1$.

Theorem 3.3.3 (Recurrence bootstrap) *We suppose that we know the* TL$_k$*-data* $Z_0^k\dots Z_k^k$ *in terms of their development into the fusion matrices* $Z_k^N=\sum_\iota \alpha_\iota^{N,k}\,N_\iota(q,k)$ *and that they satisfy the recurrence relation*

$$Z_k^N=\sum_{i=1}^k \rho_i(p,q)\,N_i(q,k)\,Z_k^{N-i}\quad \forall N\geq k$$

Then we can derive a recurrence relation for the TL$_{k+1}$ *theory*[3]

$$Z_{k+1}^N=\sum_{i=1}^{k+1} \rho_i(p,q)\,N_i(q,k+1)\,Z_{k+1}^{N-i}\quad \forall N\geq k+1$$

with the ρ_i *being the same as above for* $i=1\dots k$.

Furthermore, the development of the TL$_{k+1}$ *initial data is*

$$\begin{aligned} Z_{k+1}^i &= \textstyle\sum_\iota \alpha_\iota^{i,k}\,N_\iota(q,k+1) \qquad \textit{for } i=0..k-1 \\ Z_{k+1}^k &= \textstyle\sum_\iota \alpha_\iota^{k,k}\,N_\iota(q,k+1)+\left[(F+\beta G)^k-\sum_\iota \alpha_\iota^{k,k}\right]\,N_{k+1} \end{aligned}$$

If the $\alpha_\iota^{N,k}$ *satisfy the selection rules '*$\alpha_\iota^{N,k}=0$ *if* $|\iota|\neq N+1$*' so do the* $\alpha_\iota^{N,k+1}$.

[3] In fact, we can only derive a recurrence relation of type $\sum_\iota \rho_\iota\,N_\iota\,Z^{N-|\iota|}=0$ with $|\iota|\leq k$. However, the above argumentation keeps valid when we replace i by ι, and it seems, that only the fundamental fusion matrices appear.

Proof.

Initial values. For the Z_{k+1}^i with $i = 0 \dots k-1$ this follows from lemma 3.3.4: $[Z_{k+1}^i]^\lambda$ is non-zero only if λ has exactly $i+1$ boxes, and hence at most $i+1 \leq k$ rows. Then 3.3.4 gives that $[Z_{k+1}^i]^\lambda = [Z_k^i]^\lambda$ and hence it is already determined by the TL_k recurrence.

For $[Z_{k+1}^k]^\lambda$ the same arguments hold for all diagrams but the one with $k+1$ boxes in one column, and its coefficient is determined by lemma 3.3.5. ∘

Recurrence relation. In order to compute the recurrence relation for TL_{k+1} we proceed as in the proof of theorem 3.3.2. We express the TL_{k+1} condition in terms of the X *(1)* and multiply it out, leading to products of $X_1, \dots, X_{k+1}$ with coefficients being polynominals in A, B *(2)*. We multiply with $X_N \dots X_{k+2}$, take the trace, and simplify the terms using the Hecke algebra relations and cyclicity of the trace *(3)*. We obtain an equation of the form

$$\sum_j \varepsilon_j(p,q)\,\mathrm{tr}(X_N \dots X_{k+2} X_{i(j,1)} \dots X_{i(j,n_j)}) = 0$$

where ε_j is a rational function in its arguments and $k+2 > i(j,1) > \dots > i(j,n_j) \geq 1\ \forall j$. We break the traces at the leftmost position possible, giving

$$\sum_j \mathrm{tr}(X_N \dots X_{l_j+1} \cdot O_j) = 0$$

where the operator O_j is an ordered and minimal product of the X_i which does not contain X_{l_j}. Because $l_j \leq k+1$, the trace $\mathrm{tr}(O_j)$ breaks into a product of the partition functions $Z_{k+1}^0 \dots Z_{k+1}^k$. We proved that they are determined by the TL_k recurrence. So we insert their development in terms of the fusion matrices *(5)* and simplify the resulting expression *(6)*, thereby collecting all the terms that multiply the same fusion matrices N_ι. ∘

Independence of the ρ_i on k. Using basically the equation $[Z_k^N]^\lambda$ and $[Z_{k(\lambda)}^N]^\lambda$ of theorem 3.3.4, the independence of the universal fusion matrices N_i on k, and that $(N_i)^\lambda{}_\mu = 0$ if $k(\lambda) > i$ or $k(\mu) > k(\lambda)$ we find the two equations

$$\begin{aligned} [Z_k^N]^\lambda &= \sum_{i=0}^{k(\lambda)} \rho_{i,k}\,(N_i)^\lambda{}_\mu\,[Z_k^{N-i}]^\mu \\ [Z_k^N]^\lambda &= \sum_{i=0}^{k(\lambda)} \rho_{i,k(\lambda)}\,(N_i)^\lambda{}_\mu\,[Z_k^{N-i}]^\mu \end{aligned}$$

where $\rho_{i,k}$ denotes the recurrence coefficient for a TL_k-theory. This equality – with $k(\lambda)$ unchanged – still holds when we replace λ with a ν such that $k(\lambda) \geq k(\nu)$, and hence

$$\sum_{i=0}^{k(\lambda)} (\rho_{i,k} - \rho_{i,k(\lambda)})\,[N_i \cdot Z_{k(\lambda)}^{N-i}]^\nu = 0 \quad \forall \nu \text{being a } \mathrm{TL}_{k(\lambda)}\text{-diagram}$$

When we suppose the family of vectors $\left(N_i \cdot Z_{k(\lambda)}^{N-i}\right)_{i=1\dots k(\lambda)}$ not to be uniformly (in N) linearly dependent – this would lead to a $k-1$ step recurrence relation for the TL_k-theory – we conclude that $\rho_{i,k} - \rho_{i,k(\lambda)} = 0$ for $i = 1 \dots k(\lambda)$. ∘

Selection rules. The partition function $[Z^N]^\lambda = \mathrm{tr}^\lambda(z^N)$ is the trace of an operator in the length $N+1$ path algebra. Because we consider here the non-quotiented graphs, a path of length $N+1$ starting at 0 ends at a diagram with $N+1$ boxes. On the other hand, N_i adds i boxes, hence N_ι adds ι boxes, what finishes the proof of the selection rules. •

Lemma 3.3.4 (Hecke algebra) *Let λ be a Young diagram with N boxes in $k(\lambda)$ rows. The λ component of any TL_k partition function satisfies*

$$[Z_k^N]^\lambda = \begin{cases} [Z_{k(\lambda)}^N]^\lambda & k \geq k(\lambda) \\ 0 & k < k(\lambda) \end{cases}$$

Proof.

Structure of the Hecke algebra. The proof uses the structure of the Hecke algebra with N generators H^N. For generic q it is semi-simple [Wen88], and for q at a root of unity we consider the truncated algebra (see sect. 1.3.2) corresponding to its maximal semi-simple subalgebra. It follows that it is the direct sum of its irreducible unitary representations

$$H^N = \bigoplus_\lambda \mathbf{1}_\lambda{}^N \cdot H^N$$

where the sum goes over all the λ Young diagrams with exactly N boxes and $\mathbf{1}_\lambda{}^N \in H^N$ is the central projection onto the representation of type λ. $\mathrm{TL}_k{}^N \subset H^N$ is a subalgebra obtained out of H^N by quotienting out all those representations whose Young diagrams have more than k rows, symbolically $k(\lambda) > k$. Hence

$$\mathrm{TL}_k{}^N = \bigoplus_{\lambda:k(\lambda)\leq k} \mathbf{1}_\lambda{}^N \cdot H^N =: \bigoplus_{l=1}^{k} \Delta\mathrm{TL}_l{}^N$$

where the algebra $\Delta\mathrm{TL}_k{}^N$ collects the representations with $k(\lambda) = k$.

It follows that a representation λ of the Hecke algebra is in fact a representation of the ideal and subalgebra $\Delta\mathrm{TL}_{k(\lambda)}{}^N \subset H^N$ and hence of $\mathrm{TL}_{k(\lambda)}$.

In particular if we take U_i to be the generators of any Hecke algebra, then the $U_{i,\lambda} := \mathbf{1}_\lambda U_i$ satisfy the relation (2.10) and form hence a representation of $\mathrm{TL}_{k(\lambda)}$. ∘

Conclusion. After these preliminary remarks we can write

$$\begin{aligned} [Z_k^N]^\lambda &= \mathrm{tr}(\mathbf{1}_\lambda X_{N,k} \ldots X_{1,k}) & &= \mathrm{tr}((\mathbf{1}_\lambda X_{N,k}) \ldots (\mathbf{1}_\lambda X_{1,k})) \\ &= \mathrm{tr}((\mathbf{1}_\lambda X_{N,k(\lambda)}) \ldots (\mathbf{1}_\lambda X_{1,k(\lambda)})) & &= \mathrm{tr}(\mathbf{1}_\lambda X_{N,k(\lambda)} \ldots X_{1,k(\lambda)}) \quad = [Z_{k(\lambda)}^N]^\lambda \end{aligned}$$

where the notation $X_{i,k}$ means that $X_{i,k} \in \mathrm{TL}_k{}^N$. •

Lemma 3.3.5 (Coefficients) *Let $Z_{k+1}^k = \sum_\iota \alpha_\iota^{k,k+1} N_\iota$. Then the coefficients satisfy*

$$\sum_\iota \alpha_\iota^{k,k+1} = (F + \beta G)^k$$

Proof. Recall that $Z^k_{k+1} = \mathrm{tr}(X_k \ldots X_1)$ with $X_i \equiv X_{i,k+1} \in \mathrm{TL}_{k+1}$, and that $[Z^k_{k+1}]^\lambda$ is the trace in the irreducible Hecke algebra representation of type λ [Wen88]. Taking in particular $\lambda = (k+1)$ the diagram with $k+1$ boxes in one column then $[Z^k_{k+1}]^{(k+1)} = \mathrm{tr}(\mathbf{1}_{(k+1)} X_k \ldots X_1)$.

The representation $(k+1)$ of $\mathrm{TL}_{(k+1)}$ is one dimensional because there is only one path from $\lambda = 0$ to $\lambda = (k+1)$, and hence each $\mathbf{1}_{(k+1)} X_i$ can be identified with a c-number, its eigenvalue. The operators X_i are locally the same, hence $\mathrm{tr}(\mathbf{1}_{(k+1)} X_k \ldots X_1) = \mathrm{tr}(\mathbf{1}_{(k+1)} X)^k$.

We know that $X = F + GU = F + \beta G e$ with e the projection onto the representation $\lambda = (2)$ of the one generator Hecke algebra. Hence $\mathrm{tr}(X) = F N_1^2 + \beta G N_2$ and $\mathrm{tr}^{(k+1)}(X) = F + \beta G$ giving the right hand side of the equation.

In order to find the left hand side we use the selection rules giving $|\iota| = k+1$ and observe that $[N_\iota]^{(k+1)}{}_0 \equiv [N_{\iota_1} \ldots N_{\iota_l}]^{(k+1)}{}_0 = 1$ because of the multiplication laws of Young diagrams. •

3.3.3 The general models

Here we show[4], that the recurrence and bootstrap relations that we derived for the basic models apply for the general models as well, with the recurrence and development coefficients unchanged, so that in fact the knowledge of the basic models is already exhaustive.

Theorem 3.3.6 (General models) *Let $\tilde{Z}^N = \mathrm{tr}(\tilde{X}_N \ldots \tilde{X}_1)$ be the partition function of a general TL_k model associated with the fusion algebra representation matrices $(\tilde{N}_\mu)^t{}_s$ and let Z^N, N_μ, X_i bet the objects associated with the basic TL_k model which satisfy*

$$Z^N = \sum_{i=1}^{k} \rho_i \, N_i \, Z^{N-i} = \sum_\iota \alpha_\iota^N N_\iota$$

Then the general partition function satisfies

$$\tilde{Z}^N = \sum_{i=1}^{k} \rho_i \, \tilde{N}_i \, \tilde{Z}^{N-i} = \sum_\iota \alpha_\iota^N \tilde{N}_\iota$$

with the same coefficients ρ_i, α_ι.

Proof. In order to prove this we could have followed the same steps as in the previous section, but we want to give a direct proof here, relying on the conjectured translation invariance decomposition for the general models (3.8). Note that we have to use this conjecture in either of the two cases.

We show here that the coefficients α_ι^N describing the development of Z^N into the N_ι describe the development $\tilde{Z}^N$ into $\tilde{N}_\iota$ as well. That they satisfy the same recurrence relations follows from

[4] In fact we show it relying on the conjecture concerning the translation invariance decomposition in the general models (3.8).

the representation property (2.20).

$$\begin{aligned}[\tilde{Z}^N]^t{}_s &= \sum_\lambda (\tilde{N}_\lambda)^t{}_s \, Z^{N,\lambda} = \sum_\lambda (\tilde{N}_\lambda)^t{}_s \cdot \sum_\iota \alpha_\iota^N (N_\iota)^\lambda{}_0 = \sum_\lambda (\tilde{N}_\lambda)^t{}_s \sum_\mu \beta_\mu^N \, (N_\mu)^\lambda{}_0 \\ &= \sum_\lambda (\tilde{N}_\lambda)^t{}_s \sum_\mu \beta_\mu^N \delta_{\mu,\lambda} = \sum_\mu \beta_\mu^N (\tilde{N}_\mu)^t{}_s = \sum_\iota \alpha_\iota^N (\tilde{N}_\iota)^t{}_s\end{aligned}$$

We use above the possibility of transforming $\sum_\iota \alpha_\iota^N \, N_\iota$ into $\sum_\mu \beta_\mu^N \, N_\mu$ via the relations of the fusion algebra and the fact that we can reconstitute the α_ι^N because the $\tilde{N}_\mu$ satisfy the same relations. •

3.4 Computation of the free energy

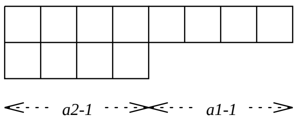
We solve here the recurrence relation of theorem 3.3.2 in order to compute the physical partition function Z^N and hence the free energy. The recurrence relation is a non-trivial vector equation involving the fusion matrices N_i. But all these matrices commute and they are hence diagonalizable in the common basis of the $|a>$ (1.27). What we finally compute is the Fourier transformed renormalized partition function $\bar{Z}^{N,a}$ defined in (3.19). That allows us to compute the free energy in each Fourier sector separately.

3.4.1 Diagonalization of the fusion graphs

The eigenvectors and eigenvalues of all the su(k)$_q$ fusion graphs are explicitly known [Zub95]. Recall from sect. 2.1.2 that the graphs defining the general Hecke algebra models form a representation of the corresponding fusion algebra, and hence their spectrum is included in the spectrum of the fusion graphs.

We denote $[a_1, a_2]$ the Young diagram

$$[a_1, a_2] =$$



(3.30)

The eigenvectors of the fusion graph of height h can hence be indexed with the vertex set, the diagrams $\{[1] \ldots [h+1]\}$ for su(2) and $\{[a_1, a_2] | a_{1,2} \geq 1, a_1 + a_2 \leq h+2\}$ for su(3).

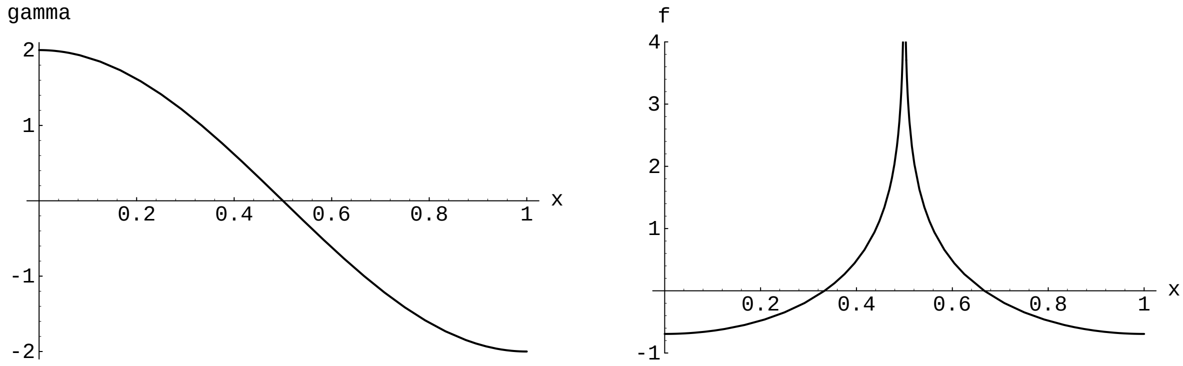
Like above, we define α and η to be the phases associated with the spectral and the deformation parameters

$$p = e^{i\alpha}, \quad q = e^{i\eta}, \quad \eta = \frac{\pi}{l}, \quad l = h + k \tag{3.31}$$

For su(2), the eigenvalue of the fusions N_i in the sector a are

$$\gamma_1^a = 2\cos(\eta \cdot a/2), \quad \gamma_0^a = \gamma_2^a \equiv 1 \tag{3.32}$$

The value of γ_1^a is the partition function per lattice site of the reduced lattice in the sector a for a infinite temperature system on the path algebra. The logarithm of its modulus is the free energy per site. The phase factor is due to a symmetry of the graph as explained in sect. 1.4.1. In order to visualize this eigenvalue distribution, we plotted in figure 3.2 the eigenvalue and the free energy associated against the reduced coordinate $x = a\eta/2\pi$.



Eigenvalue γ_1^a and the entropy $f = -\log|\gamma_1^a|$ associated as a function of the reduced sector coordinate $x(a) = a\eta/2\pi$

Figure 3.2: Diagonalization of the su(2) graph

For su(3), the only non-trivial fusion matrices are N_1 and N_2 and they are conjugate to each other. The eigenvalue in the sector $[a_1, a_2]$ of the fusion N_i is given by

$$\gamma_1^{a_1,a_2} = q^{(2a_1+a_2)/3} + q^{(a_2-a_1)/3} + q^{(-a_1-2a_2)/3}, \quad \gamma_2 = \gamma_1^*, \quad \gamma_0 = \gamma_3 \equiv 1 \tag{3.33}$$

In order to visualize the eigenvalues of the N_1 fusion graph, we mapped it into the euclidean plane. The basis vector $a_{1,2}$ maps onto $(x, y) = (\sqrt{3}/2l, \pm 1/2l)$, hence the mapping range is roughly independent of l. We plotted in figure 3.3 the modulus of the eigenvalue for a small l. For huge l we plotted the modulus $|\gamma_1(x, y)|$, its phase angle and the free energy associated $-\log|\gamma_1|$ against the reduced coordinates (x, y). This is even for small l a good approximation when we evaluate it at the points $x(a_1, a_2), y(a_1, a_2)$.

3.4.2 Solving the recurrence

Because the recurrence relation of theorem 3.3.2 is linear in the Z^N, we can solve it with a kind of generalized transfer matrix formalism:

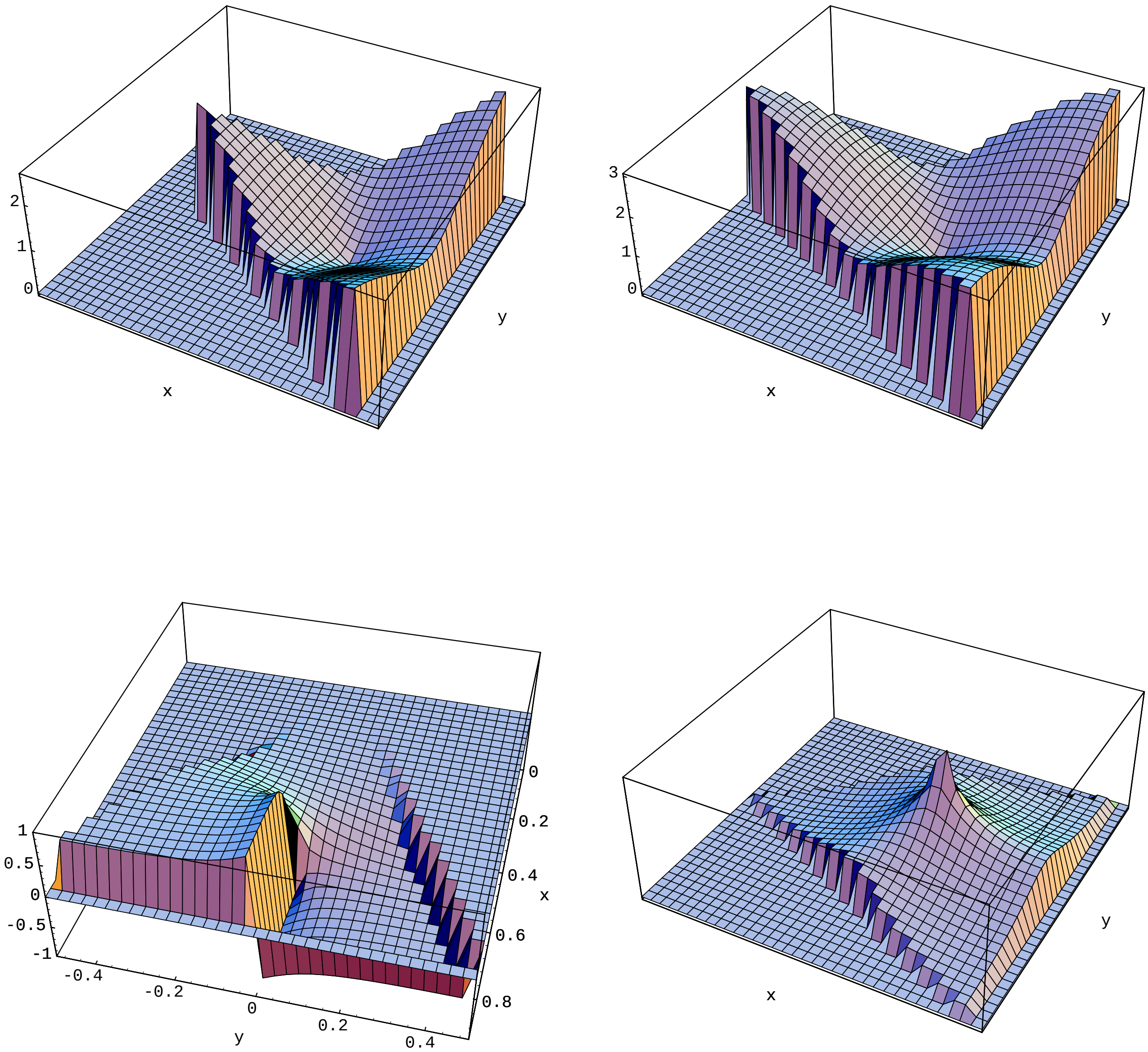
When we define $\tilde{Z}^N := (\sin(\eta)/\sin(\alpha))^N Z^N$, we get the su(3) relations in matrix form

$$\begin{pmatrix} \tilde{Z}^{N,a} \\ \tilde{Z}^{N-1,a} \\ \tilde{Z}^{N-2,a} \end{pmatrix} = \begin{pmatrix} \frac{\sin(\eta-\alpha)}{\sin(\alpha)}\gamma_1^a & \frac{\sin(2\eta-\alpha)}{\sin(\alpha)}\gamma_2^a & \frac{\sin(3\eta)\cos(\eta-\alpha)}{\sin(\alpha)\cos(\eta)}\gamma_3^a \\ 1 & & \\ & 1 & \end{pmatrix} \begin{pmatrix} \tilde{Z}^{N-1,a} \\ \tilde{Z}^{N-2,a} \\ \tilde{Z}^{N-3,a} \end{pmatrix} \tag{3.34}$$

and for su(2) they reduce to the system with the third row and column deleted.

For su(2) we obtain the eigenvalues

$$\lambda_\pm = \cos(a\,\eta/2)\,\frac{\sin(\eta-\alpha)}{\sin(\alpha)}\left(1 \pm \sqrt{1 + \frac{\sin(2\,\eta-\alpha)\,\sin(\alpha)}{\sin^2(\eta-\alpha)\,\cos^2(a\,\eta/2)}}\right) \tag{3.35}$$



Eigenvalues of the su(3) fusion graph: modulus for small l (NW); modulus (NE), phase (SW) and entropy associated (SE) for huge l

Figure 3.3: Diagonalization of the su(3) graph

The initial conditions are $Z^0 = N_1$ and $Z^1 = \mathrm{tr}(X) = \frac{\sin(\eta-\alpha)}{\sin(\eta)} N_1^2 + \frac{\sin(\alpha) 2\cos(\eta)}{\sin(\eta)} N_2$ (see (3.27)) and hence

$$\tilde{Z}^{a,0} = 2\cos(a\eta/2), \quad \tilde{Z}^{a,1} = 4\cos^2(a\eta/2)\,\frac{\sin(\eta-\alpha)}{\sin(\alpha)} + 2\cos(\eta) \tag{3.36}$$

In order to compute the recurrence, we have to know the eigenvectors of the recurrence matrix. If we denote $|\Omega, a\rangle$ this vector of initial conditions and $\lambda_{\pm,a}$ the eigenvalues above with eigenvectors $|\pm, a\rangle$ then

$$\tilde{Z}^{a,N} = (\lambda_{+,a})^{N-1} |\langle a, +|\Omega, a\rangle|^2 + (\lambda_{-,a})^{N-1} |\langle a, -|\Omega, a\rangle|^2 \tag{3.37}$$

In the thermodynamic limit, we have to distinguish two cases: either, both $\lambda_\pm$ are real and hence the recurrence is – without accident due to the boundary conditions – determined by the modulus of the largest eigenvalue $\lambda_{\max} := \max(|\lambda_\pm|)$. If not, they are by force conjugate to one another and the asymptotic behaviour contains an oscillating term proportional to $|\lambda|^N \cos((N-N_0)\lambda)$. However, the free energy is in both cases determined by the modulus. Defining

$$A := \cos(a\,\eta/2)\,\frac{\sin(\eta-\alpha)}{\sin(\alpha)}, \quad B := \frac{\sin(2\,\eta-\alpha)\,\sin(\alpha)}{\sin^2(\eta-\alpha)\,\cos^2(a\,\eta/2)} \tag{3.38}$$

the modulus of the largest eigenvalue (3.35) becomes

$$\lambda_{\max} = \begin{cases} |A|\,(1+\sqrt{1+B}) & B \geq -1 \quad \lambda \text{ real} \\ |AB| & B < -1 \quad \lambda \text{ complex} \end{cases} \tag{3.39}$$

We multiply this value with $\sin(\alpha)/\sin(\eta)$ in order to obtain Z and with $(\gamma_1^a)^{-1}$ in order to obtained the renormalized partition function $\bar{Z}$. We get the expression for the renormalized specific free energy

$$\bar{f}^a = -\log\,\left|\frac{\sin(\eta-\alpha)}{2\sin(\eta)}\right| \cdot \begin{cases} 1+\sqrt{1+B(\eta,\alpha,a)} & B \geq -1 \\ |B(\eta,\alpha,a)| & B < -1 \end{cases} \tag{3.40}$$

and the non-analycity occurs transverse to $B = -1$. In order to take the continuum limit, we replace the generalized Fourier index a with the reduced Fourier coordinate $x := a\eta/2\pi \in [0,1]$ that we already used for figure 3.2. The condition $B \geq -1$ then gives

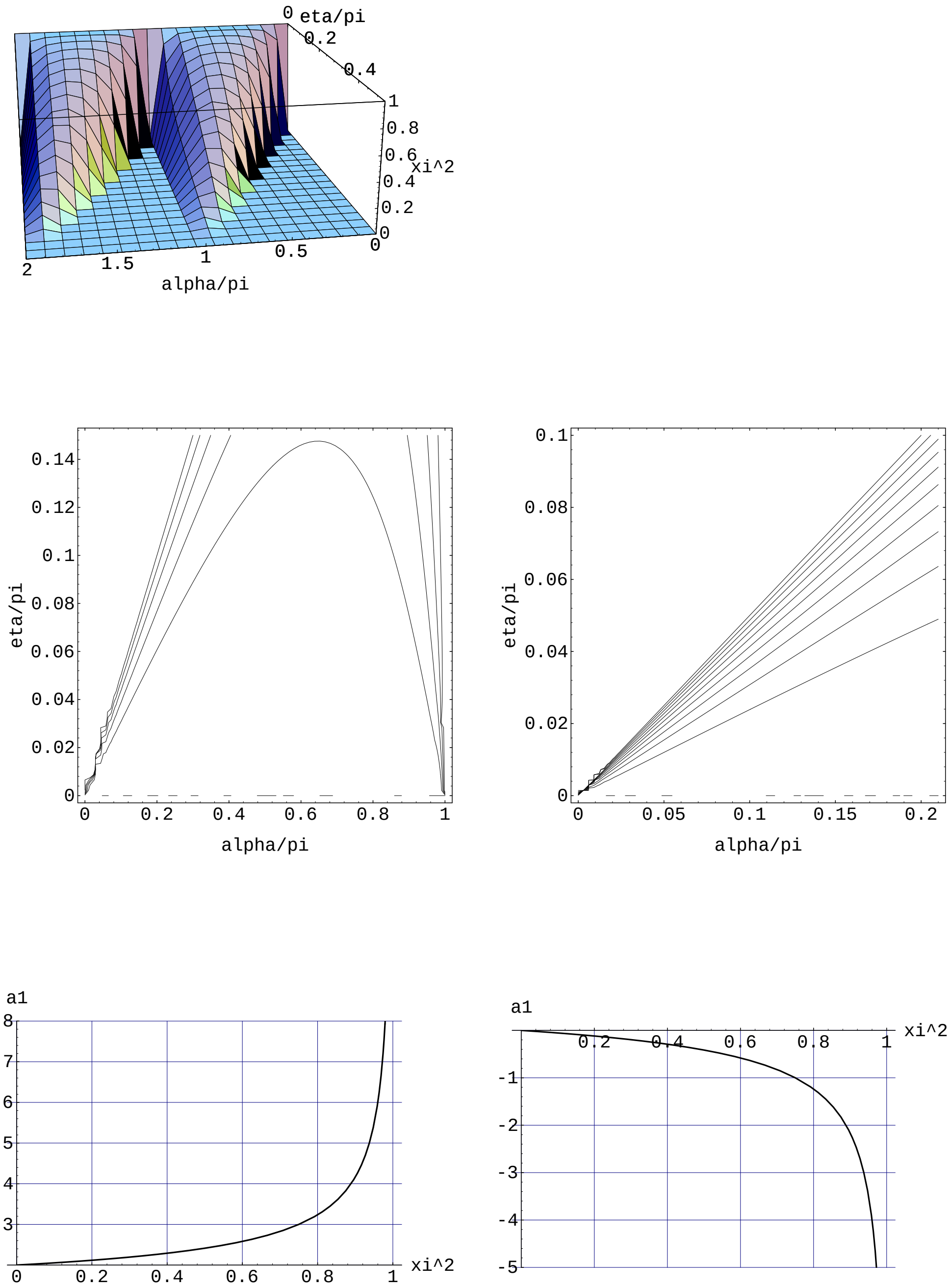
$$\sin(2\eta-\alpha)\,\sin(\alpha) + \sin^2(\eta-\alpha)\,\cos^2(\pi x) \geq 0 \tag{3.41}$$

The critical surface is plotted in figure 3.4, where we noted $\xi^2 := \cos^2(\pi x) \in [0,1]$. In the limit, where η becomes small (see (3.31)), we can develop around the solution $\alpha = \eta = 0$. So taking the ansatz $\alpha(\eta,\xi) = a_1(\xi)\eta + O(\eta^2)$, we compute $a_1(\xi)$ on the critical surface, and the condition $B \geq -1$ becomes

$$0 \leq \alpha \leq \left(1+\sqrt{1+\frac{\xi^2}{1-\xi^2}}\right)\eta + O(\eta^2) \leq 2\eta + O(\eta^2) \tag{3.42}$$

for the domain $\alpha \geq 0$. Verifying the contourplots of figure 3.4 we see that the linear approximation is valid for $\eta < 0.1$ corresponding to the models whose underlying graphs contain at least 10 vertices. In the domain $\alpha \leq 0$ we obtain

$$0 \geq \alpha \geq \left(1 - \sqrt{1 + \frac{\xi^2}{1 - \xi^2}}\right) \eta + O(\eta^2) \tag{3.43}$$



Plot of the transition surface in (α, η, ξ^2) space; contours $\xi^2 = (0, 0.2, \dots 0.8)$ and $\xi^2 = (0, 0.1, \dots 0.9)$ in (α, η) space (0 is the outest one); the linear coefficient $\alpha(\eta, \xi) = a_1(\xi)\eta + O(\eta^2)$ on the surface for $\alpha \geq 0, \alpha \leq 0$.

Figure 3.4: The su(2) critical surface

Chapter 4

Additional remarks

4.1 Path algebra systems and gauge symmetries

We establish here a relation between multi-particle gauge theories and some theories based on the path algebra of a graph, the latter being determined by the gauge algebra. This relation generalizes the duality between Spin and IRF-models described in sect. 2.

Coproduct and Gauge symmetry. Let $\mathcal{A}$ be a Hopf algebra which is the local symmetry algebra of a physical system. So an N particle system having a $\mathcal{A}^{\otimes N}$-gauge symmetry is described by an observable algebra[1]

$$\mathcal{A}^N_{\mathrm{obs}} = \bigotimes_{i=1}^{N} r_i(\mathcal{A}) \tag{4.1}$$

with r_i being the representation of $\mathcal{A}$ used at the lattice point i[2]. When we suppose the uniformity of the configuration space, then all the r_i are equal to some standard representation r characterizing the local configuration space.

The global action of the gauge algebra $\Delta_N : \mathcal{A} \to \mathcal{A}^{\otimes N}$ is inductively determined by the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ once we have chosen an ordering of the local spaces. In the case of a cocommutative gauge algebra — e.g. a gauge group in the classical meaning — Δ_N does not depend on this arbitrary choice. In the generic case, however, it does.

[1] If the local behaviour of the system is not too singular, as it is mormally the case for discrete systems, the whole symmetry algebra is observable.

[2] The geometry of the underlying lattice is not important here. We can however continue thinking of a two dimensional square lattice with $i \in Z \times Z$ and an arbitrary ordering chosen.

4.1.1 Duality between spin and path algebra systems

In this section we establish the duality generalizing the one between Spin- and IRF-models. One the one side, this duality involves a gauge theory, describing a homogeneous[3] N-particle system. The local degrees of freedom, the spins, form a representation of the local gauge algebra $r(\mathcal{A})$, with $\mathcal{A}$ a Hopf algebra. The global gauge transformations $\Delta_N(\mathcal{A})$ are induced by the coproduct, and we suppose the partition operator z to be invariant under global gauge transformations, and hence to commute with $\Delta_N(\mathcal{A})$.

On the other side of the duality, there is a theory whose partition operator is an element of a path algebra. More precisely, it is in the path algebra of $\mathcal{G}_r$, the graph associated with the fusion with r. The 'sausage geometry' partition function $Z = \mathrm{tr}(z)$ of (3.1) is a matrix in the space of boundary conditions. However, the two boundary conditions are not independent. So the general partition function $Z^\nu{}_\mu$ can be computed out of the partition function with one boundary trivially fixed $Z^\lambda{}_0$, see (3.6). For $U(1)$-theories this dependence is $Z^\nu{}_\mu = Z^{\nu-\mu}{}_0$, see (4.5), justifying the name 'translation invariance' for this type of symmetry.

To start with, we define the objects that we will need throughout this section

Definition 4.1.1 *Let $\mathcal{A}$ be a Hopf algra with central decomposition[4] $\mathcal{A} = \bigoplus_\lambda \mathcal{A}_\lambda$ and fusion coefficients $N_{\lambda\mu}{}^\nu$ and let $r^\lambda : \mathcal{A} \to \mathcal{A}_\lambda$ denote the standard unitary representation of type λ. Let r be an arbitrary unitary representation of $\mathcal{A}$ and let $\mathcal{G}_r$ be the associated fusion graph[5] '$r\otimes$'.*

We denote $\Delta_{1\ldots M}$ the global representation of $\mathcal{A}$ on $r(\mathcal{A})^{\otimes M}$ induced by the coproduct Δ. Similarly $\Delta_{0\ldots M}$ is the representation on $\mathcal{A} \otimes r(\mathcal{A})^{\otimes M}$.

We denote $(\cdots)'$ the commutant of an algebra. So with a slight abuse of notation we denote '$\Delta'_{1\ldots M} \subset r(\mathcal{A})^{\otimes M}$' the set of all operators z in the general linear algebra of the space where $r(\mathcal{A})^{\otimes M}$ is represented on that commute with each operator of $\Delta_{1\ldots M}(\mathcal{A})$.

The last objects to define are the path projections $\mathbf{1}_\gamma$. At first, we suppose $\mathcal{G}_r$ to be simply laced. We associate to a path γ of length M on $\mathcal{G}_r$ (an *M-path*) the path projection

$$\mathbf{1}_\gamma \in \mathcal{A} \otimes r(\mathcal{A})^{\otimes M}, \quad \mathbf{1}_\gamma := \Delta_0(\mathbf{1}_{\lambda_0})\Delta_{01}(\mathbf{1}_{\lambda_1})\ldots\Delta_{0\ldots M}(\mathbf{1}_{\lambda_M}) \tag{4.2}$$

where λ_i is the vertex that γ passes at i. Note that like in (2.23) all these projections commute because of the coassociativity of Δ and because $\mathbf{1}_\lambda \in \mathcal{A}$ is central. In the case that γ starts at the

[3]We consider here only homogeneous systems, which means systems that look locally the same. However, the generalization to the non-homogeneous case is straight-forward, the overall graph $\mathcal{G}_r$ being replaced by a graph dependent on the path coordinate $\mathcal{G}_{r_i}$, see below.

[4]For non-semi-simple Hopf algebras like quantum groups at a root of unity we define $\mathcal{A}$ to be the truncated algebra, which means its maximal semi-simple subalgebra. In either of the two cases, the $\mathcal{A}_\lambda$ are exactly the possible unitary representations.

[5]If r has the multiplicity vector n_λ such that $r = \oplus n_\lambda\, r^\lambda$ then $(\mathcal{G}_r)^\nu{}_\mu = \sum_\lambda n_\lambda N_{\lambda\mu}{}^\nu$.

vertex $'0'$ associated with the trivial representation then we can interpret the projection to be in $\mathbf{1}_\gamma \in r(\mathcal{A})^{\otimes M} \equiv \mathcal{A}_0 \otimes r(\mathcal{A})^{\otimes M} \subset \mathcal{A} \otimes r(\mathcal{A})^{\otimes M}$ as well.

If $\mathcal{G}_r$ is not simply laced, then the paths carry supplementary link variables. In terms of representations, that signifies that the fusion '$r\otimes$' may produce multiplicities, so there are μ, ν such that $r \otimes \lambda = \mu \oplus \mu \oplus \cdots$. In this case the path projections onto paths differing only in the link variables pick out the different representation spaces of the same type. Suppose for example that $r \otimes \lambda = \mu \oplus \mu$. Hence $(\mathcal{G}_r)^\mu{}_\lambda = 2$ and one of the two paths is associated with $\mathbf{1}_\mu \oplus 0$, and the other one to $0 \oplus \mathbf{1}_\mu$.

In the following theorem, we identify the commutant $\Delta_{1\ldots M}(\mathcal{A}) \subset r(\mathcal{A})^{\otimes M}$ as a subalgebra of the length M path algebra on $\mathcal{G}_r$.

Theorem 4.1.2 (Duality) *The representation $\Delta_{1\ldots M} : \mathcal{A} \to r(\mathcal{A})^{\otimes M}$ induces a decomposition*

$$r(\mathcal{A})^{\otimes M} \supset \bigoplus_\lambda \mathcal{C}_\lambda \otimes \mathcal{A}_\lambda, \quad \mathcal{A} \cong \bigoplus \mathrm{id}_\lambda \otimes \mathcal{A}_\lambda$$

The algebra $\mathcal{C}_\lambda$, the commutant of type λ representations, is isomorphic to the length M path algebra of $\mathcal{G}_r$ in the sector of paths from 0 to λ. For a $z \in \Delta_{1\ldots M}(\mathcal{A})'$ this isomorphism induces that the path algebra matrix elements between two M-pathes γ, γ' ending at λ, λ' are given as

$$z^\gamma{}_{\gamma'} = \frac{\delta_{\lambda\lambda'}}{\mathrm{tr}_{\mathcal{A}_\lambda}(\mathrm{id})} \cdot \mathrm{tr}_{r(\mathcal{A})^{\otimes M}}(\hat{\mathbf{1}} \cdot \mathbf{1}_\gamma \, z \, \mathbf{1}_\gamma{}')$$

with $\hat{\mathbf{1}} := \sum |\gamma_1\rangle\langle\gamma_2|$, so $(\hat{\mathbf{1}})^{\gamma_1}{}_{\gamma_2} \equiv 1 \forall \gamma_1, \gamma_2$.

Proof. The decomposition follows from representation theory of simple algebras or factors [GdlHJ89, Fuc92]: in finite dimension they are always of the type $\mathcal{A}_\lambda \ni x \mapsto \mathrm{id}_n \otimes x \in M_n \otimes \mathcal{A}_\lambda$ where id_n is the identity matrix of the n dimensional multiplicity space. Hence each representation of $\mathcal{A}$, and in particular $\Delta_{1\ldots M}(\mathcal{A})$, is of the form $x \mapsto \oplus \mathrm{id}_\lambda \otimes x_\lambda \in \bigoplus \mathcal{C}_\lambda \otimes \mathcal{A}_\lambda \subset r(\mathcal{A})^{\otimes N}$. It is obvious that every operator $\oplus c_\lambda \otimes \mathrm{id}_\lambda \in \bigoplus \mathcal{C}_\lambda \otimes \mathcal{A}_\lambda$ commutes with $\Delta_{1\ldots M}(\mathcal{A})$. That these are the only ones follows from Schur's lemma.

Now we show that $\mathcal{C}_\lambda$ is isomorphic to the path algebra: each M-path γ_i starting at 0 and ending at one fixed representation λ describes one possibility how to obtain the representation λ by the M-fold tensorization of the representation r. So the space V on which $r^{\otimes M}$ is represented contains different copies of type λ representation spaces, one for each path. Hence $V = \bigoplus_{\gamma_i} V_{\lambda,\gamma_i} \cong M_{\#\{\gamma_i\}} \otimes V_\lambda$. Via construction, the path projections are the canonical one dimensional projections in $M_{\#\{\gamma_i\}} \otimes \mathrm{id}_\lambda$. The formula for the matrix elements then follows when we remark that $\mathrm{tr}(\hat{\mathbf{1}} \cdot \mathbf{1}_i \, z \, \mathbf{1}_j) = \langle i| \, z \, |j\rangle$ if $(\mathbf{1}_i) = (\, |i\rangle\langle i| \,)$ is a complete set of orthogonal and one dimensional projections and $\langle i| \, \hat{\mathbf{1}} \, |j\rangle = 1$. •

The next two theorems concern the behaviour of the different partition operators z and their associated partition function $Z = \mathrm{tr}(z)$ under the tower inclusions $r(\mathcal{A})^{\otimes M} \subset r(\mathcal{A})^{\otimes M+1}$.

Theorem 4.1.3 (Inclusions) *Let $z \in \Delta_{1\dots M}(A)' \subset r(\mathcal{A})^{\otimes M}$ and interpret $r(\mathcal{A})^{\otimes M} \subset \mathcal{A}\otimes r(\mathcal{A})^{\otimes M}$ respectively $r(\mathcal{A})^{\otimes M} \subset r(\mathcal{A})^{\otimes(L+M+N)}$ as canonically embedded. Let γ, γ' be $L+M+N$-paths starting at zero, and $\gamma = \gamma_L \times \gamma_M \times \gamma_N$ the decomposition of γ into (L, M, N)-paths. Then the matrix elements in the length M and the length $L+M+N$ path algebra are related as*

$$(z_{L+M+N})^{\gamma}{}_{\gamma'} = \delta_{\gamma_L\gamma_L{}'}\,\delta_{\gamma_N\gamma_N{}'}\ (z_M)^{\gamma_M}{}_{\gamma_M{}'}$$

Before we prove this, we remark that this is not the strongest theorem possible: We related the $L+M+N$ matrix elements to the M matrix elements with the M-pathes starting at arbitrary vertices. In theorem 4.1.2 we saw that the commutant is isomorphic to the path algebra starting at 0 and hence it should be possible to express the $L+M+N$ matrix elements in terms of these. However, this problem is much more complicated and we do not know how to solve it.

Proof. We call $\bar{z} := \mathrm{id}\otimes z \in \mathcal{A}\otimes r(\mathcal{A})^{\otimes M}$ and $\tilde{z} := \mathrm{id}\otimes z\otimes \mathrm{id} \in r(\mathcal{A})^{\otimes L}\otimes r(\mathcal{A})^{\otimes M}\otimes r(\mathcal{A})^{\otimes N}$. For the sake of simplicity we restrict ourselves to simply laced $\mathcal{G}_r$, but the generalization to the general case is straight forward.

In order to prove the factor $\delta_{\gamma_L\gamma_L'}\delta\gamma_N\gamma_N'$ we remark that $\tilde{z}$ commutes with $\Delta_{1\dots i}(\mathbf{1}_\lambda)$ for $i = 1\dots L$ and $i = L+M+1\dots L+M+N$. The first case is obvious, and for the second we remark that, with the notations of sect. 1.3.1, $\Delta_{1\dots L+M+N}(x) = \Delta_{1\dots L+M}(x^{(1)})\otimes\Delta_{L+M+1\dots L+M+N}(x^{(2)})$ and because z is in the commutant of $\Delta_{1\dots L+M}(\mathcal{A})$, it commutes in particular with $\Delta_{1\dots L+M}(\mathbf{1}_\lambda{}^{(1)})$.

Now we call $\mathcal{C}^L_\lambda$ the commutant of $\mathcal{A}$ in $r(\mathcal{A})^{\otimes L}$, so $r(\mathcal{A})^{\otimes L} \supset \bigoplus \mathcal{C}^L_\lambda \otimes \mathcal{A}_\lambda$. The path projection onto an L-path $\mathbf{1}_{\gamma_L}$ is in the commutant of $\Delta_{1\dots L}$ and can be decomposed at $\mathbf{1}_{\gamma_L} = c_{\lambda_L}\otimes \mathrm{id}_{\lambda_L}$ where λ_L is the vertex of γ_L at L. The decomposition formula follows using that $\mathrm{tr}_{\mathcal{A}^{\otimes(L+M+N)}} = \sum_\lambda \mathrm{tr}_{\mathcal{C}_\lambda}\cdot \mathrm{tr}_{\mathcal{A}_\lambda\otimes\mathcal{A}^{\otimes(M+N)}}$ for a decomposing element as $\mathrm{id}_L\otimes z_M\otimes \mathrm{id}_N$. •

The next theorem gives relations between the partition functions that can be obtained under the tower inclusion out of a given operator z when we take the thermodynamic limit of the algebras.

Theorem 4.1.4 (Partition functions) *Let $z \in \Delta_{1\dots M}(\mathcal{A})'$ and define the partition functions*

$$(Z_M[z])^{\nu}{}_{\mu} := \frac{1}{\mathrm{tr}_{\mathcal{A}}(\mathbf{1}_\nu)}\cdot \mathrm{tr}_{\mathcal{A}\otimes r(\mathcal{A})^{\otimes M}}(\Delta_0(\mathbf{1}_\mu)\Delta_{0\dots M}(\mathbf{1}_\nu)\cdot z)$$

Then

(i). For each μ,ν the mapping $z \mapsto (Z_M[z])^{\nu}{}_{\mu}$ defines a trace on $\Delta_{1\dots M}(\mathcal{A})'$, which means it is linear and cyclic $Z_M[z_1\cdot z_2] = Z_M[z_2\cdot z_1]$.

(ii). The partition function is translation invariant[6] in the sense that

$$(Z_M[z])^{\nu}{}_{\mu} = \sum_\lambda N_{\lambda\mu}{}^{\nu}\cdot (Z_M[z])^{\lambda}{}_0 =: \sum_\lambda (N_\lambda)^{\nu}{}_{\mu}\cdot (Z_M[z])^{\lambda}$$

[6] Evidently this is a symmetry in the target space of graph vertices and not of the lattice. Compare (3.6) and (4.5).

(iii). Under the tower inclusion Z behaves as

$$(Z_{L+M+N}[z])^{\nu}{}_{\mu} = (\mathcal{G}_r{}^L \cdot Z \cdot \mathcal{G}_r{}^N)^{\nu}{}_{\mu} = (\mathcal{G}_r{}^{L+N} \cdot Z)^{\nu}{}_{\mu}$$

in the sense of matrices.

Proof.

(i) Is obvious when we know that $z, \Delta_0(\mathbf{1}_\mu)$ and $\Delta_{0\ldots M}(\mathbf{1}_\nu)$ all commute, what can be proven as in theorem 4.1.3 with the decomposition of $\Delta_{0\ldots M}(\mathbf{1}_\nu)$.

(ii) Like above we decompose $r(\mathcal{A})^{\otimes M} \supset \bigoplus \mathcal{C}_\lambda \otimes \mathcal{A}_\lambda \ni \oplus z_\lambda \otimes \mathrm{id}_\lambda = z$. Computing the trace over $\mathcal{A} \otimes r(\mathcal{A})^{\otimes M}$ with this decomposition, it factorizes to

$$(Z_M[z])^{\nu}{}_{\mu} = \sum_\lambda \mathrm{tr}_{\mathcal{A}_\mu \otimes \mathcal{A}_\lambda}(\Delta(\mathbf{1}_\nu)) \cdot \mathrm{tr}_{\mathcal{C}_\lambda}(z_\lambda)$$

The left expression is the trace over the space of all type ν representations contained in $\mathcal{A}_\mu \otimes \mathcal{A}_\lambda$ and hence $d_\nu \cdot N_{\lambda\mu}{}^{\nu}$. Taking in particular $\nu = \lambda$ and $\mu = 0$ we get the desired result.

(iii) To show this we use in a similar argument that for z_1, z_2 being in the commutant of two different representations r_1, r_2

$$\mathrm{tr}(\Delta(\mathbf{1}_\nu) \cdot z_1 \otimes z_2) = \sum_{\lambda\mu} N_{\lambda\mu}{}^{\nu} \cdot \frac{\mathrm{tr}(r_1(\mathbf{1}_\lambda)\, z_1)}{d_\lambda} \cdot \frac{\mathrm{tr}(r_2(\mathbf{1}_\mu)\, z_2)}{d_\mu}$$

and that $\mathrm{tr}_{r(\mathcal{A})}(\mathbf{1}_\nu) = d_\nu(\mathcal{G}_r{}^{\nu}{}_{\mu})$. •

We showed here a duality between some Hopf algebra gauge theories and path algebra theories whose underlying graph is related to the comultiplicative structure of the Hopf algebra. Examples of such gauge theories are numerous. The most intuitive ones are gauge theories in the classical sense, based on a classical group such as $U(1)$, $SO(3)$. The case of $U(1)$ is interesting because of its relation to Fourier transform as we will show in sect. 4.1.2. The Hopf algebra in this case is either the group algebra, which means the algebra of functions over the group with the product being the convolution with respect to the Haar measure [FH91, Hum72] or it is the enveloping algebra of the Lie algebra. Note that the second one can be formally considered as containing the first one, remarking that the group algebra is spanned as a vector space by the Dirac distributions over the group elements which satisfy the group multiplication law $\delta_{g_1} \star \delta_{g_2} = \delta_{g_1 g_2}$ and which can hence be identified with the exponentiated Lie algebra generators.

Studying the systems on the path algebra has advantages when we take the thermodynamic limits and consider the tower inclusions. They are trivial of the type $x \mapsto x \otimes \mathrm{id}$ in the gauge theories. However, once the global symmetry is divided out, the tower inclusions mix the different symmetry sectors in a manner which is coded in the associated graph $\mathcal{G}_r$, leading to algebras whose thermodynamic limits may be non-trivial containing type II or III factors [GdlHJ89].

4.1.2 Example: the Fourier transformation

As an example we show that the Fourier transformation is related to the group algebra of $U(1)$ [FH91, Hum72] and compute some objects important in our formalism.

The group algebra $C[U(1)]$ is the set of functions over the group with the algebra product being the convolution with respect to the Haar measure. So $f \in C[U(1)]$ is given by its values $f(\alpha)$ on the $\alpha \in U(1)$, and the product $f \star g$ is

$$(f \star g)(\alpha) = \int \mathrm{d}\beta\, f(\beta)\, g(\alpha - \beta) \tag{4.3}$$

with respect to the normalized Haar measure $\mathrm{d}\beta$. The group algebra is formally spanned as a vector space by the Dirac functions over the group elements δ_α with $\delta_\alpha(\beta) = \delta(\beta - \alpha)$. They satisfy $\delta_\alpha \star \delta_\beta = \delta_{\alpha+\beta}$ and hence the set of Dirac functions is isomorphic to the group itself. The unitary representations of the algebra are induced by the ones of the group. They are all one dimensional and they are labelled by an integer λ. Explicitly, the representation of type λ is $r^\lambda(\alpha) = e^{i\lambda\alpha} \in \mathrm{End}C$ and the central projection onto the representations of type λ in an arbitrary representation r is

$$\mathbf{1}_\lambda = \int \mathrm{d}\alpha\, e^{-i\lambda\alpha}\, r(\alpha) \tag{4.4}$$

The group algebra is a cocommutative Hopf algebra under the coproduct $\Delta(\delta_\alpha) = \delta_\alpha \otimes \delta_\alpha$. For $U(1)$, it induces the fusion rules

$$r^\mu \otimes r^\nu = r^{\mu+\nu} \iff N_{\mu\nu}{}^\lambda = \delta_{\mu+\nu,\lambda} \tag{4.5}$$

and hence translation invariance (3.6) becomes $Z^\nu{}_\mu = Z^{\nu-\mu}$.

We suppose now a physical system where the degrees of freedom are $U(1)$ variables on a square lattice. Using a transfer matrix formalism, a vector in the reduced configuration space $\varphi(\alpha_1 \ldots \alpha_N)$ is a function of N $U(1)$ variables α_i, representing one line one the lattice. The transfer matrix is a function of $2N$ variables $T(\alpha_i|\beta_i)$, and it acts on the configuration space as

$$(T\varphi)(\alpha_1 \ldots \alpha_N) = \int \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_N\, T(\alpha_1 \ldots \alpha_N|\beta_1 \ldots \beta_N)\, \varphi(\beta_1 \ldots \beta_N) \tag{4.6}$$

For the XY model in zero magnetic field [ID89] the transfer matrix is for example

$$T(\alpha_i|\beta_i) = e^{-J\left(\sum_i \cos(\alpha_i - \beta_i) + \sum_i \cos(\beta_{i+1} - \beta_i)\right)} \tag{4.7}$$

The global action Δ_N is

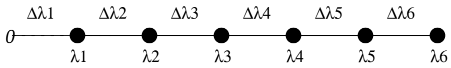
$$(\Delta_N(\gamma))(\alpha_i|\beta_i) = \prod_i \delta(\alpha_i - \beta_i - \gamma) \tag{4.8}$$

leading to

$$\begin{aligned}
(\Delta_N(\gamma)\,\varphi)(\alpha_i) &= \varphi(\alpha_i - \gamma) \\
(\Delta_N(\gamma)\,z)(\alpha_i|\beta_i) &= z(\alpha_i - \gamma|\beta_i) \\
(z\,\Delta_N(\gamma))(\alpha_i|\beta_i) &= z(\alpha_i|\beta_i + \gamma) \\
(\Delta_N(\gamma)\,z\,\Delta_N(\gamma)^{-1})(\alpha_i|\beta_i) &= z(\alpha_i - \gamma|\beta_i - \gamma)
\end{aligned} \tag{4.9}$$

So an operator $z(\alpha_i|\beta_i)$ commutes with the global action of $C[U(1)]$ iff it depends only on differences of the variables. Note that this does not mean that z depends only on the differences $\alpha_i - \beta_i$, see for example (4.7).

Each local space carries the whole algebra and hence all representations λ. The associated fusion graph is $(Z, +Z)$, so each vertex is connected to each vertex exactly once. Any sequence of vertices $(\lambda_1, \ldots, \lambda_N)$ defines a path γ. We add a formal vertex $\lambda_0 = 0$ and call $\Delta\lambda_i := \lambda_i - \lambda_{i-1}$.



Then

$$(\mathbf{1}_\gamma\, \varphi)\,(\alpha_i) = e^{-i(\Delta\lambda_1\alpha_1 + \cdots + \Delta\lambda_N\alpha_N)}\, \hat{\varphi}(\Delta\lambda_1, \ldots, \Delta\lambda_N) \tag{4.10}$$

where $\hat{\varphi}$ denotes the Fourier transformation of the configuration vector φ.

For an operator $z(\alpha_i|\beta_i)$ we introduce the new coordinate $\delta_i = \beta_i - \alpha_i$ and note $\tilde{z}(\delta_i; \alpha_i) := z(\alpha_i|\alpha_i + \delta_i)$. With the paths $\gamma = (\lambda_i)$ and $\gamma' = (\mu_i)$ we compute

$$(\mathbf{1}_\gamma\, z\, \mathbf{1}_{\gamma'})\,(\alpha_i|\beta_i) = e^{i(\Delta\mu_i\beta_i - \Delta\lambda_i\alpha_i)} \times \int \mathrm{d}\eta\, \mathrm{d}\delta\; e^{-i[(\Delta\lambda_i - \Delta\mu_i)\eta_i - \Delta\mu_i\delta_i]}\, \tilde{z}(\delta_i; \eta_i) \tag{4.11}$$

Note that the right factor is the Fourier transformation $\widehat{\tilde{z}}(\Delta\mu_i; \Delta\mu_i - \Delta\lambda_i)$, and if $\tilde{z}$ depends only on the δ_i it becomes $\prod_j \delta_{\Delta\mu_j, \Delta\lambda_j} \times \widehat{\tilde{z}}(\Delta\mu_1 \ldots \Delta\mu_N)$.

Furthermore we find that the reduced partition function is

$$Z^\lambda = \mathrm{tr}({\mathbf{1}_\lambda}^N z) = \int \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_N \int \mathrm{d}\alpha\; e^{-i\lambda\alpha}\, z(\beta_i - \alpha|\beta_i) \tag{4.12}$$

or in terms of $\tilde{z}(\delta_i; \beta_i)$

$$Z^\lambda = \int \mathrm{d}\alpha\; e^{-i\lambda\alpha} \left(\int \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_N\; \tilde{z}(\alpha \ldots \alpha; \beta_1 \ldots \beta_N) \right) =: \int \mathrm{d}\alpha\; e^{-i\lambda\alpha} z_{\mathrm{ft}}(\alpha) \tag{4.13}$$

as the inverse Fourier transformation of a function $z_{\mathrm{ft}}(\alpha)$. Using that the fusion eigenvalues are $\gamma(a, \lambda) = e^{ia\lambda}$ we can identify the latter with the Fourier transformed reduced partition function

$$Z^a = \sum_\lambda \gamma(a, \lambda)\, Z^\lambda = z_{\mathrm{ft}}(a) = \int \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_N\; \tilde{z}(a \ldots a; \beta_1 \ldots \beta_N) \tag{4.14}$$

The renormalized partition function $\bar{Z}$ is not well defined in this example: The adjacency matrix is $\mathcal{G}_r = \sum_{\lambda=-\infty}^{\infty} \mathcal{G}_\lambda$, leading to eigenvalues $\mathcal{G}_r\,|a\rangle = 2\pi\delta(a)\,|a\rangle$. This singularity reflects the fact that here the local system has a infinity of degrees of freedom, leading to an infinite entropy.

4.1.3 Translation invariant processes

Here we consider the duality the other way around. Being inspired by the importance of the Fourier transformation for the solution of (classically) translation invariant processes one could try to associate a globally invariant gauge theory to a given translation invariant[7] process on a representation graph[8].

The physical interpretation here is close to the Feynmann picture used in string theories: the physical space is the graph, and the system's length coordinate is a time (for a one dimensional system) or a worldsheet coordinate. The expression

$$(Z_N)^{\nu}{}_{\mu} = \sum_{\gamma} z^{\gamma}{}_{\gamma} \text{ with } |\gamma| = N,\ \partial\gamma = (\mu, \nu) \tag{4.15}$$

resembles a discrete version of the Feynmann definition of the two point function as summing over all the paths with a given length connecting the two points. In this interpretation, the diagonal elements $z^{\gamma}{}_{\gamma}$ are the Feynmann weights of the path γ, the non-diagonal elements only appear in a string type theory.

The accessible target spaces are quite exhaustive, for example all the representation graphs of Hopf algebras. From the geometric point of view, perhaps the most interesting ones are the graphs of $u(1)$, su(2) and su$(2)_q$ for q at a root of unity, leading to the discrete straight line with $0, 1$ or 2 boundaries respectively. Taking direct products of these algebras, we can obtains objects of arbitrary high dimensions. The possible paths are constrained by the adjacency conditions that are controlled by the choice of the local representations. In table 4.1 we gave some graphs and the local symmetry algebras related.

Algebra	Graph	is related to
$u(1)_{1\oplus\bar{1}}$	$(Z, \pm 1)$	distance 1 random walk on the line
$u(1)_{1\oplus\bar{1}\oplus 0}$	$(Z, \pm 1, 0)$	distance 0,1 random walk on the line
$\mathrm{su}(2)_1$	$(N, \pm 1)$	distance 1 random walk with one boundary
$\mathrm{su}(k)_1$	$(N^k, +1)/(1, \ldots, 1) = 0$	
$\mathrm{su}(2)_{q,1}$	$(\{1 \ldots l\}, \pm 1)$	distance 1 random walk with two boundaries
$\mathrm{su}(2)_{q,1\oplus 0}$	$(\{1 \ldots l\}, \pm 1)$	distance 0,1 random walk with two boundaries
$\mathrm{su(k)}_{q,1}$	$(N_l^k, +1)/(1, \ldots, 1) = 0$	

$q = e^{2\pi i/l}$ is a simple root of unity

$(\cdot)_0$ is the trivial and $(\cdot)_1$ the fundamental representation

Table 4.1: Graphs and Algebras

Unfortunately, the condition $Z^{\nu}{}_{\mu} = \sum N_{\lambda\mu}{}^{\nu} Z^{\lambda}$ that generalizes the translation invariance one the graphs is for other algebras as $u(1)$ – where it becomes $Z^{\nu}{}_{\mu} = Z^{\nu-\mu}$ – quite complicated and it is not at all clear if there are interesting processes whose partition function satisfy these relations.

[7]In the sense of theorem 4.1.4.

[8]For our purposes a random process is the set of paths on a graph together with a Boltzmann weight.

4.2 Conclusion

In this work, we studied the Hecke algebra models. Apart from their intrinsic interestingness in relation to conformal field theories, they can serve as an example of how statistical mechanics theories are described in the formalism of operator algebras. One accent in the presentation of this work was to identify the physical meaning of the apparent mathematical objects.

We pointed out that for the basic Hecke algebra models two dual descriptions exist. These two descriptions are quite different at first sight, relating two not obviously connected domains of modern physics. In the Spin-description, the models are related to quantum groups, the transfer matrix being constructed out of the quantum R-matrix. In the IRF-description, the transfer matrix lives in the path algebra of some graph, and the thermodynamic limit is related to C^* and W^*-algebra theory, in particular to the Jones-theory of subfactors.

We showed that the duality in the basic models consists of dividing out the global action of the symmetry algebra. This allows us to construct the Hecke algebra models as a real generalization, because they are no longer constrained by the requirement of having a quantum group representation.

We studied as basic objects the partition functions whose underlying geometry is that of a tube of variable length N and radius 1. We established a symmetry of these partition functions $(Z^N)^t{}_s$ which are dependent on the boundary conditions s, t that we called 'generalized translation invariance'. It is a symmetry in the vertex space of the graph, relating the boundary conditions (s, t) for the general models with the aid of the representation matrices of the fusion algebra to the partition functions of the basic models with the lower boundary fixed to 0. Mathematically these last objects are the characters of the Hecke algebra.

Our main results are a recurrence relation for the partition functions of the general TL_2 and TL_3-theories and a finite algorithm which allows to establish a k-step recurrence relation for the general TL_k models. This algorithm delivers the initial condition for the recurrence as well, and hence it allows us to solve the systems for any fixed k in a finite time.

The recurrence that we found is universal: When we describe all apparent objects in terms of the underlying representation of the fusion algebra, then the coefficients depend only on the deformation and the spectral parameters. It should be pointed out that these coefficients do not depend on k either. The difference between theories for different k is the one with the greater k contains more coefficients.

The fusion algebra shares an important property with the action of translations: it is the commutative. Hence with an appropriate choice of the basis in the vertex space, all the fusion matrices are diagonal. Like for a translation invariant theory in the classical sense, where it is advantageous to consider the Fourier transformation of the reduced two point function, a natural object to study here is the partition function in the diagonalizing basis Z^a. In this basis the recurrence relation becomes scalar in any sector a, the fusion matrices being replaced with their eigenvalues. It follows from the representation theory of commutative algebras that these are included in the spectrum of the fundamental fusion matrices, and hence the knowledge of the basic models is already exhaustive.

We solved the recurrence relation for the TL_k-theory explicitly and we found a non-analytic behaviour (cusps) of the free energy in dependence on the spectral and deformation parameters and the sector coordinate.

Possible extensions For future works, there may be some interest in computing the recurrence relations for higher k. However, from su(4) on it is necessary to make an extensive use of computer technology in order to factorize the resulting polynominals in p and q. Because of the complexity of the algorithm even powerful machines may saturate for small values of k.

Another extension is to reconstitute conformal invariance taking the thermodynamic limit in time direction as well, for example diagonalizing the transfer matrix with Bethe ansatz or related techniques that have been developed for non-cyclic boundary conditions.

Another way of taking the thermodynamic limit in time direction is to compute the partition function for a square of size (N, M) explicitly and then taking $M \to \infty$. Using the Hecke algebra relations and the cyclicity of the trace, the partition function of a tube of an arbitrary radius M – with either of the two boundaries uniformly fixed – can be expressed in terms of the partition functions Z^N that we computed in this work. However, the apparent coefficients will be complicated polynominals in both p and q, so finding a general formula is a complicated combinatorical problem.

Finally it could be possible to generalize the described duality in order to study stochastic processes in a gauge theory formalism – despite the fact that the notion of a generalized translation invariant system is not obvious – and it may be interesting to give an interpretation of the apparent objects in terms of non-commutative geometry.

4.3 Acknowledgement

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Appendix A

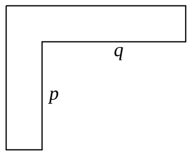
A recurrence for $q = 1$

We prove here the recurrence relation (3.26) at $q = 1$

$$\sum_{i=0}^{k}(-1)^i N_i \cdot Z^{N-i} = 0 \tag{A.1}$$

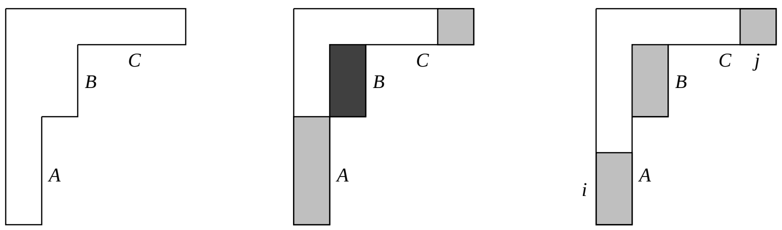
for the non-physical partition functions $Z^N = \mathrm{tr}(g_N \dots g_1)$. The partition operator is hence the cyclic permutation $g_N \dots g_1$ acting on the $N+1$-fold tensor product of fundamental su(k) representation spaces. We note that $Z^{N,\lambda}$ is the S_{N+1} character of the cyclic permutation. Its expression is explicitly known [ZI90]. It is non-vanishing only over the hook diagrams

$$Z^{N,\lambda} \equiv \mathrm{ch}^\lambda(g_N \dots g_1) = \sum_{p+q+1=N} (-1)^p\, \delta_{\lambda,\Lambda_{p,q}} \text{ with } \Lambda_{p,q} := \tag{A.2}$$



and we show that the sequence of Z^N given here satisfies the recurrence relation (A.1).

We recall that the rules for the multiplication of Young diagrams imply that the product of a diagram with the one of the fundamental representations (i) is the sum over all diagrams obtained by adding i boxes in different rows. The diagrams obtained out of the Hook diagrams are exactly the ABC-diagrams defined in figure A.1. The constraints for su(k) make that $A + B \leq k - 1$.



The ABC diagrams (L); the boxes certainly (darkgrey) and maybe added (lightgrey) when obtained out of a hook diagram (C); how to obtain a certain diagram out of a hook diagram (R)

Figure A.1: ABC-diagrams

So we keep one diagram (A, B, C) fixed and study how it can be produced inserting (A.2) into (A.1). Regarding figure A.1 (R) we see that there are exactly $2A$ possibilities to obtain the given diagram (A, B, C) out of the set of hook diagrams by multiplying with a fundamental representation.

We label these possibilities with $i = 0 \dots A-1$ and $j = 0, 1$, the pair (i, j) standing for the product

$$(B + i + j) \cdot \Lambda_{A+B-i,C-j} \tag{A.3}$$

It catches the sign $(-1)^{B+i+j}$. So the contributions of the terms $(i, \pm 1)$ have opposite sign and hence cancel two by two, which proves (A.1).

Appendix B

A toy model

Here we show that the formulas obtained in sect. 3.2 allow an interpretation of a translation invariant path algebra system as a composite system built out of non-interacting components, each of them associated to one of the eigenvectors $|a\rangle$ and having the dimension $|\gamma_a|$[1], and we give an interpretation of the renormalised partition function $\bar{Z}$ in terms of an entropy normalisation.

We compare our system based on a path algebra with the example of a toy system consisting of C independent 'differently coloured' components. The hypothesis of independence implies that the observable algebra — we call it $\Gamma_{m,n}$ as well — is the direct sum over the component algebras

$$\Gamma_{m,n} = \bigoplus_{c=1}^{C} \left(\bigotimes_{i=m}^{n} \mathcal{A}_{c,i} \right) \qquad \text{with } \mathcal{A}_{c,i} \cong \mathcal{A}_c \ \forall i \tag{B.1}$$

where $\mathcal{A}_c$ is the local observable algebra of the system with colour c. We take the local systems as having a very simple dynamics: $\mathcal{A}_c$ is the d_c dimensional matrix algebra M_{d_c} and the Hamiltonian acts 'hyperlocally' on each lattice site separately. Explicitly we assume the Hamiltonian

$$H_{m,n} = \bigoplus_{c=1}^{C} \sum_{i=m}^{n} H_{c,i}, \quad H_{c,i} = \mathrm{id}_c \otimes \cdots \otimes \begin{pmatrix} \varepsilon_{1,c} & & \\ & \ddots & \\ & & \varepsilon_{d_c,c} \end{pmatrix}_c \otimes \cdots \otimes \mathrm{id}_c \tag{B.2}$$

and the partition operator depending on the inverse temperature β becomes

$$z^N(\beta) = e^{-\beta \sum_{i=1}^{N} H_i} = e^{-\beta H_N} \cdots e^{-\beta H_1} \tag{B.3}$$

In the infinite temperature limit $\beta \to 0$, the operator $z^N(\beta)$ converges to the identity operator. The z^N here have a product form similar to those of the Hecke partition operator $X_N \dots X_1$ with the difference, that here the local operators over different points i commute[2].

[1] In the case that the γ_a are not positive, the phase factor is due to the symmetry of the graph (sect. 1.4.1) and we ignore it here.

[2] This remains true for any classical system even if the interaction couples adjacent sites. However, for quantum systems, this is not the case; even computing the factorised form out of the exponential form like in (B.3) is in general impossible.

The algebras $\Gamma_{m,n}$ satisfy the same tower relations (3.9) as the path algebras. We can follow literally the same argumentation as in the preceding section: when we define

$$Z^{n-m,a} := \mathrm{tr}_{m,n}^{c=a}(z) \tag{B.4}$$

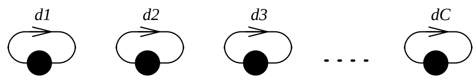
the natural partition function with the natural trace in the colour sector $c = a$, then the partition function associated to the same z but taken in the path sector $(m - m', b), (n + n', a)$ is

$$Z^{n+n',a}{}_{m-m',b} = \delta_{a,b}\, d_a^{n'+m'}\, Z^{n-m,a} \tag{B.5}$$

and hence the invariant trace is

$$\mathrm{Tr}^b{}_a = \delta_{a,b}\, d_a^{-N} \cdot Z^{N,a} \tag{B.6}$$

We could have obtained the same result observing that in the case that $\mathcal{A}_{loc,c} = M_{d_c}$ is the full matrix algebra, then $\Gamma_{m,n}$ is the path algebra of the graph



(B.7)

We note however the important difference that here the decomposition even takes place on the algebra level while for the path algebra only partition functions decompose.

Renormalised partition function. The factor $\delta_{a,b}$ in (B.6) indicates, that the different components of the system do not interact, and hence we study them separately in what follows. Like in (3.17) we define the renormalised partition function $\bar{Z}^a$ with the aid of the invariant trace (3.19) as

$$\bar{Z}^{N,a} := \mathrm{Tr}^{M,a}(z^N) \tag{B.8}$$

independent of $M \geq N$. Using that $\bar{Z}(\beta = 0) = \mathrm{id}$ we see that

$$\bar{Z}^{N,a}(\beta) = Z^a(\beta)/Z^a(\beta = 0) \tag{B.9}$$

Hence $\bar{Z}$ is normalised to $\bar{Z}^a(\beta = 0) = 1$ and the associated *renormalised specific free energy* $\bar{f}^a = -\lim_N(\frac{1}{N}\log \bar{Z}^{N,a})$ is zero for $T = \infty$. The difference $f^a - \bar{f}^a$ is the entropy contribution to the free energy in the sector a and $\bar{f}$ is hence normalised to entropy zero. In the toy model, all these objects are easy to compute and reproduce well known formulas of classical thermodynamics

$$\begin{array}{lcl|lcl}
Z^{N,a}(\beta) & = & \left(\sum_{i=1}^{d_a} e^{-\beta\varepsilon_{a,i}}\right)^N & f^a(\beta) & = & -\log\left(\sum_{i=1}^{d_a} e^{-\beta\varepsilon_{a,i}}\right) \\
Z^{N,a}(\beta = 0) & = & d_a^N & f^a(\beta = 0) & = & -\log d_a \\
\bar{Z}^{N,a}(\beta) & = & \left(\sum_{i=1}^{d_a} e^{-(\beta\varepsilon_{a,i} + \log d_a)}\right)^N & \bar{f}^a & = & f^a(\beta) - f^a(\beta = 0)
\end{array} \tag{B.10}$$

The entropy contribution is related to Jones subfactor theory ([Wen88, GdlHJ89]; see sect. 1.4.2). The algebra $\mathbf{1}_a\Gamma_{n,\infty}$[3] is a factor[4] and $\mathbf{1}_a\Gamma_{n,\infty} \subset \mathbf{1}_b\Gamma_{n+1,\infty}$ is a subfactor with index

$$[\mathbf{1}_a\Gamma_{n,\infty} : \mathbf{1}_b\Gamma_{n+1,\infty}] = \delta_{a,b} \cdot \begin{cases} d_a^2 & \text{(coloured algebras)} \\ \gamma_a^2 & \text{(path algebras)} \end{cases} \tag{B.11}$$

and hence the entropy contribution is

$$f^a - \bar{f}^a = \frac{1}{2} \log[\mathbf{1}_a\Gamma_{n,\infty} : \mathbf{1}_b\Gamma_{n+1,\infty}] \tag{B.12}$$

[3] Or, equally, the algebra obtained out of $(\Gamma_{n,\infty}, \mathrm{tr}^a)$ with the GNS-construction.

[4] At least for the toy model, where it is a type I-factor, and for the path model for the Perron-Frobenius sector where it is a type II_1-factor.

Appendix C

Hecke generators

In the literature two different definitions of q can be found. They are related as

$$q_R = q^2 \tag{C.1}$$

where q_R stands for Reshetikhin. We adopted the other convention that we called q throughout the work.

Which convention an author applies can be most easily seen by which definition of β he uses

$$\beta = q_R^{\frac{1}{2}} + q_R^{-\frac{1}{2}} = q + q^{-1} \tag{C.2}$$

Furthermore, there are different sets of generators commonly used: g, U in the papers of Di Francesco, Saleur/Zuber etc. et e, h in the papers of Jones and Wenzl (h is called g there). X_p is the face transfer matrix depending on the spectral parameter p.

	U	g	h	e
$U =$		$q-g$	$q-q^{-1}h$	$\beta e = (q+q^{-1})e$
$g =$	$q-U$		$q^{-1}h$	$q-\beta e = q(1-e)-q^{-1}e$
$h =$	q^2-qU	qg		$q^2(1-e)-e$
$e =$	$\beta^{-1}U$ $\frac{U}{q+q^{-1}}$	$\beta^{-1}(q-g)$ $\frac{q-g}{q+q^{-1}}$	$\beta^{-1}(1-q^{-1})h$ $\frac{q^2-h}{q^2+1}$	
$X =$	$F_{p,q}\mathrm{id}+G_{p,q}\,U_i, \quad F_{p,q}=\frac{qp^{-1}-q^{-1}p}{q-q^{-1}}, \quad G_{p,q}=\frac{p-p^{-1}}{q-q^{-1}}$ $q=e^{i\eta},\ p=e^{i\alpha} \quad F_{\alpha,\eta}=\frac{\sin(\eta-\alpha)}{\sin\eta}, \quad G_{\alpha,\eta}=\frac{\sin(\alpha)}{\sin\eta}$			

$A=U,g,h,e,X$	$A_iA_j=A_jA_i, \quad \lvert i-j\rvert\geq 2$
e	$e_i^2=e_i$ $e_i\,e_{i+1}\,e_i-\frac{q}{(1+q)^2}e_i=e_{i+1}\,e_i\,e_{i+1}-\frac{q}{(1+q)^2}e_{i+1}$
U	$U_i^2=\beta U_i$ $U_i\,U_{i+1}\,U_i-U_i=U_{i+1}\,U_i\,U_{i+1}-U_{i+1}$
g	$g_i^2=1+(q-q^{-1})\,g_i$ $g_i\,g_{i+1}\,g_i=g_{i+1}\,g_i\,g_{i+1}$
h	$h_i^2=q+(q-1)h_i$ $h_i\,h_{i+1}\,h_i=h_{i+1}\,h_i\,h_{i+1}$
X	$X_i^2=(p+p^{-1})X_i+\left(\frac{p-p^{-1}}{q-q^{-1}}\right)^2-1$ $X_i\,X_{i+1}\,X_i-X_i=X_{i+1}\,X_i\,X_{i+1}-X_{i+1}$

Appendix D

Notations

μ, ν, λ	Young diagrams, su(k)$_q$ irreducible representations, vertices of the fusion graphs.
s, t, u	Vertices of the general graphs.
a, b, c	Indices of the eigenbasis of the graphs etc..
γ	A path, as index a label for the canonical basis in the path space.
$\Gamma_{m,n}, \Gamma_{n,\infty}$	The path algebra on paths indexed from m to n / the inductive limits.
z	The *partition operator* or *weight operator*.
Z	$=\mathrm{tr}(z)$ *partition function*, seen abstractly as operator in boundary condition space.
$\bar{Z}$	$=\mathrm{Tr}(z)$ *renormalized partition function*, normalized to entropy zero.
$z^N, Z^N, \bar{Z}^N$	The object related to paths of length $N+1$.
$z_{m,n}, Z_{m,n}, \bar{Z}_{m,n}$	The object related to paths indexed with $m \dots n$.
$<\gamma' \vert\, z \,\vert \gamma> = z^{\gamma'}{}_{\gamma}$	A matrix element in the path space.
$\mathrm{tr}^{\mu,M}{}_{\nu,N}, \mathrm{Tr}^{\mu,M}{}_{\nu,N}$	Trace in the sector of paths indexed from N to M running from μ to ν.
$\mathrm{tr}^{\mu}{}_{\nu}, \mathrm{Tr}^{\mu}{}_{\nu}, \mathrm{tr}^{s}{}_{t}, \mathrm{Tr}^{s}{}_{t}$	Natural and invariant trace in the sector (μ, ν) or (s, t).
$Z^{\mu}{}_{\nu}, \bar{Z}^{\mu}{}_{\nu}, Z^{s}{}_{t}, \bar{Z}^{s}{}_{t}$	The partition function in the sector (μ, ν) or (s, t) .
$Z^{a}{}_{b}, \bar{Z}^{a}{}_{b}$	The partition function seen as operator in boundary condition space in the eigenbasis of the graphs.
$Z^{\lambda}, \bar{Z}^{\lambda}$	The *reduced partition function*, reduced by translational invariance.
$Z^{a}, \bar{Z}^{a}$	The *Fourier transformed partition function* seen as vector in the eigenbasis of the graphs.

${N_{\mu\nu}}^{\lambda}$	The $\mathrm{su(k)}_q$ *fusion coefficients.*
(μ)	An abstract fusion algebra element.
$N_\mu \equiv (N_\mu)^{\lambda}{}_{\nu}$	The fusion matrices or graphs, sometimes used as $\tilde{N}_\mu$.
$(i), N_i$	The fundamental fusion objects related to the one column Young diagrams.
$\tilde{N}_\mu, \tilde{N}_i$	The image of the fusion objects under a representation on a graph.
$\mathcal{G}$	The graph on which the model is based.
$\gamma(a,\lambda), \gamma_a$	The eigenvalues of the fusion matrices, the graph.
$\mathbf{1}, \mathbf{1}_\lambda, {\mathbf{1}_\lambda}^{1\ldots N}$	Identity operator, central projection, acting in the spaces $1\ldots N$.
$\hat{\mathbf{1}}$	$= \sum_{\gamma\gamma'} \lvert\gamma\rangle\langle\gamma'\rvert$ such that $\langle\gamma\rvert\, \hat{\mathbf{1}}\, \lvert\gamma'\rangle = 1$.

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