Mirror Symmetry among (2,2) and (0,2) String Vacua

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Chapter 1

Introduction

Seven years after its discovery it has become clear that mirror symmetry is an indispensible tool for extracting the physics (2,2) supersymmetric vacua of the string. Among the more important issues which benefit extensively from mirror symmetry is the computation of Yukawa couplings, not only at special points of the moduli space but as functions of the moduli. This result in essence preempties to a certain degree one of the major motivations for considering string vacua constructed with exactly solvable models (which themselves however are at present, important for establishing mirror symmetry to begin with). These computations represent one of the few instances where physical methods turn out to be vastly superior to standard algebraic methods. Other problems to which mirror symmetry has been successfully applied include topology change of a mild kind as well as the recent developments in Type IIA/Type IIB duality and Heterotic/Type IIA duality.

Useful as it is, there always has been the question whether this symmetry is in fact an artefact of the restriction to (2,2) vacua but not a property of more general (0,2) supersymmetric ground states. In such a situation mirror symmetry would not be a true property of the N=1 spacetime supersymmetric strings but would only be valid in a small corner of its moduli space. This long-standing problem has been solved recently [26] and it has been shown that (0,2) mirror symmetry has the same status as (2,2) mirror symmetry: it is a valid symmetry for certain classes of exactly solvable theories as well as more general (0,2) Landau-Ginzburg theories and certain stable vector bundles over Calabi-Yau manifolds, succinctly called (0,2) Calabi-Yau manifolds.

It is the purpose of the present manuscript to review the present status of mirror symmetry in both the (2,2) framework as well as the very recent developments in (0,2) compactifications.

To begin with, it is not easy, perhaps surprisingly, to say in full generality what mirror symmetry actually is without reference to any particular framework for string compactification (not to mention provide a proof). As a first step one may consider the conventional framework, formulated in [39], in which left–right symmetric compactifications without torsion are described by internal spaces which are compact complex Kähler manifolds and admit a covariantly constant spinor, i.e. have vanishing first Chern class, so–called Calabi–Yau manifolds [1]. Such manifolds are particularly simple, a fact that is encoded concisely in the spectrum of the theory described, in part, by the cohomology of the space.

Because the space is complex the real cohomology can be decomposed, via the Hodge decomposition, into complex cohomology groups. Thus the Betti numbers $b_i = \dim H^i(\mathcal{M}, \mathbb{R})$ can be expressed in terms of the Hodge numbers $h^{(p,q)} = \dim H^{(p,q)}(\mathcal{M}, \mathbb{C})$: $b_i = \sum_{p+q=i} h^{(p,q)}$. Because the manifold is Kähler the Hodge numbers are symmetric, $h^{(p,q)} = h^{(q,p)}$, and because the first Chern class vanishes it follows that $h^{(1,0)} = 0 = h^{(0,1)}$ and $h^{(3,0)} = 1 = h^{(0,3)}$. Hence the cohomology of the internal space, summarized in the Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{(1,1)} & & 0 & \\
1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\
 & 0 & & h^{(1,1)} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

contains only two independent elements $h^{(1,1)} = h^{(2,2)}$ and $h^{(2,1)} = h^{(1,2)}$ which parametrize the number of antigenerations and generations, respectively, that we observe in low energy physics.

It has become clear through the work of Gepner [86][87] [88] on the one hand and Martinec [144] and Vafa and Warner [176] on the other, that an alternative way of thinking about the *same* conformal fixed points of the heterotic string is via exactly solvable models and their mean field theories. The virtue of these formulations lies in the fact that they are easier to deal with since they are formulated in a nongeometric fashion. They do, in particular, allow for a detailed analysis of the massless spectrum[2] which naturally falls in the same pattern as the Hodge diamond above when vacua with central charge nine are being considered. More insight into the relation between Landau-Ginzburg theories and Calabi-Yau was provided by Witten using the linear σ-model [179].

It is in the context of Landau-Ginzburg theories and Calabi-Yau manifolds that mirror symmetry was discovered in [41]. It was observed in that paper that for the majority of string vacua described by Landau–Ginzburg potentials with an isolated singularity there exists another vacuum

[1] A certain emphasis on the geometrical picture may be justified in virtue of the fact that not only is the manifold framework at present the most general but also the least understood.

[2] Attention here is focused on the generations and antigenerations. The gauge singlets are not considered.

for which the number of generations and antigenerations is reversed, or put differently, the Hodge diamond is flipped along the offdiagonal. Since the class of theories constructed in [41] can also be understood as a special class of Calabi–Yau manifolds this result can be interpreted as an explicit demonstration of a symmetry on the space of all Calabi–Yau manifolds, a property that came as quite a surprise to mathematicians.

Even though almost inconceivable, in principle the observation of [41] might have been considered an accidental pairing of a priori independent theories. That this is not the case was shown in [141] via a technique called fractional transformations. It was demonstrated in there, and in more detail in [143], that mirror maps could indeed be established between many different types of Landau–Ginzburg theories with mirror spectra.

Contemporaneously with the discovery of [41] an alternative route to mirror symmetry was found in [97]. While constructing orbifolds of Gepner models these authors observed that upon repeated orbifolding a pattern emerged which did not only produce the mirror spectrum of the original model but also mirror symmetric theories to orbifolds constructed along the way. In this exactly solvable framework mirror symmetry can be described as a Kramers–Wannier type symmetry of the underlying parafermionic theories. Both of these approaches are complementary. Whereas the Kramers–Wannier symmetry shows that the original theory and it mirror are in fact isomorphic as conformal field theories, and differ only in their large radius expansions, the Landau-Ginzburg/Calabi-Yau constructions are vastly more general in practice and also allow to explicitly construct mirror maps between moduli spaces.

The discovery of mirror symmetry per these two constructions just described provided new impulse to string compactification and a number of different aspects have been investigated vigorously by many groups. One of the obvious question to be addressed was whether whether mirror symmetry is an artefact of the particular framework in which it was discovered and if not, then what the natural context is in which to properly define and possible show the existence of this symmetry in a satisfactory way. The problem has been discussed in a number of papers [16, 18, 122, 126, 127, 130, 128, 131, 132, 161, 36]. On the other hand the practical question arises what mirror symmetry can teach us about string theory. Simply put, the answer to the latter is that mirror symmetry allows us to do things that were impossible before its discovery. Main foci in this directions have been problems concerning a mild type of topology change [6, 7, 8], questions regarding the action of the modular group on moduli space [30, 33, 45, 50], and the computation of Yukawa couplings, pioneered in [33]. The latter has sparked an immense interest in mathematics and has put mirror symmetry to the forefront of algebraic geometry. It is presently being used as a hypothesis to obtain results in enumerative algebraic geometry and has been shown to lead to correct results in all computations that have been performed so far for genus zero instantons [13, 15, 17, 33, 34, 37, 44, 29, 69, 75, 72, 73, 105, 116, 123, 124, 136, 147] as well as higher genus instantons [21, 22] in Calabi–Yau manifolds of complex dimension three and for genus zero instantons in Brieskorn–Pham Calabi–Yau manifolds for $dim_{\mathbb{C}} M > 3$ [99, 117].

More recently mirror symmetry has become important in the context of nonperturbative dualities in string theory i [28, 109, 112, 74, 168, 175, 181] and the physics of severe phase transitions which change not only the Yukawa couplings but the massless spectrum [31, 143, 171, 100].

These remarks indicate that at present a thorough discussion of all aspects of mirror symmetry is beyond the scope of any one review. Instead, as indicated already, in the present work we focus mostly on the question 'what is the proper framework for mirror symmetry'? The individual chapters follow roughly the historical development of our understanding of the structure of the moduli space of string theory and the discovery of mirror symmetry. Fortuitiously, at the same time a certain logic emerges according to which the configurations discussed in later chapters are more general than the ones discussed earlier, thereby expanding our horizon step wise. This is a feature for which a price has to be paid: our detailed understanding of the models becomes more and more incomplete. Ascending from exactly solvable models to Landau–Ginzburg models and from there to Calabi–Yau manifolds which in turn will be superseded by a special class of Fano–folds will bring us from a complete knowledge of the partition function to theories where it is even difficult to compute the massless sector alone. It will become that at present the class of Landau-Ginzburg theories is the one that appears to be a rather useful framework: it is rather general and it allows good control of the spectrum. Supplementing this with the linear σ-model we also gain insight into the structure of Calabi-Yau manifolds. However, the space of all Calabi–Yau manifolds has a disconcertening property: it contains spaces which are rigid, i.e. they do not have string modes corresponding to complex deformations of the manifold, fields that describe generations in the low energy theory. Since mirror symmetry exchanges complex deformations and Kähler deformations of a manifold, the latter describing the antigenerations seen by a four–dimensional observer, it would seem that the mirror of a rigid Calabi–Yau manifold cannot be Kähler and hence does not describe a consistent string vacuum. In fact, it appears that the mirror vacuum cannot even be $N = 1$ spacetime supersymmetric [188]. This shows that the class of Calabi–Yau manifolds is not the appropriate setting for mirror symmetry and the question arises what structure should replace the space of Calabi–Yau manifolds. To lead the way to this structure is one of the goals of this review. Finally, we discuss the generalizations of mirror symmetry to the (0,2) framework.

As a final technical remark it should be mentioned that because any reasonably detailed discussion of mirror symmetry needs many different techniques it is occasionally useful to jump ahead and use material of later chapters. This is awkward but is the only way to break this long exposition into individual readable parts. Trying to avoid this at all cost would make the presentation even more awkward.

Chapter 2

ADE Vacua

2.1 Overview

Mirror symmetry is most easily discussed in the framework of exactly solvable compactifications. Among these Gepner's framework [86] of (2,2)–vacua in terms of minimal N=2 superconformal models stands out because of its simplicity. It is also in the context of these realizations of (2,2) supersymmetric heterotic string compactifications that the relation between exactly solvable models, Landau–Ginzburg theories and Calabi–Yau compactifications is most easily discussed. This relation was uncovered by investigating the spectrum and discrete symmetries of particular exact models and Calabi–Yau manifolds at special points in their moduli space. The correspondence between exactly solvable models, Landau–Ginzburg theories and Calabi–Yau manifolds will be discussed in detail in later chapters but for illustration we mention here two examples. The first is the perhaps simplest Calabi-Yau space, simply called the 'quintic'

$$(3^5)_{A^5} \;\sim\; \mathbb{C}_{(1,1,1,1,1)}[5] \;\sim\; \mathbb{P}_4[5] \qquad (2.1.0.1)$$

whereas the second is the only exactly solvable Calabi-Yau 3-generation models that has been found. This 3-generation groundstate is obtained by an appropriate modding of the cover theory

$$(1 \cdot 16^3)_{A_2 \otimes E_7^3} \sim \mathbb{C}_{(2,2,2,3,3,3,3)}[9] \sim \begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_3 \end{array} \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} \qquad (2.1.0.2)$$

by the discrete group $\mathbb{Z}_3^2$. The notation will become clear as we proceed. Not only can it be shown that these models lead to the same massless spectrum and exhibit the same discrete symmetries but it can also be shown that the fields transform in the same way under discrete symmetry transformations. It was shown in [158] that the Calabi–Yau manifold on the rhs of (2.1.0.2) can be used to make a three–generation model with a two–step construction by first modding out a freely acting discrete symmetry group and then modding out a discrete action with fixed points. These fixed points need to be blown up and therefore introduce additional modes. It was realized by Gepner [88] that each step of this construction can be repeated in the conformal field theory model leading again to a three–generation model. The modes that previously originated from the blow–up are now created by the twisted sector.

This result, i.e. the rediscovery of a phenomenologically interesting model in a language that made the computation of a number of physical quantities such as Yukawa couplings, much easier, is sufficient motivation to investigate this class of theories in more detail. In this chapter, following [140] (see also [83]) we describe the complete construction of the subseries of Kazama and Suzuki corresponding to the minimal discrete series of N=2 models considered originally in [58, 59, 60, 27, 187]. Jumping slightly ahead we should remark that by using Vafa's prescription [173] to compute the cohomology groups of an $N = 2$ Landau–Ginzburg theory the spectrum of all LG–models corresponding to the ADE exact models can be computed and complete agreement is found betweeen the LG–results and the exact results.

As we will see in this chapter the original expectations for this class of models as far as its richness is concerned were rather exaggerated. The effort invested in understanding these theories is far from wasted, however, since this class is a very important stepping stone toward further generalizations. In particular as far as mirror symmetry is concerned it is the framework which is under most precise control.

2.2 Parafermions

The essential building block of the degenerate discrete series of conformal field theories with $N = 2$ supersymmetry are the parafermionic $\mathbb{Z}_k$ theories [187].

In a sense these theories are somewhat more general than standard conformal field theories, which are defined via a closed associative operator algebra with respect to OPEs. The parafermionic theories contain special non–local fields in the sense that the operator product of two operators

$$\mathcal{O}_1(z_1)\mathcal{O}_2(z_2) \tag{2.2.0.3}$$

acquires a phase

$$e^{2\pi i\gamma} \tag{2.2.0.4}$$

by continuation, e.g., in the variable z_1 along a closed contour around the point z_2 (counterclockwise). The basic difference compared to strictly local algebras is that the structure coefficients of the OPEs are no longer single valued function of $x \in \mathbb{R}$.

Such γ–local fields appear naturally in statistical systems with $\mathbb{Z}_k$ symmetries [80]. In such systems (in the continuum limit) there are $k-1$ fields σ_l of the "order parameter" corresponding to $k-1$ representations of $\mathbb{Z}_k$:

$$\sigma_l \longrightarrow \alpha^l \sigma_l \qquad (2.2.0.5)$$

where α is the generator of the cyclic group. Besides the σ_l there are $k-1$ fields μ_l which are "disorder parameters" [80]. These fields μ_l are mutually local but they are not local with respect to the $\sigma_l' s$. They are γ–local with the phase

$$\gamma = \frac{lm}{k}. \qquad (2.2.0.6)$$

A "dual" group $\tilde{\mathbb{Z}}_k$ acts on the fields μ_l

$$\mu_l \longrightarrow \alpha^l \mu_l \qquad (2.2.0.7)$$

A closed operator algebra can be constructed by considering OPEs between the $\sigma_l' s$ and the $\mu_l' s$. The fields in this theory can be classified according to their $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ charges; clearly the σ_l have charges $(l,0)$ and the μ_l fields have charges $(0,l)$.

The algebra generated by OPEs of the order and disorder parameters can be decomposed into fields with charges (l,m)

$$\mathcal{A} = \oplus_l \oplus_m \mathcal{A}_{(l,m)} \qquad (2.2.0.8)$$

which generate themselves a closed algebra

$$\mathcal{A}_{(l,m)} \mathcal{A}_{(l',m')} \subset \mathcal{A}_{(l+l',m+m')} \qquad (2.2.0.9)$$

Zamolodchikov and Fateev [186] have constructed $\mathbb{Z}_k \times \tilde{\mathbb{Z}}_k$ theories in terms of special "parafermionic symmetries". These theories can be defined for any level $2 \leq k \in \mathbb{N}$ and generalize the Ising model (which they describe for $k=2$). The level k theory is defined in terms of k conformal fields with fractional spin which satisfy the OPEs

$$\begin{aligned}
\psi_p(z)\psi_q(w) &= \frac{c_{p,q}}{(z-w)^{2pq/k}}\left[\psi_{p+q}(w) + O(z-w)\right], \qquad p+q<k && (2.2.0.10)\\
\psi_p(z)\psi_q^+(w) &= \frac{c_{p,k-q}}{(z-w)^{2p(k-q)/k}}\left[\psi_{p-q}(w) + O(z-w)\right], \qquad p>q && (2.2.0.11)\\
\psi_p(z)\psi_p(w) &= \frac{1}{(z-w)^{2p(k-p)/k}}\left[1 + \frac{2\Delta_p}{c}(z-w)^2 T(w) + O(z-w)^3\right] && (2.2.0.12)
\end{aligned}$$

where the c is the central charge of the theory and

$$\Delta_p := p(k-p)/k = \Delta_{k-p} \qquad (2.2.0.13)$$

is the spin of the p^{th} parafermion. Furthermore $\psi_0 = 1$ and $\psi_p^+ = \psi_{k-p}$. Because of eq. (2.2.0.12) the parafermion ψ_1 carries the essential information of the theory.

These parafermionic currents satify the equations

$$\partial_{\bar{z}}\psi_l = 0 = \partial_z\bar{\psi}_l \tag{2.2.0.14}$$

Demanding associativity of the above OPEs completely determines the structure coefficients $c_{p,q}$ appearing in these expressions [186]

$$c_{p,q} = \frac{(p+q)!(n-p)!(n-q)!}{(n-p-q)!p!q!} \tag{2.2.0.15}$$

as well as the central charges

$$c = 2(k-1)/(k+2). \tag{2.2.0.16}$$

The parafermionic currents have discrete charges

$$\begin{aligned} \psi_l &\in \mathcal{A}_{(l,l)} \\ \bar{\psi}_l &\in \mathcal{A}_{(l,-l)}. \end{aligned} \tag{2.2.0.17}$$

The total space $\mathcal{A}$ can be decomposed into n parafermionic families generated by spinless primary fields

$$\sigma_l \in \mathcal{A}_{(l,0)} \quad , \quad \Delta_l = \frac{k(k-l)}{k(k+2)}. \tag{2.2.0.18}$$

with discrete charge $(l, 0)$. Each of these fields generates a block of fields

$$[\sigma_l]_\psi \tag{2.2.0.19}$$

appearing in the OPEs of the field σ_l with all parafermionic currents $\psi_m, \bar{\psi}_m$.

Each of the subspaces $[\sigma_l]_\psi$ contains a series of fields

$$\phi_l^m \in \mathcal{A}_{(l-m,-m)} \tag{2.2.0.20}$$

of dimension

$$\Delta_l^m = \Delta_l + \frac{m(l-m)}{k}. \tag{2.2.0.21}$$

This formula shows that the field

$$\mu_l^+ = \mu_{n-l} = \phi_l^l \in \mathcal{A}_{(0,n-l)} \tag{2.2.0.22}$$

is of spin 0 and has the same dimension Δ_l as the order parameters σ_l. The fields μ_l are the components of the disorder parameter. It is shown in [186] that the space generated by the $[\sigma_l]_\psi$ generates an associative operator algebra. The corresponding structure constants and some correlation functions are comuted as well. It is also shown that all $\mathbb{Z}_n$ models are unitary.

The primary fields $\phi^{l,\bar{l}}_{m,\bar{m}}$, (where $l,\bar{l} \in \{0,1,2,...,k\}$ and $-l \leq m,\bar{m} \leq 2k-l$, $l-m \in 2\mathbb{Z}$, $\bar{l}-\bar{m} \in 2\mathbb{Z}$) of the theory have conformal dimensions $(\Delta^l_m, \Delta^{\bar{l}}_{\bar{m}})$ with

$$\begin{aligned} \Delta^l_m &= \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} \quad \text{for} \quad |m| \leq l \\ \Delta^l_m &= \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k} + \frac{m-l}{2} \quad \text{for} \quad l \leq m \leq 2k-l \end{aligned} \tag{2.2.0.23}$$

(analogous for $\Delta^{\bar{l}}_{\bar{m}}$). The independent weights can be found by identifying

$$\phi^l_m \sim \phi^l_{m+2k} \sim \phi^{k-l}_{m+k}. \tag{2.2.0.24}$$

The partition function of the parafermionic theory

$$Z = \sum_{l,\bar{l},m,\bar{m}} N^{l,\bar{l}}_{m,\bar{m}} c^m_l, \bar{c}^{\bar{m}}_{\bar{l}} \tag{2.2.0.25}$$

has two ingredients, the affine part and the theta function part. The affine invariants are listed in [91, 43]. There exist three infinite series: the trivial diagonal invariant exists for every level k; the second series starts at level 4 for $k = 4m$, $m \in N$ and the third one only exists beyond level 6 for $k = 4m-2$, $N \ni m \geq 2$. In addition to these infinite series there exist three exceptional invariants at level 10, 16 and 28.

For the theta function system the invariants have been discussed in ref. [90]. Consider the partition function

$$Z(\tau,\bar{\tau}) \sim \sum A_{m,n} \Theta_{m,k}(\tau) \Theta^*_{n,k}(\bar{\tau}) \tag{2.2.0.26}$$

of theta functions at level k and demand modular invariance. The most general solution is obtained by considering orbifolds of the diagonal invariant, i.e. it is parametrized by the divisors α of the level k; with $\beta = k/\alpha$ the solutions can be written as:

$$A_{m,n} = \frac{1}{2} \sum_{x \in \mathbb{Z}_{2\beta}, y \in \mathbb{Z}_{2\alpha}} \delta_{m,\alpha x + \beta y} \delta_{n,\alpha x - \beta y}. \tag{2.2.0.27}$$

There is one solution for every divisor α. For $\alpha = 1$ this defines the left–right symmetric solution

$$A_{m,n} = \delta_{m,n} \tag{2.2.0.28}$$

whereas for $\alpha = k$ one finds the theta function modular invariant

$$A_{m,n} = \delta_{m,-n}. \tag{2.2.0.29}$$

This result is of importance. Since the parafermionic theory is effectively a product of an affine theory and a theta function system, modding out with the full $\mathbb{Z}_k$ symmetry group will lead to the same theory described in terms of the dual spin variables μ_l [90].

Given a modular invariant solution A of the theta function system and an affine invariant L a modular invariant solution for the parafermionic system can be defined as

$$N_{l,\bar{l}}^{m,\bar{m}} = \frac{1}{2} L_{l,\bar{l}} A^{m,\bar{m}} \frac{1+(-1)^{l-m}}{2}. \tag{2.2.0.30}$$

Using the diagonal invariant for the partition function the Ising model is recovered as the parafermionic $\mathbb{Z}_2$ theory.

For $k = 2$ the affine index l and the theta–function index m run like $0 \leq l \leq 2$, $-l \leq m \leq 4-l$ with $l - m \in 2\mathbb{N}$. Hence the spectrum of the unitary model consists of the primary fields

$$\begin{array}{ll} l=0: & \phi_0^0,\ \phi_2^0,\ \phi_4^0 \\ l=1: & \phi_{-1}^1,\ \phi_1^1,\ \phi_3^1 \\ l=2: & \phi_{-2}^2,\ \phi_0^2,\ \phi_2^2\ . \end{array} \tag{2.2.0.31}$$

Using the diagonal affine invariant and the diagonal theta function invariant and expanding out the partition function

$$\begin{aligned} Z &= \frac{1}{4} \sum \delta_{l,\bar{l}} \delta^{m,\bar{m}} (1+(-1)^{l-m}) c_m^l \bar{c}_{\bar{m}}^{\bar{l}} \\ &= \frac{1}{4} \sum (1+(-1)^{l-m}) c_m^l \bar{c}_m^l \end{aligned} \tag{2.2.0.32}$$

one finds the following complete primary fields

$$\begin{array}{ll} l=0: & \phi_{0,0}^{0,0},\ \phi_{2,2}^{0,0},\ \phi_{4,4}^{0,0} \\ l=1: & \phi_{-1,-1}^{1,1},\ \phi_{1,1}^{1,1},\ \phi_{3,3}^{1,1} \\ l=2: & \phi_{-2,-2}^{2,2},\ \phi_{0,0}^{2,2},\ \phi_{2,2}^{2,2}\ . \end{array} \tag{2.2.0.33}$$

Not all these fields are independent however because of the field identifications. In the case at hand these reduce to

$$\phi_m^l \sim \phi_{m+4}^l \sim \phi_{m+2}^{2-l} \tag{2.2.0.34}$$

and hence there are only three independent fields

$$\begin{aligned}\phi^{0,0}_{0,0} = \phi^{0,0}_{4,4} = \phi^{2,2}_{2,2} &= \phi^{2,2}_{-2,-2} &: \Delta^0_0 = \overline{\Delta}^0_0 = 1\\ \phi^{0,0}_{2,2} &= \phi^{2,2}_{0,0} &: \Delta^2_0 = \overline{\Delta}^0_0 = \frac{1}{2}\\ \phi^{1,1}_{1,1} = \phi^{1,1}_{-1,-1} &= \phi^{1,1}_{3,3} &: \Delta^1_1 = \overline{\Delta}^0_0 = \frac{1}{16}.\end{aligned} \tag{2.2.0.35}$$

But these are precisely the weights of the Ising model fields.

Now consider the $\mathbb{Z}_2$ orbifold of the theta system, i.e. $\alpha = 2$ and hence $\beta = 1$. In this case the invariant becomes

$$\begin{aligned} A_{m,n} &= \frac{1}{2} \sum_{x\in\mathbb{Z}_2,\ y\in\mathbb{Z}_4} \delta_{m,2x+y}\ \delta_{n,2x-y}\\ &= \frac{1}{2}\sum_{x\in\mathbb{Z}_2} \delta_{m,4x-n}\\ &= \delta_{m,-n}\ , \end{aligned} \tag{2.2.0.36}$$

since in the level $k = 2$ parafermionic theory we have the identifications

$$m \ \sim m + 2k = m + 4. \tag{2.2.0.37}$$

Using this theta function modular invariant the parafermion partition function becomes

$$\begin{aligned} Z &= \frac{1}{4}\sum \delta_{l,\bar{l}}\delta^{m,-\bar{m}}(1+(-1)^{l-m})c^l_m\bar{c}^{\bar{l}}_{\bar{m}}\\ &= \frac{1}{4}\sum(1+(-1)^{l-m})c^l_m\bar{c}^l_{-m} \end{aligned} \tag{2.2.0.38}$$

and hence the primary fields of this theory are

$$\begin{aligned} l = 0 : &\quad \phi^{0,0}_{0,0}\\ l = 1 : &\quad \phi^{1,1}_{-1,1},\ \phi^{1,1}_{1,-1}\\ l = 2 : &\quad \phi^{2,2}_{-2,2},\ \phi^{2,2}_{0,0},\ \phi^{2,2}_{2,-2}\ . \end{aligned} \tag{2.2.0.39}$$

Because of the equivalences however the are again only three independent fields

$$\begin{aligned} \phi^{0,0}_{0,0} &= \phi^{2,2}_{2,2} = \phi^{2,2}_{-2,-2} = \phi^{2,2}_{2,-2} = \phi^{2,2}_{-2,2}\\ \phi^{1,1}_{1,-1} &= \phi^{1,1}_{1,3} = \phi^{1,1}_{3,1}\\ \phi^{2,2}_{0,0} &= \phi^{0,0}_{2,2} \end{aligned} \tag{2.2.0.40}$$

with the same weights $1, \frac{1}{16}, \frac{1}{2}$ respectively as in the diagonal theory. These are conventionally called

$$1 = \phi^{0,0}_{0,0}\ , \quad \mu = \phi^{1,1}_{1,-1}\ , \quad \epsilon = \phi^{2,2}_{0,0} \tag{2.2.0.41}$$

i.e. they are identified with the identity 1, the dual spin operator μ (also called disorder parameter) and the energy operator ϵ.

2.3 The N=2 Minimal Discrete Series

The N=2 superconformal discrete series can be constructed by combining a level k parafermion theory with a free scalar field. Set

$$\begin{aligned} T(z) &= T_{PF} + T_{\varphi}\,, \qquad J(z) = \frac{i}{\alpha}\partial\varphi \\ G(z) &= \frac{1}{\alpha}\psi_1(z)e^{i\alpha\varphi(z)}\,, \qquad \bar{G}(z) = \frac{1}{\alpha}\psi_1^{+}(z)e^{-i\alpha\varphi(z)} \end{aligned} \tag{2.3.0.42}$$

where $\alpha = \sqrt{k+2/k}$. These operators satisfy the N=2 superconformal OPEs at

$$c(k) = \frac{3k}{k+2} \tag{2.3.0.43}$$

at level k. The primary fields of the supersymmetric theory can then be related to the parafermions by

$$\Phi^{l}_{q,s}(z) = \phi^{l}_{q-s}(z)e^{i\alpha_{q,s}\varphi(z)} \tag{2.3.0.44}$$

where

$$\alpha_{q,s} = \frac{-q+\frac{s}{2}(k+2)}{\sqrt{k(k+2)}}. \tag{2.3.0.45}$$

Here $l \in \{0, ..., k\}$ and $l-q+s \in 2\mathbb{Z}$ and $|q-s| \leq l$ because of the relations of the parafermionic primary fields. The conformal weights and the charges of the $\Phi^{l}_{q,s}$ are then given by

$$\Delta^{l}_{q,s} = \frac{l(l+2)-q^2}{4(k+2)} + \frac{s^2}{8} \tag{2.3.0.46}$$

$$Q^{l}_{q,s} = -\frac{q}{k+2} + \frac{s}{2}. \tag{2.3.0.47}$$

The relations discussed above for these fields lead to the following identifications of N=2 primary fields:

$$\Phi^{l}_{q,s} \sim \Phi^{l}_{q+2(k+2),s} \sim \Phi^{l}_{q,s+4} \sim \Phi^{k-l}_{q+k+2,s+2}. \tag{2.3.0.48}$$

The most useful representation of the characters of the primary fields is in terms of string and theta functions [86, 152]

$$\chi^{l}_{q,s} = \sum_{j} c^{l}_{q-s+4j}\Theta_{2q+(4j-s)(k+2),2k(k+2)} \tag{2.3.0.49}$$

since it follows easily from the modular behaviour of the string functions c^{l}_{m} and the theta functions $\Theta_{m,k}$ that these characters transform like SU(2)–affine characters in their l index and as theta functions in their q and s index. This makes it easy to classify modular invariant partition functions for these theories because it follows that in

$$Z_{N=2} = \sum N_{l\bar{l}}^{q\bar{q}s\bar{s}} \chi_{q,s}^{l} \chi_{\bar{q},\bar{s}}^{\bar{l}} \tag{2.3.0.50}$$

the multiplicities $N_{l\bar{l}}^{q\bar{q}s\bar{s}}$ decompose into

$$N_{l\bar{l}}^{q\bar{q}s\bar{s}} = A_{l,\bar{l}} B^{q,\bar{q}} C^{s,\bar{s}}. \tag{2.3.0.51}$$

The primary fields of the $N = 2$ theory endowed with the diagonal theta function invariant are of the form

$$\Phi_{q,q;s,s}^{l,\bar{l}} = \phi_{m,m}^{l,\bar{l}} e^{i(\alpha_{q,s}\varphi + \alpha_{q,s}\bar{\varphi})} \tag{2.3.0.52}$$

with $m = q - s$. Among these fields are $k + 1$ chiral fields

$$\Sigma_l = \Phi_{l,l;0,0}^{l,l} = \phi_{l,l}^{l,l} e^{i\alpha_{l,0}(\varphi + \bar{\varphi})} \tag{2.3.0.53}$$

with anomalous dimensions $d_l = \frac{l}{k+2}$. These are the order parameters of the theory which are expressed in terms of the spin fields

$$\sigma_l = \phi_{l,l}^{l,l} \tag{2.3.0.54}$$

of the parafermionic theory. Similarly if the theory is endowed with the "anti–diagonal" invariant the primary fields are of the form

$$\Phi_{q,-q;s,-s}^{l,\bar{l}} = \phi_{m,-m}^{l,\bar{l}} e^{i(\alpha_{q,s}\varphi + \alpha_{-q,-s}\bar{\varphi})} \tag{2.3.0.55}$$

and the fields with the anomalous dimensions d_l

$$\bar{\Sigma}_l = \Phi_{l,-l;0,0}^{l,l} = \phi_{l,-l}^{l,l} e^{i\alpha_{l,0}(\varphi - \bar{\varphi})}. \tag{2.3.0.56}$$

are expressed in terms of the dual spin fields (disorder operator)

$$\mu_l = \phi_{l,-l}^{l,l} \tag{2.3.0.57}$$

of the parafermionic theory. The charges of the fields σ_l and μ_l are however reversed in the right sector.

The complete list of affine invariants can be found in [43, 91, 85, 90]. There exist three infinite series: the trivial diagonal invariant exists for every level k; the second series starts at level 4 for $k = 4m$, $m \in N$ and the third one only exists beyond level 6 for $k = 4m - 2$, $N \ni m \geq 2$. In addition to these infinite series there exist three exceptional invariants at level 10, 16 and 28. All these nontrivial invariants play an important role in that they generically change the spectrum of the compactified models quite dramatically. We list these partition functions in table 1.

Level	Partition Function	ADE–type
$k \geq 1,$	$Z = \sum_{l=0}^{k} \chi^l \bar{\chi}^l$	A_{k+1}
$k = 4j, j \geq 1$	$Z = \sum_{l=0}^{j-1} \mid \chi_{2l} + \chi_{k-2l} \mid^2 + 2\chi_{k/2}\bar{\chi}_{k/2}$	$D_{2(j+1)}$
$k = 4j-2, j \geq 2$	$Z = \sum_{l=0}^{k/2} \mid \chi_{2l} \mid^2 + \sum_{l=0}^{2j-2} \chi_{2l+1}\bar{\chi}_{k-2l-1}$	D_{2j+1}
$k = 10,$	$Z = \mid \chi_0 + \chi_6 \mid^2 + \mid \chi_3 + \chi_7 \mid^2 + \mid \chi_4 + \chi_{10} \mid^2$	E_6
$k = 16,$	$Z = \mid \chi_0 + \chi_{16} \mid^2 + \mid \chi_4 + \chi_{12} \mid^2 + \mid \chi_6 + \chi_{10} \mid^2 + \mid \chi_8 \mid^2$	
	$+ (\chi_2 + \chi_{14})\overline{\chi}_8 + \chi_8(\overline{\chi}_2 + \overline{\chi}_{14})$	E_7
$k = 28,$	$Z = \mid \chi_0 + \chi_{10} + \chi_{28} \mid^2 + \mid \chi_6 + \chi_{12} + \chi_{16} + \chi_{22} \mid^2$	E_8

Table 2.1 ADE–*partition functions for the minimal discrete series*

For the theta function system the invariants have been discussed in refs. [114, 146]. Consider the partition function

$$Z(\tau, \bar{\tau}) \sim \sum A_{m,n} \Theta_{m,k}(\tau) \Theta^*_{n,k}(\bar{\tau}) \tag{2.3.0.58}$$

of theta functions at level k and demand modular invariance. The most general solution is parametrized by a divisor α of the level k; with $\beta = k/\alpha$ the following solution is modular invariant:

$$A_{m,n} = \frac{1}{2} \sum_{x \in \mathbf{Z}_{2\beta}, y \in \mathbf{Z}_{2\alpha}} \delta_{m,\alpha x + \beta y} \delta_{n, \alpha x - \beta y}. \tag{2.3.0.59}$$

There is one solution for every divisor α. For $\alpha = 1$ this defines the left–right symmetric solution.

2.4 Tensor Products

The next step is to tensor the individual $N = 2$ superconformal theories together to saturate the necessary central charge $c_D = 15 - \frac{3}{2}D$ for a compactification from a 10– to a D–dimensional spacetime,

$$c_D = 15 - \frac{3}{2}D = \sum_{i=1}^{r} \frac{3k_i}{k_i + 2}, \tag{2.4.0.60}$$

where r is the number of models tensored together. All states in such a tensor product can be labeled by the $3r$ quantum numbers

$$|\text{state}> \sim (l^1, q^1, s^1) \dots (l^r, q^r, s^r) \tag{2.4.0.61}$$

(where s distinguishes between the various sectors).

Not all combinations however are allowed to occur in a physically sensible theory. In the following we will be only interested in the D–dimensional massless sector. BRST–invariance then requires the condition
(A): physical states have conformal weight (1,1). Furthermore, to ensure locality of the $U(1)$ current $J(z)$ with respect to the gravitino vertex [11, 61] one demands condition
(B): the $U(1)$ charge Q_{tot} must be quantized. In fact it can be normalized to take the values ± 1 in both, the left moving and the right moving sector. Finally the boundary conditions for both the internal part and the 4–dimensional part of the fermionic partner T_F of T have to be the same (either both anti–periodic or both periodic) because of the coupling of the world sheet gravitino to the total T_F [149]. This leads to condition
(C): in each sector only the appropriate sectors should be multiplied together, i.e. the Neveu–Schwarz sector only multiplies the Neveu–Schwarz sector and the Ramond fields only multiply Ramond fields. (The analogous conditions also hold for the antiholomorphic sector.) Thus the physical modes we are after are marginal operators in the theory which carry U(1)–charges of equal absolute value – something that will be important later on [1]

The spectrum is determined by the partition function (in the light cone gauge)

$$Z_{het} = | (Im\tau)^{-1} \eta(\tau)^{-4} | \sum_{\lambda \in \{o,v,s,\bar{s}\}} B_{\lambda}^{SO(2)}(\tau) B_{\bar{\lambda}}^{E_8 \times SO(10)}(\bar{\tau}) Z_{int}(\tau, \bar{\tau}) \tag{2.4.0.62}$$

where the $4d$–part comes from the spacetime variables, the $SO(2)$ characters $B_{\lambda}^{SO(2)}$ come from the spacetime fermions, whereas the characters $B_{\bar{\lambda}}^{E_8 \times SO(10)}$ are determined by the internal fermions and the internal partition function Z_{int} is determined by the tensor models. The internal part of the partition function will be given by products $\prod_{i=1}^{r} \chi_{q_i, s_i}^{l_i}$. Collecting the affine indices into an $\vec{l}$–index and the charge and the sector indices (q_i, s_i) as well as the external index (coming from the level 2 theta function of the 4–d fermions) into one $(2r+1)$ vector, we can write the partition function (modulo the external transverse bosons) as

$$\chi_{\vec{\mu}}^{\vec{l}} = B_{\mu_0}^{G} \prod_{i=1}^{r} \chi_{\mu_i, \mu_{r+i}}^{l_i} \tag{2.4.0.63}$$

[1]It is of interest to note that this relation is *not* preserved in the topological version of this theory when coupled ot 2D topological gravity. It was shown in [148] that in such a system neither the the complex deformations cannot be described by marginal operators.

where G stands for SO(2) in the supersymmetric sector and SO(26) or $E_8 \times SO(10)$ in the Kac–Moody sector and μ_0 describes the highest weight of a given representation of G. The standard fermionic invariant for the theta–function products then leads to

$$Z \sim \sum_{\vec{l},\vec{\bar{l}}} A_{\vec{l},\vec{\bar{l}}} \sum_{\mu,m_i,n_i} (-1)^{m_0+n_0} \chi^{\vec{l}}_{\vec{\mu}+m_i\vec{\beta}^i} \chi^{\vec{\bar{l}}*}_{\vec{\mu}'+n_i\vec{\beta}^i}. \tag{2.4.0.64}$$

for the compactified theory. Consider eq. (13) in more detail. The multimatrix $A_{\vec{l},\vec{\bar{l}}} = \prod_{i=1}^r A_{l_i,\bar{l}_i}$ represents a choice of the affine invariant for each of the individual factors in the internal sector. The $(2r+1)$–vectors $\vec{\beta}^i$, $\vec{\mu}$, $\vec{\mu}'$ are chosen in such a way as to implement requirements (B) and (C). To that account define $\vec{\beta}^0 = (\bar{s}|1,\ldots,1)$ and $\vec{\beta}^i = (v|0,\ldots,0,2,0,\ldots,0)$, $\;i=1,\ldots,r$ where 2 is in the $(r+1+i)^{th}$ entry and $\bar{s}, v$ are highest weights for the group G. The vectors $\vec{\mu}$ are then constrained by

$$\beta^i \cdot \mu \in 2\mathbb{Z} + \delta^{i0} \tag{2.4.0.65}$$

where the scalar product is defined as follows. Recall that the charge indices q_i belong to level $(k+2)$ theta functions whereas the external and the sector indices belong to level 2 theta functions. It is then natural to define

$$\vec{\beta}^i \cdot \mu = \frac{\beta_0^i \mu_0}{2} + \sum_{j=1}^{r} \left(-\frac{\beta_j^i \mu_j}{k+2} + \frac{\beta_{r+j}^i \mu_{r+j}}{2} \right). \tag{2.4.0.66}$$

For $i=0$ this condition then implements requirement (B) and for $i \neq 0$ requirement (C) is implemented in the partition function. The integers m_i and n_i run over the integers modulo α where α is such that $\beta_i\alpha/2(k_i+2)$ are integers for all i. The vectors $\vec{\mu}$, $\vec{\mu}'$ only differ in their external entry in such a way as to implement the transition from the superstring to the heterotic string. This is accomplished by permuting the highest weights $(o,v,s,\bar{s})$ of the gauge group G into $(v,o,s,\bar{s})$.

2.5 The Minimal $N=2$, $c=9$ Series

It is easy to see that there can be only a finite number of solutions to eq. (7), even though the discrete series is infinite, because there are finite gaps at the lower end of the sequence of central charges and the sequence converges at the upper end. In fact there are surprisingly few, just 168. Since the central charge of each of the factors is between $1 \leq c < 3$ there cannot be a tensor product with more than 9 factors and there can be only one product of nine models, namely 1^9. Here we denote a model by the levels of its factors, i.e. $k_1 \cdots k_r$ denotes a model which consists of r individual minimal $N=2$ theories. There must be at least four factors in order to achieve c=9. It is also easy to see that $k=1804$ is the highest level possible for any single $N=2$ superconformal theory in this series. For $r=5,6,7,8,9$ the highest possible levels are 154, 40, 10, 4, 1 respectively.

The total number of models in this class is of course much larger as each even factor, $k > 2$, admits at least two affine invariants, the diagonal and a D–invariant, whereas levels 10,16,28 admit yet another invariant – an E–invariant (see table 1). Therefore the majority of the 168 combinations leads to several sister models leading to a total of 836 models. In the following we discuss the spectrum of all these theories. We restrict ourselves to the computation of the generations and anti–generations since these are the most interesting for the discussion of the following sections and because their computation is uncontroversial. (For results regarding the discrepancy between E_6–singlets in the three–generation model constructed by the second author and its exact version found by Gepner see reference [68]).

We now discuss some properties of the spectrum of these models that are universal to this class. First we consider fermions. The emission vertices for fermions are best described in the –1/2 picture. In D=10 the holomorphic part of the fermion vertex is described by spin fields S^α of the RNS fermions ψ^μ and the spin fields ϕ of the (1,0) superconformal ghosts β, γ. At zero momentum one finds [81, 125]

$$V^{\alpha}_{-\frac{1}{2}}(z) = e^{-\frac{\phi(z)}{2}} S^{\alpha}(z) \qquad (2.5.0.67)$$

$$V^{\dot\alpha}_{-\frac{1}{2}}(z) = e^{-\frac{\phi(z)}{2}} S^{\dot\alpha}(z). \qquad (2.5.0.68)$$

In the compactified theory (i.e. D=4) these vertices can be written as [167, 62]

$$V^{\alpha}_{-\frac{1}{2}}(z) = e^{-\frac{\phi(z)}{2}} S^{\alpha}(z)\Sigma(z) \qquad (2.5.0.69)$$

$$V^{\dot\alpha}_{-\frac{1}{2}}(z) = e^{-\frac{\phi(z)}{2}} S^{\dot\alpha}(z)\Sigma^{+}(z) \qquad (2.5.0.70)$$

where again S^α are the spin fields but now there are only 4 left and Ψ is a conformal field in the internal sector of weight (3/8,0). N=1 supersymmetry charges can be constructed using the (anti–) holomorphic parts of these operators by defining

$$Q_{\alpha} = \oint dz e^{-\frac{\phi(z)}{2}} S_{\alpha}(z)\Sigma(z) \qquad (2.5.0.71)$$

$$Q_{\dot\alpha} = \oint dz e^{-\frac{\phi(z)}{2}} S_{\dot\alpha}(z)\Sigma^{+}(z) \qquad (2.5.0.72)$$

if Σ, Σ^+ etc. satisfy a certain set of OPEs.

There exists a simple representation of Σ in terms of a free scalar field $H(z)$. The starting point of this construction is the fact that the U(1) current $J(z)$ decouples from the other fields in the theory [167, 62]. One can write the current in terms of a free chiral scalar field $H(z)$

$$J(z) = i\sqrt{\frac{c}{3}}\partial_z H(z), \qquad (2.5.0.73)$$

where $H(z)$ is normalized such that $< H(z)H(w) >= -ln(z-w)$. Then any operator $\Phi(z,\bar{z})$ with U(1) charge q, i.e. such that

$$J(z)\Phi^q(w,\bar{w}) \sim \frac{q\Phi(w,\bar{w})}{(z-w)} \tag{2.5.0.74}$$

can be written as

$$\Phi^q(z,\bar{z}) =: e^{iq\sqrt{\frac{3}{c}}H(z)}P(J) : \Phi^0(z,\bar{z}) \tag{2.5.0.75}$$

where H commutes with Φ^0 and $P(J)$ is a polynomial in J and its derivatives. In particular it can be shown that

$$\Sigma =: e^{\frac{i}{2}\sqrt{\frac{c}{3}}H(z)} : . \tag{2.5.0.76}$$

Therefore every conformal field theory compactification which preserves N=1 spacetime supersymmetry has a field with $(\Delta, Q) = (3/8, \pm 3/2)$. This field is the conformal field theory analog of the covariantly constant spinor on a Calabi–Yau manifold. This spinor is essential for the existence of the holomorphic three–form which plays a central role in the Calabi–Yau compactification scheme. Analogously the conformal field theory has a field Ω in its spectrum which is essentially the square of the Ramond field Σ

$$\Omega =: e^{i\sqrt{\frac{c}{3}}H(z)} : \tag{2.5.0.77}$$

and which corresponds to the holomorphic three–form [27].

Next we turn to the generations and anti–generations. In the context of Calabi–Yau compactification it is well known that generations and anti– generations are associated to moduli of the internal space. The origin of this relation is the embedding of the gauge connection in the spin connection of the compact space. This leads to a decomposition of the adjoint representation of the E_8 under its maximal subgroup $E_6 \times SU(3)$:

$$\mathbf{248} = (\mathbf{78},\mathbf{1}) \oplus (\mathbf{27},\mathbf{3}) \oplus (\overline{\mathbf{27}},\overline{\mathbf{3}}) \oplus (\mathbf{1},\mathbf{8}). \tag{2.5.0.78}$$

In the (2,2)–compactification on the $N=2$, $c=9$ tensor series only an SO(10) symmetry is manifest in the non–supersymmetric anti–holomorphic (right moving) sector. However, by incorporating the internal superconformal U(1) it is possible to build representations of the E_6 group e.g.

$$\mathbf{27} = (\mathbf{16},\mathbf{1/2}) \oplus (\mathbf{10},-\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \tag{2.5.0.79}$$

$$\overline{\mathbf{27}} = (\mathbf{16'},-\mathbf{1/2}) \oplus (\mathbf{10},\mathbf{1}) \oplus (\mathbf{1},-\mathbf{2}) \tag{2.5.0.80}$$

where the numbers on the right refer to the $SO(10) \times U(1)$ group. Therefore one can identify the generations in a given model as those fields that have

$$(\bar{\Delta}, \bar{Q}) = (1, 2) \tag{2.5.0.81}$$

whereas the anti–generation could be identified as those fields that have

$$(\bar{\Delta}, \bar{Q}) = (1, -2). \tag{2.5.0.82}$$

(Here we take the Kac–Moody sector to be antiholomorphic).

By expanding the partition function one can enumerate all the fields of these theories. The fields of the internal exactly solvable theory will be build from the primary fields of the various factors, i.e. they are of the form

$$\prod_{i=1}^{4} \Phi^{l_i,\bar{l}_i}_{q_i,\bar{q}_i;s_i,\bar{s}_i} \tag{2.5.0.83}$$

for which we will use the notation

$$\prod_{i=1}^{4} \begin{pmatrix} l_i & q_i & s_i \\ \bar{l}_i & \bar{q}_i & \bar{s}_i \end{pmatrix}. \tag{2.5.0.84}$$

This internal field is the internal part of a vertex operator describing the total field. To find the specific form we have to specify the spacetime properties of the fields we want to consider. We will restrict our considerations to the generations and antigenerations.

The generations in this theory decompose into multiplets of the manifest group $SO(10)\times U(1)$ building **27** of the E_6

$$\mathbf{27} = (\mathbf{16}, \frac{\mathbf{1}}{\mathbf{2}}) \oplus (\mathbf{10}, -\mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \tag{2.5.0.85}$$

whereas the antigenerations decompose as

$$\overline{\mathbf{27}} = (\mathbf{16}', -\frac{\mathbf{1}}{\mathbf{2}}) \oplus (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{1}, -\mathbf{2}). \tag{2.5.0.86}$$

These fields appear in spacetime supersymmetry multiplet and a specific picture has to be chosen. In terms of the bosonized supersymmetry ghost ρ, the fermions λ^a generating the SO(10) current algebra a spacetime scalar field which transforms in the **10** of the SO(10) can be written in the form

$$V^a_{-1}(z,\bar{z}) = e^{-\rho(z)}\mathcal{O}(z,\bar{z})\lambda^a \tag{2.5.0.87}$$

where $\mathcal{O}$ is an operator in the internal theory with

$$\Delta = \frac{1}{2} = \bar{\Delta}\ , \quad Q = -1 = \bar{Q}. \tag{2.5.0.88}$$

Also of importance in the following are the spacetime spinors which are simplest in the -1/2 picture:

$$V^{\alpha\,\cdot}_{-1/2}(z,\bar{z}) = e^{-\rho(z)}S^\alpha\mathcal{O}(z,\bar{z})R^{\cdot} \tag{2.5.0.89}$$

where the S^α are the spinfields of the RNS fermions ψ^μ, $\mathcal{O}$ is again the purely internal part and $R^{\cdot}$ is the vertex operator of a primary field of the rightmoving SO(10) current algebra, depending on a particular representation $\cdot \in \{singlet, vector, spinor, antispinor\}$.

The components of the multiplets in the left and right sector are related via the supersymmetry charges

$$\begin{aligned} Q^{\alpha} &= \int \frac{dz}{2\pi i} e^{-\rho(z)/2} S^{\alpha} \Sigma(z,\bar{z}) \\ \bar{Q}^{\alpha} &= \int \frac{d\bar{z}}{2\pi i} e^{-\rho(z)/2} \bar{S}^{\alpha} \bar{\Sigma}(z,\bar{z}) \end{aligned} \tag{2.5.0.90}$$

Here Σ ($\bar{\Sigma}$) is an operator with $(\Delta, Q) = (3/8, 3/2)$ ($(\Delta, Q) = (0,0)$) and $(\bar{\Delta}, \bar{Q}) = (0,0)$ ($(\bar{\Delta}, \bar{Q}) = (3/8, 3/2)$).

In a tensor model with r factors the internal operator of the supercharges is

$$\Sigma = \prod_{i=1}^{r} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5.0.91}$$

for the left moving sector and

$$\bar{\Sigma} = \prod_{i=1}^{r} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{2.5.0.92}$$

for the right moving sector. The adjoint operators with opposite charge are

$$\Sigma^{+} = \prod_{i=1}^{r} \begin{pmatrix} k & k+1 & 1 \\ k & k+2 & 2 \end{pmatrix} = \prod_{i=1}^{r} \begin{pmatrix} 0 & 2k+3 & 3 \\ 0 & 2k+4 & 4 \end{pmatrix} = \prod_{i=1}^{r} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.5.0.93}$$

and

$$\bar{\Sigma}^{+} = \prod_{i=1}^{r} \begin{pmatrix} k & k+2 & 2 \\ k & k+1 & 1 \end{pmatrix}. = \prod_{i=1}^{r} \begin{pmatrix} 2k & 2k+4 & 4 \\ 2k & 2k+3 & 3 \end{pmatrix}. = \prod_{i=1}^{r} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}. \tag{2.5.0.94}$$

respectively. With these operators the components in the left sector are related via Σ as

$$\mathrm{SO}(2): scalar \quad \longrightarrow \quad spinor \longrightarrow \quad scalar\ (aux) \tag{2.5.0.95}$$

whereas the right moving $\bar{\Sigma}$ relates the SO(10) representations

$$\mathrm{SO}(10): \mathbf{10_{-1}} \quad \longrightarrow \quad \mathbf{16_{1/2}} \longrightarrow \quad \mathbf{1_2} \tag{2.5.0.96}$$

of the generation multiplet and the adjoint operator $\bar{\Sigma}^{+}$ relates the antigeneration multiplet

$$\mathrm{SO}(10): \mathbf{10_{1}} \quad \longrightarrow \quad \mathbf{16_{-1/2}} \longrightarrow \quad \mathbf{1_{-2}} \tag{2.5.0.97}$$

Note that if one chooses to represent the generations by spacetime scalars that transform in the **10** (fundamental) representation of the SO(10) then all internal fields (in a diagonal theory?) are expressed in terms of the form (chiral)

$$\prod_{i=1}^{r} \begin{pmatrix} l_i & l_i & 0 \\ l_i & l_i & 0 \end{pmatrix}. \tag{2.5.0.98}$$

whereas antigenerations are fields of the form

$$\prod_{i=1}^{r} \begin{pmatrix} l_i & l_i & 0 \\ l_i & -l_i & 0 \end{pmatrix}. \tag{2.5.0.99}$$

One way to identify the generations in a given model is those fields that are spacetime spinors and SO(10) singlets, i.e. fields with

$$(\Delta, Q, \bar{\Delta}, \bar{Q}) = (\frac{3}{8}, \frac{1}{2}, 1, 2) \tag{2.5.0.100}$$

and the antigenerations as those fields that have

$$(\Delta, Q, \bar{\Delta}, \bar{Q}) = (\frac{3}{8}, \frac{1}{2}, 1, -2). \tag{2.5.0.101}$$

For the first of the models one finds 57 antigenerations and 81 generations whereas for the dual companion one finds 81 antigenerations and 57 generations as necessary.

Next we turn to the behaviour of couplings under the change of the affine invariants. Most interesting are the Yukawa couplings between the generations and antigenerations. These two sets of couplings behave very different in the manifold interpretation of the model, as the couplings between generations are not corrected by nonperturbative effects whereas the couplings between antigenerations are corrected by instantons.

We first discuss the couplings between the generations. As is clear from the decomposition of the **27** we need to compute the threepoint function of either one spacetime scalar in the vector representation of the SO(10) and two spacetime spinors in the spinor representation or, alternatively, the coupling of one spacetime scalar in the SO(10) singlet representation and two spacetime spinors in the vector representation. We consider the former choice.

With the supercharge operators mentioned above these coupling can be expressed purely in terms of chiral and antichiral primary fields. The spacetime scalar in the vector representation is already expressed in terms of chiral primary fields of the factors. To express the spacetime spinor in the spinor representation similarly we write it first as in terms of a spinor in the vector representation as

$$V^{16_{1/2}}_{-1/2} = \bar{\Sigma}\; V^{10_{-1}}_{-1/2} \tag{2.5.0.102}$$

and then the spinor representation in terms of the scalar as

$$V^{16_{1/2}}_{-1} = \Sigma\; V^{16_{1/2}}_{-1/2}. \tag{2.5.0.103}$$

Hence the Yukawa coupling can be written as

$$< V_{-1}^{10_{-1}} V_{-1/2}^{16_{1/2}} V_{-1/2}^{16_{1/2}} >=< \Sigma^2\ \bar{\Sigma}^2\ V_{-1}^{10_{-1}} V_{-1}^{10_{-1}} V_{-1}^{10_{-1}} > . \tag{2.5.0.104}$$

Since the operator

$$\Sigma^2\bar{\Sigma}^2 = \prod_{i=1}^{r} \begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} = \prod_{i=1}^{r} \begin{pmatrix} k_i & k_i+4 & 4 \\ k & k_i+4 & 4 \end{pmatrix} = \prod_{i=1}^{r} \begin{pmatrix} k_i & -k_i & 0 \\ k_i & -k_i & 0 \end{pmatrix} \tag{2.5.0.105}$$

is the unique antichiral field of hightest charge $(Q, \bar{Q}) = (3, 3)$ this is an expression purely in terms of chiral and antichiral primary fields. The topweight field is composed of the highest weight fields in each of the factors of charge $\frac{k_i}{k_i+2} = \frac{c}{3}$.

The nonvanishing Yukawa couplings can therefore be easily seen to satisfy the selection rule

$$\sum_{l=1}^{3} Q_i^{(l)} = -\frac{k_i}{k_i+2} = -\frac{c_i}{3} \tag{2.5.0.106}$$

where $Q_i^{(l)}$ is the i^{th} component of the l^{th} generation. The same selection rule holds in the right sector as well.

Similarly one can discuss the couplings between the antigenerations. Again we compute the threepoint function of one spacetime scalar in the vector representation of the SO(10) and two spacetime spinors in the spinor representation or.

With the adjoint of the supercharge operators mentioned above these coupling can be again be expressed purely in terms of chiral/antichiral and antichiral/chiral primary fields. The spacetime scalar in the vector representation is now expressed in terms of antichiral primary fields of the factors. To express the spacetime spinor in the spinor representation similarly we write it first as in terms of a spinor in the vector representation as

$$V_{-1/2}^{16_{-1/2}} = \bar{\Sigma}^{+}\ V_{-1/2}^{10_1} \tag{2.5.0.107}$$

and then the spinor representation in terms of the scalar as

$$V_{-1}^{16_{-1/2}} = \Sigma\ V_{-1/2}^{16_{-1/2}}. \tag{2.5.0.108}$$

Hence the Yukawa coupling can be written as

$$< V_{-1}^{10_1} V_{-1/2}^{16_{-1/2}} V_{-1/2}^{16_{-1/2}} >=< \Sigma^2\ \bar{\Sigma}^{+^2}\ V_{-1}^{10_1} V_{-1}^{10_1} V_{-1}^{10_1} > . \tag{2.5.0.109}$$

Since the operator

$$\Sigma^2\bar{\Sigma}^{+^2} = \prod_{i=1}^{r} \begin{pmatrix} 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix} = \prod_{i=1}^{r} \begin{pmatrix} k_i & k_i+4 & 4 \\ k_i & k_i & 0 \end{pmatrix} = \prod_{i=1}^{r} \begin{pmatrix} k_i & -k_i & 0 \\ k_i & k_i & 0 \end{pmatrix} \tag{2.5.0.110}$$

is the unique (c, a) field of hightest charge $(Q, \bar{Q}) = (3, -3)$.

The nonvanishing Yukawa couplings can therefore be easily seen to satisfy the selection rule

$$\sum_{l=1}^{3} Q_i^{(l)} = -\frac{k_i}{k_i + 2} = -\frac{c_i}{3} \; , \quad \text{for} \;\; i = 1, 2, \cdots r \tag{2.5.0.111}$$

where $Q_i^{(l)}$ is the i^{th} component of the l^{th} generation. In the right sector as well the selection rule is

$$\sum_{l=1}^{3} \bar{Q}_i^{(l)} = \frac{k_i}{k_i + 2} = \frac{c_i}{3} \; , \quad \text{for} \;\; i = 1, 2, \cdots r \tag{2.5.0.112}$$

In table 2 we list all occuring combinations of $(\#\overline{\mathbf{27}}, \#\mathbf{27})$ of the tensor models, i.e. the number of antigenerations and the number of generations.

# of $\overline{\mathbf{27}}$	# of $\mathbf{27}$
0	84
1	53,61,73,77,85,101,103,145,149
2	56,58,62,74,86,106,110,122,128,272
3	51,57,69,75,99,105,165,243
4	40,76,94,112,148,208
5	41,53,65,77,89,101,161,185,251
6	60,66,90,114,120,180
7	55,61,63,79,91,103,127,143,151,271
8	35,44,68,80,86,104,116,140,164
9	33,39,57,69,81,111,117,129,153,321
10	46,70,106,178,214
11	35,51,53,59,83,91,107,131,155,167,227,491
12	48,96,126
13	37,49,61,73,85,109,121,157,181,229
14	50,98,242
15	39,63,147,183
16	52,76,100,112
17	29,41,65,77,101,155,173,377
18	42,102,126,222
19	27,43,67,91
20	32,56,74,104,176
21	21,117
22	58,70,82,130
23	23,35,47,59,71,143,263,335
24	36,84,204
25	37,85,97
26	86,98,158
27	27,59
28	52,160
29	29,41,53,221
30	78,96
31	23,31,55,91,103
33	21,69,105,141
34	46,58,118,190
35	35,59
36	144
37	49,85
38	74

#of$\overline{\mathbf{27}}$	#of$\mathbf{27}$
39	39,79,87,99
40	76
41	41,53,77
42	150
43	43,67
44	62,80
45	45,93
46	106
47	47,287
48	36,180
49	49,73
50	134
53	53,89,173
55	55
57	81
59	59,83,131
65	125
66	174
68	80,128
69	69
71	71,143
74	86
79	79
81	57
83	83
88	124
89	89
95	263
97	97
101	149
107	107
119	119,167
131	59
137	257
143	143
167	119
251	251
257	137

Table 2.2 *The spectra of all* ADE*–models with c=9.*

Some comments about these results are in order. As already mentioned above the implementation of the ADE–invariants for the 168 different tensor models

$$k_1 \cdot k_2 \cdots k_r \ , \qquad 4 \leq r \leq 9 \tag{2.5.0.113}$$

leads to 836 models. Clearly not all these theories are independent. The well known identities

$$\begin{aligned} 10_{\mathrm{E}} &= 1_{\mathrm{A}} \otimes 2_{\mathrm{A}} \\ 28_{\mathrm{E}} &= 1_{\mathrm{A}} \otimes 3_{\mathrm{A}} \end{aligned} \tag{2.5.0.114}$$

are reflected in the list of spectra. In addition to these relations the list suggests another identity

$$4_{\mathrm{D}} = 1_{\mathrm{A}} \otimes 1_{\mathrm{A}} \tag{2.5.0.115}$$

which is easily checked in the Landau–Ginzburg formulation of the two theories involved. (This identity has also been noted in [165] as a relation between the associated LG–potentials).

As a consequence of these identities there are only 19 independent models in the tensor series with more than four factors (18 in the tensor combinations with $r=5$ and only one more in tensor 6). This reduces the 836 models to at most 569 independent models. There is still plenty of room however for more non–trivial identities as one can see from the fact that there are only 248 different combinations of Hodge numbers. In Table 3 of the appendix of [140] all the independent models are listed. The other models can be easily found using the identities. There are a number of ways in which this list can be reduced by using (2.5.0.114) and (2.5.0.115). In [140] the models are ordered according to the number of their k–factors and write for a given model only the representative with the smallest number of factors. Using our identity (2.5.0.115) we therefore list only the last model of

$$1^9_{A^9} \sim (1^7 \cdot 4)_{A^7 \otimes D} \sim \cdots \sim (1 \cdot 4^4)_{AD^4}. \tag{2.5.0.116}$$

Similarly we write only the first of the two models

$$(2 \cdot 10^2)_{A^2E^2} \sim (2_{A_3})^3 \cdot 4_{D_4} \cdot 10_{A_{11}}. \tag{2.5.0.117}$$

This reduces the number of models with $r > 4$ factors to 19 and to 618 in the 108 combinations with $r=4$. However it still leaves us with relations in the tensor four models. Now we choose to delete models with E–invariants in favour of D–invariants, i.e. of the two models

$$(2 \cdot 4 \cdot 11 \cdot 154)_{ADA^2} \sim (1 \cdot 10 \cdot 11 \cdot 154)_{AEA^2} \tag{2.5.0.118}$$

we list only the first one etc.

It is clear from table 2 that there are no phenomenologically particularly exciting models in our list (even though these models are useful tools to understand the configuration space of the

Heterotic String vacua better as we will discuss later in a bit more detail). This is particularly obvious from our fig. 1 where we have plotted the Euler number versus the total number of light fields ($h^{(1,1)} + h^{(2,1)}$). A good candidate should lie very close to zero, but this area is rather empty.

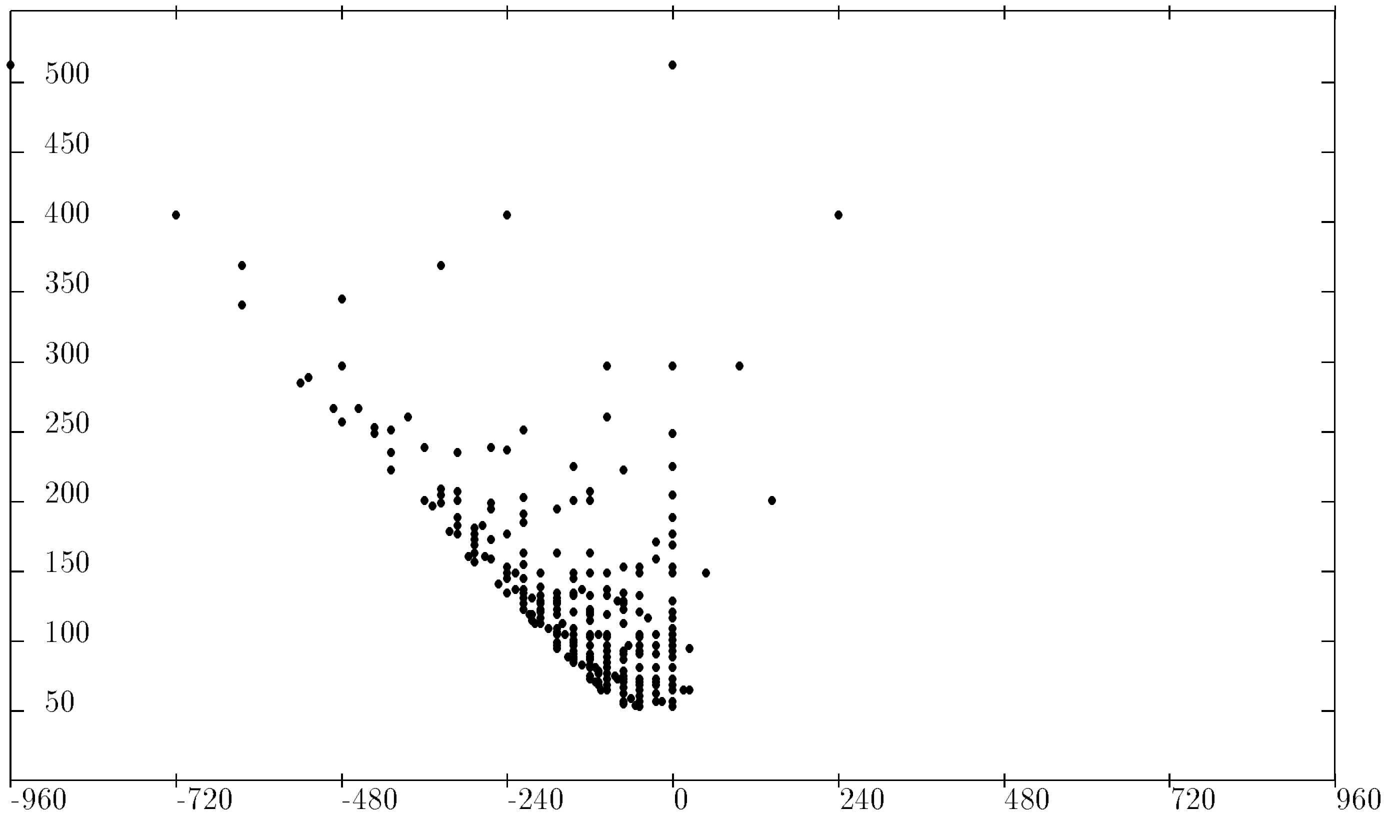

Fig. 2.1 *A plot of the Euler number versus* $(h^{(1,1)} + h^{(2,1)})$ *for the ADE* $N = 2$ *minimal series of* $c = 9$ *theories.*

What is striking however from Fig.2.1 is that there is a strong asymmetry between positive and negative Euler numbers. We will come to this point in later chapters. Here we just want to draw attention to the fact that there are not only non–trivial models with positive Euler number (i.e. models with $h^{(1,1)}$ *and* $h^{(2,1)}$ non–zero) but that there are also pairs among the ADE list, i.e. models where the number of generations and anti–generations is interchanged.

When comparing the spectrum of the ADE models with what was known at the time in manifold theory it becomes clear that there is some overlap with the completely constructed class of complete intersection Calabi–Yau manifolds [31] even though the overlap is not extensive. In a later chapter we will come back to a detailed discussion of the relation of these models to Calabi–Yau manifolds.

Chapter 3

All Landau–Ginzburg Vacua of the Superstring

It is clear from the results in the previous chapter that the space of exactly solvable string vacua based on ADE minimal models is rather small, even though it does contain a phenomenologically interesting model. As already mentioned this model was constructed in [88] by repeating in the exactly solvable theory the construction originally described in the manifold picture in [158].

The somewhat mysterious relation, observed in [87, 88], between two particular Calabi–Yau manifolds and exactly solvable models was clarified once it was realized that Zamolodchikov's generalization [184] of the Landau–Ginzburg (LG) description of the Ising model to the N=0 and N=1 supersymmetric discrete series can be generalized to N=2 superconformal models as well [115, 144, 173]. A more precise understanding of the Landau-Ginzburg theory/Calabi-Yau relation was finally obtained by Witten [179] who pointed out that these two formulations represent two different phases of the linear σ-model rather than equivalent descriptions of the same theory. Even though there are aspects of the Landau–Ginzburg formulation of conformal field theories which at present still remain to be enhanced, it is clear, that it does represent an extremely useful tool in analyzing the relation between these rather different frameworks.

Independently of the relation of Landau–Ginzburg theories to exactly solvable models the question arises what this class of theories can teach us about string theory and whether it contains new phenomenologically interesting examples. The purpose of this chapter is to present, following [41] and [122], see also [130], the complete class of Landau–Ginzburg theories. Our construction presented in [41], even though it lead to the original discovery of mirror symmetry, was constrained by the fact that it focused attention on potentials, which contained only five scaling variables and were of a special form such as to guarantee the existence of an isolated singularity, i.e. that they define so–called catastrophes or singularities. The motivation for this latter condition comes from the fact that such theories have a finite number of physical states and are therefore particularly simple. It turns out that this class of polynomials is also of considerable interest to mathematicians who have made an effort to classify the explicit form that such potentials can have [3]. The complete construction along the lines started in [41] has been achieved in [129].

Even though the explicit form of the Landau–Ginzburg potential is essential for a detailed analysis of a given model, it does not have to be specified in order to gain partial information about the theory based on it. If, for example, we are interested only in the spectrum of the model we only need to know the anomalous weights of the scaling variables and to check that the particular combination of weights allows for the existence of a polynomial with an isolated singularity. This fact has been used in [122] in order to provide for the first complete construction of this class of models.

This chapter is organized as follows: in section 1 we briefly review the constraints that a Landau–Ginzburg potential has to satisfy in order to define a string vacuum. We also discuss the relation between LG theories and Calabi–Yau manifolds since this will turn out to be useful for certain aspects of the construction. In section 2 we explain how it is possible to complete the construction of all LG vacua without an explicit classification, via an indirect construction, based on a Bertini–type theorem. In section 3 we discuss some of the main features of this class of vacua and in section 4 we close this chapter with some general remarks and comments on other compactification schemes.

3.1 Landau–Ginzburg Vacua

In this section we briefly review the properties of the Landau–Ginzburg vacua we wish to investigate [144, 176, 135, 89]. Consider a string ground state based on a Landau–Ginzburg theory which we assume to be $N = 2$ supersymmetric since we demand $N = 1$ spacetime supersymmetry. Using a superspace formulation in terms of the coordinates $(z, \bar{z}, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$ the ingredients of the N=2 supersymmetric action are chiral superfields, i.e. superfields satisfying

$$D^+\Phi = 0 = \bar{D}^+\Phi \qquad (3.1.0.1)$$

where

$$D^\pm := \frac{\partial}{\partial\theta^\pm} + \theta^\mp\partial_z \qquad (3.1.0.2)$$

(and similar for the right moving sector). The solution of the chirality conditions can be written in the form

$$\begin{aligned} \Phi(z,\bar{z},\theta^{\pm},\bar{\theta}^{\pm}) &= \Phi(z+\theta^{+}\theta^{-},\bar{z}+\bar{\theta}^{+}\bar{\theta}^{-},\theta^{-},\bar{\theta}^{-}) \\ &= \phi(y,\bar{y})+\theta^{-}\psi(y,\bar{y})+\bar{\theta}^{-}\lambda(y,\bar{y})+\theta^{-}\bar{\theta}^{-}F(y,\bar{y}) \end{aligned} \tag{3.1.0.3}$$

where $y = z - \theta^{+}\theta^{-}$.

Consider a set of such chiral superfields Φ_i on the worldsheet. The action for the chiral superfields will be taken to be of the form

$$\mathcal{A} = \int d^2z d^4\theta \; K(\Phi_i,\bar{\Phi}_i) + \int d^2z d^2\theta^{-} \; W(\Phi_i) + \int d^2z d^2\theta^{+} \; W(\bar{\Phi}_i) \tag{3.1.0.4}$$

where K–term, involving the Kähler potential, is called the D–term and the second term, involving the superpotential W, which is a holomorphic function of the chiral superfields Φ_i, is called the F–term. It is assumed that the nonrenormalization theorems of N=2 supersymmetric theories hold, implying that the F–term does not get renormalized whereas the D–term does receive corrections. Since the only relevant operators appear in the F–term this implies that the universality class is determined only by the superpotential.

Since the ground states of the bosonic potential are the critical points of the superpotential of the LG theory we demand the existence of the latter. The type of critical points we need is determined by the fact that we wish to keep the fermions in the theory massless; hence we assume that the critical points are completely degenerate. Furthermore we require that all critical points be isolated, since we wish to relate the finite dimensional ring of monomials associated to such a singularity with the chiral ring of physical states in the Landau–Ginzburg theory, in order to construct the spectrum of the corresponding string vacuum. Finally we demand that the theory is conformally invariant; from this follows, relying on some assumptions regarding the renormalization properties of the theory, that the Landau–Ginzburg potential is quasihomogeneous. In other words, we require to be able to assign, to each field Φ_i, a weight q_i such that for any non–zero complex number $\lambda \in \mathbb{C}^{\star}$

$$W(\lambda^{q_1}\Phi_1,\ldots,\lambda^{q_n}\Phi_n) = \lambda W(\Phi_1,\ldots,\Phi_n). \tag{3.1.0.5}$$

Thus we have formulated the class of potentials that we need to consider: quasihomogeneous polynomials that have an isolated, completely degenerate singularity (which we can always shift to the origin).

Putting these physical requirements together one learns that mathematicians have found such structures of interest as well and they have given them a name: associated to each of the superpotentials, $W(\Phi_i)$ is a so–called catastrophe which is obtained by first truncating the superfield Φ_i

to its lowest bosonic component $\phi_i(z, \bar{z})$, and then going to the field theoretic limit of the string by assuming ϕ_i to be constant $\phi_i = z_i$. Writing the weights as $q_i = k_i/d$, we will denote by

$$\mathbb{C}_{(k_1,k_2,...,k_n)}[d] \tag{3.1.0.6}$$

the set of all catastrophes described by the zero locus of polynomials $p(z_1, ..., z_n)$ of degree d in variables z_i of weight k_i. The basic structure of such a quasi–homogeneous singularity is encoded in a ring generated by the power series of the coordinate space modulo the ideal generated by the partials of the superpotential

$$\mathcal{R} = \frac{\mathbb{C}[\Phi_1, ..., \Phi_n]}{\mathcal{J}[\partial_1 W, ..., \partial_n W]}. \tag{3.1.0.7}$$

Many properties of the Landau–Ginzburg theory can be read off directly from this ring. In particular it allows for a simple identification of the central charge of a given LG–Theory [145]. Note first that there exists precisely one generator of the ring with maximal degree. According to a result of Arnold this maximal degree for a quasi–homogeneous singularity of type $(q_1, ..., q_r)$ is given by

$$d_{max} = \sum_{i=1}^{r} (1 - 2q_i) \tag{3.1.0.8}$$

Further note that the projective weights of the fields are essentially the U(1)–charges discussed in the previous chapter. Therefore the maximal projective weight of the generators of the local ring is precisely the maximal U(1)–charge in the $N = 2$ SCFT which is determined by unitarity to be [27]

$$Q_{max} = c/3. \tag{3.1.0.9}$$

Since we are interested in chiral fields we also have $2\Delta_{max} = Q_{max}$. Therefore one finds [145]

$$c = 3\sum_{i=1}^{r} (1 - 2q_i)\,. \tag{3.1.0.10}$$

Also of interest is the dimension of the ring $\mathcal{R}$ which is easily computed by evaluating the Poincaré polynomial of the catastrophe type (refcats)

$$P(t) = \prod_{i=1}^{n} \left(\frac{1 - t^{d-k_i}}{1 - t^{k_i}}\right) \tag{3.1.0.11}$$

at $t = 1$, i.e. by computing

$$\mu = P(1) = \prod_{i=1}^{n} \frac{d - k_i}{k_i}. \tag{3.1.0.12}$$

which is usually called the multiplicity of the singularity defined by the potential. (In these formulas the weights have been made integers $k_i = q_i d$ where d is the smallest possible integer.)

It must be emphasized that neither does the Milnor number determine any physical quantity nor is the Landau–Ginzburg theory as such of interest for string theory. What is missing is the GSO projection which leads to the U(1)–charge quantization. To compute the physically relevant spectrum of the LG–model this projection has to be implemented. This is just what has been done in ref. [173] where a prescription has been developed for computing the complete Hodge–diamond for the projected LG–theory – more precisely the *orbifoldized* LG–theory – taking into account also the twisted sectors, which are entirely responsible for the anti–generation sector.

This can be put slightly different. The affine varieties described by the catastrophe polynomials are not compact and what the GSO projection does is to compactify these spaces. In Landau–Ginzburg language, this amounts to an orbifolding of the theory with respect to a discrete group $\mathbb{Z}_d$ the order of which is the degree of the LG potential [173]. The spectrum of the orbifold theory will contain twisted states which, together with the monomial ring of the potential, describe the complete spectrum of the corresponding Calabi–Yau manifold. We will denote the orbifold of a Landau–Ginzburg theory by

$$\mathbb{C}^{\star}_{(k_1,k_2,\ldots,k_n)}[d] \tag{3.1.0.13}$$

and call it a configuration.

Vafa's result for the LG analog of the Euler number for the GSO–projected LG theory is

$$\chi = \frac{1}{d}\sum_{l,r=0}^{d-1}\ \prod_{lq_i,rq_i\in\mathbb{Z}}\frac{d-k_i}{k_i}. \tag{3.1.0.14}$$

The Euler numbers for all configurations range between $-960 \leq \chi \leq 960$, among them the numbers, such as 2,3,4, that have been difficult to obtain previously.

However, one can do better and actually compute the Hodge numbers as well. To this end Vafa constructs a Poincaré–type polynomial for the $l^{\rm th}$ twisted sector

$$Tr_l\left((t\bar{t})^{dJ_0}\right) = t^{d\left(Q_l+\frac{1}{6}c_T\right)}\bar{t}^{d\left(-Q_l+\frac{1}{6}c_T\right)}\prod_{lq_i\in\mathbb{Z}}\left(\frac{1-(t\bar{t})^{d-k_i}}{1-(t\bar{t})^{k_i}}\right) \tag{3.1.0.15}$$

with

$$\begin{aligned} Q_l &= \sum_{lq_i\notin\mathbb{Z}}\left(lq_i-[lq_i]-\frac{1}{2}\right)\ , \\ \frac{1}{6}c_T &= \sum_{lq_i\ \notin\mathbb{Z}}\left(\frac{1}{2}-q_i\right) \end{aligned} \tag{3.1.0.16}$$

where t and $\bar{t}$ are formal variables, d is the degree of the Landau–Ginzburg potential, the $q_i = k_i/d$ are the normalized weights of the fields and $[lq_i]$ is the integer part of lq_i. Expanding this polynomial in powers in t and $\bar{t}$ it is possible to read off the contributions to the various cohomology groups from the different sectors of the twisted LG–theory. The (2,1)–forms for example are given by the number of fields of charge (1,1), i.e. the coefficient of $(t\bar{t})^d$. In general, the number of (p,q)–forms are given by the coefficient of $t^{(3-p)d}\bar{t}^{qd}$ in the Poincare polynomials summed over all sectors $l = 0, ..., d-1$.

Put slightly different, the basic observation is that in the l^{th} twisted sector the charges of the states are of the form

$$(Q_l, -Q_l) + (r, r) \tag{3.1.0.17}$$

where

$$Q_l = \sum_{lq_i \in \mathbb{Z}} \left(lq_i - [lq_i] - \frac{1}{2} \right) \tag{3.1.0.18}$$

is the contribution of the charge coming from the twisted fields and r is any of the charges generated by the subring of those fields that are invariant under the l–twist. These charges are of course again generated by the Poincaré polynomial, only now it is restricted to the invariant fields

$$P(t) = \prod_{lq_i \in \mathbb{Z}} \left(\frac{1 - t^{d-k_i}}{1 - t^{k_i}} \right). \tag{3.1.0.19}$$

cking out the fields with integral charge will generate to the cohomology classes: fields with charges $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ will generate the cohomology groups $H^{(D-p,q)}$ (with our conventions).

To illustrate this method we consider an example to which we will come back below. Consider the exact model

$$(4^2 \cdot 10^2)_{A_5^2 \otimes A_{11} \otimes D_7} \tag{3.1.0.20}$$

whose LG–potential according to table 3 is given by

$$W = \Phi_1^6 + \Phi_2^6 + \Phi_3^{12} + \Phi_4^6 + \Phi_4 \Phi_5^2. \tag{3.1.0.21}$$

The degree of the potential is $d = 12$ and the weights are $(k_1, ..., k_5) = (2, 2, 1, 2, 5)$. The integrally charged fields in the invariant sector ($l = 0$) lead to one (3,0)–form, 84 (1,2)–forms, 84 (2,1)–forms and one (0,3)–form. The sectors $l = 1$ ($l = 11$) contribute one (3,3)–form ((0,0)–form). As mentioned above all anti–generations come from the twisted sectors: each of the sectors 4,5,9,10 contributes one (1,1)–form whereas each of the sectors 2,3,7,8 contributes one (2,2)–form. But the twisted sectors can also contribut to the generations: the sector $l = 6$ in our example contributes 10 (1,2)–forms and 10 (2,1)–forms. Summing the generations and anti–generations we find that this LG–model has 94 generations and 4 anti–generations, in agreement with our previous exact result.

Similarly the appropriate cohomology groups for all ADE–LG models can be computed, thereby providing a check on the exact computation described in the previous chapter.

Before turning to the problem of constructing LG configurations satisfying the constraints described above, it is useful to make some remarks regarding the validity of the requirements formulated in the previous paragraph.

Even though the assumptions formulated in refs. [144, 176] and [135, 89] and reviewed above seem rather reasonable, and previous work shows that the set of such Landau–Ginzburg theories certainly is an interesting and quite extensive class of models, it is clear that it is not the most general class of (2,2) vacua. Although it provides a rather large set of different models [1], which contains many classes of previously constructed vacua [2], there are known vacua which cannot be described in this framework.

Perhaps the simplest example that does not fit into the classification above is that of the Calabi–Yau manifolds in

$$\mathbb{P}_5[4\ \ 2], \tag{3.1.0.22}$$

described by the intersection of two hypersurfaces defined by a quartic and a quadratic polynomial in a five–dimensional projective space $\mathbb{P}_5$ because of the purely quadratic polynomial that appears as one of the constraints defining the hypersurface. The requirement that the singularity be completely degenerate seems, in fact, to exclude a great many of the CICY manifolds, the complete class of which was constructed in ref. [31]. An important set of manifolds in that class that does not fit into the present framework either is defined by polynomials of bidegree (1,4) and (1,1)

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}. \tag{3.1.0.23}$$

The superpotential $W = p_1 + p_2$ is not quasihomogeneous, nor does it have an isolated singularity [3]. Thus it appears that there ought to be a generalization of the framework described above, which allows a modified LG description of these and other string vacua. This generalization has yet to be found.

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[1]The rather extravagant values that have been mentioned in the literature as the number of possible (2,2) vacua are based on extrapolations that do not take into account the problem of overcounting that is generic to all of these different constructions.

[2]Such as vacua constructed tensor models based on the ADE minimal models [144, 176, 98, 138, 139, 140, 82] and level–1 Kazama–Suzuki models [135, 89, 77, 155], as well as G/H LG theories [142] related to Kazama–Suzuki models of higher levels.

[3]These manifolds are important in the context of possible phase transitions between Calabi–Yau string vacua [38] via the process of splitting and contraction introduced in [31].

3.2 Transversality of Catastrophes

The most explicit way of constructing a Landau–Ginzburg vacuum is, of course, to exhibit a specific potential that satisfies all the conditions imposed by the requirement that it ought to describe a consistent ground state of the string. Even though much effort has gone into the classification of singularities of the type described in the previous section, such polynomials have not been classified yet. The mathematicians have classified polynomials with at most three variables [3], which is two short of the lowest number of variables that is needed in order to construct a vacuum that allows a formulation of a four–dimensional low–energy effective theory [4].

In ref. [41] a set of potentials in five variables was constructed which represents an obvious generalization of the polynomials that appear in two–dimensional catastrophes. After imposing the conditions for these theories to describe string vacua, only a finite number of the infinite number of LG theories survive and all these solutions were constructed. It was clear however this construction of singularities is not complete, even after restricting to five variables. It is indeed easy to construct polynomials that are not among the types of polynomials considered in [41], a simple one being furnished by the example [151]

$$\mathbb{C}_{(15,10,6,1)}[45] \ni \left\{ z_1(z_1^2 + z_2^3 + z_3^5) + z_4^{45} + z_2^2 z_3^4 z_4 = 0 \right\}. \tag{3.2.0.24}$$

This polynomial is not of any of the types analyzed in [41] but it is nevertheless transverse.

Knowledge of the explicit form of the potential of a LG theory is very useful information when it comes to the detailed analysis of such a model. It is however not necessary if only limited knowledge, such as the computation of the spectrum of the theory, is required. In fact the only ingredients necessary for the computation of the spectrum of a LG vacuum [173] are the anomalous dimensions of the scaling fields as well as the fact that in a configuration of weights there exists a polynomial of appropriate degree with an isolated singularity. However, it is much easier to check whether there exists such a polynomial in a configuration than to actually construct such a potential. The reason is a theorem by Bertini [101, 102], which asserts that if a polynomial does have an isolated singularity on the base locus then, even though this potential may have worse singularities away from the base locus, there exists a deformation of the original polynomial that only admits an isolated singularity anywhere. Hence we only have to find criteria that guarantee at worst an isolated singularity on the base locus. It is precisely this problem that was addressed in the mathematics literature [71] at the same time as the explicit construction of LG vacua was started in ref. [41]. We briefly review the main point of the argument in [71] for the sake of completeness.

[4] The complete list of K3 representations embedded in $dim_{\mathbb{C}} = 3$ weighted projective space $\mathbb{P}_{(k_1,k_2,k_3,k_4)}$ has been obtained in [71].

Suppose we wish to check whether a polynomial in n variables z_i with weights k_i has an isolated singularity, i.e. whether the condition

$$dp = \sum_i \frac{\partial p}{\partial z_i} dz_i = 0 \qquad (3.2.0.25)$$

can be solved at the origin $z_1 = \cdots = z_n = 0$. According to Bertini's theorem, the singularities of a general element in $\mathbb{C}_{(k_1,\ldots,k_n)}[d]$ will lie on the base locus, i.e., the intersection of the hypersurface and all the components of the base locus, described by coordinate planes of dimension $k = 1, ..., n$. Let $\mathcal{P}_k$ such a k–plane, which we may assume to be described by setting to zero the coordinates $z_{k+1} = \cdots = z_n$. Expand the polynomials in terms of the non–vanishing coordinates $z_1, ..., z_k$

$$p(z_1, ..., z_n) = q_0(z_1, ..., z_k) \ + \sum_{j=k+1}^{n} q_j(z_1, ...,_k)z_j + h.o.t. \qquad (3.2.0.26)$$

Clearly, if $q_0 \neq 0$ then $\mathcal{P}_k$ is not part of the base locus and hence the hypersurface is transverse. If on the other hand $q_j = 0$, then $\mathcal{P}_k$ is part of the base locus and singular points can occur on the intersection of the hypersurfaces defined by $\mathcal{H}_j = \{q_j = 0\}$. If, however, we can arrange this intersection to be empty, then the potential is smooth on the base locus.

Thus we have found that the conditions for transversality in any number of variables is the existence for any index set $\mathcal{I}_k = \{1, ..., k\}$ of

1. either a monomial $z_1^{a_1} \cdots z_k^{a_k}$ of degree d
2. or of at least k monomials $z_1^{a_1} \cdots z_k^{a_k} z_{e_i}$ with distinct e_i.

Assume on the other hand that neither of these conditions holds for all index sets and let $\mathcal{I}_k$ be the subset for which they fail. Then the potential has the form

$$p(z_1, ..., z_n) = \sum_{j=k+1}^{n} q_j(z_1, ..., z_k)z_j \ + \ \cdots \qquad (3.2.0.27)$$

with at most $k-1$ non–vanishing q_j. In this case the intersection of the hypersurfaces $\mathcal{H}_j$ will be positive and hence the polynomial p will not be transverse.

As an example for the considerable ease with which one can check whether a given configuration allows for the existence of a potential with an isolated singularity, consider the polynomial of Orlik and Randall

$$p = z_1^3 + z_1 z_2^3 + z_1 z_3^5 + z_4^{45} + z_2^2 z_3^4 z_4. \qquad (3.2.0.28)$$

Condition (3.2.0.25) is equivalent to the system of equations

$$\begin{array}{rclrcl} 0 & = & 3z_1^2 + z_2^3 + z_3^5, & 0 & = & 3z_1 z_2^2 + 2z_2 z_3^4 z_4 \\ 0 & = & 5z_1 z_3^4 + 4z_2^2 z_3^3 z_4, & 0 & = & z_2^2 z_3^4 + 45 z_4^{44}. \end{array} \qquad (3.2.0.29)$$

which, on the base locus, collapses to the trivial pair of equations $z_2z_3 = 0 = z_2^3 + z_5^5$. Hence this configuration allows for such a polynomial. To check the system away from the base locus clearly is much more complicated.

By adding a fifth variable z_5 of weight 13 it is possible to define a singularity of type $\mathbb{C}_{(1,6,10,13,15)}[45]$, a configuration not considered in [41].

3.3 Finiteness Considerations

The problem of finiteness has two parts: first one has to put a constraint on the number of scaling fields that can appear in the LG theory and then one has to determine limits on the exponents with which the variables occur in the superpotential. Both of these constraints follow from the fact that the central charge of a Landau–Ginzburg theory with fields of charge q_i

$$c = 3\sum(1-2q_i) =: \sum c_i \tag{3.3.0.30}$$

has to be $c = 9$ in order to describe a string vacuum.

It should be clear that without any additional input the number of Landau–Ginzburg vacuum configurations that can be exhibited is infinite. This is to be expected simply because it is known from the construction of CICYs [41] that it is often possible to rewrite a manifold in an infinite number of ways and we ought to encounter similar things in the LG framework. A trivial way to do this is to simply add mass terms that do not contribute to the central charge. Even though trivial such mass terms are important and necessary for LG theories, not only in orbifold constructions [128] but also in order to relate them to CY manifolds. Consider e.g. the codimension–four Calabi–Yau manifold

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{array}\left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \tag{3.3.0.31}$$

with the defining polynomials

$$\begin{array}{rclrcl} p_1 &=& \sum_{i=1}^2 u_i^2 v_i, & p_2 &=& \sum_{i=1}^3 v_i^2 w_i \\ p_3 &=& \sum_{i=1}^3 w_i^2 x_i, & p_4 &=& \sum_{i=1}^3 x_i^2 \end{array} \tag{3.3.0.32}$$

the superpotential $W = \sum p_i$ of which lives in

$$\mathbb{C}^{\star}_{(5,5,6,6,6,4,4,4,8,8,8)}[16]^{10}_{-32} \tag{3.3.0.33}$$

and has an isolated singularity at the origin. All eleven variables are coupled and hence this example appears to involve three fields with zero central charge in a nontrivial way.

It turns out, however, that the manifold (3.3.0.31) is equivalent to a manifold with nine variables. One way to see this is by making use of some topological identities introduced in [31]. First consider the well–known isomorphism

$$\mathbb{P}_2[2] \equiv \mathbb{P}_1, \tag{3.3.0.34}$$

which allows us to rewrite the manifold above as

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}. \tag{3.3.0.35}$$

Using the surface identity [31]

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \end{array} \tag{3.3.0.36}$$

applied via the rule

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ X \end{array} \begin{bmatrix} 2 & 0 \\ 1 & a \\ 0 & M \end{bmatrix} = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ X \end{array} \begin{bmatrix} a \\ a \\ M \end{bmatrix} \tag{3.3.0.37}$$

shows that this space is in turn equivalent to

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 1 & 2 \end{bmatrix}, \tag{3.3.0.38}$$

a manifold with only nine homogeneous coordinates. It should be noted that the LG potential of (3.3.0.38), defined by the sum of the two polynomials, certainly does not have an isolated singularity. Furthermore it is not possible to even assign weights to the fields such that the central charge comes out to be nine! It is thus possible, by applying topological identities, to extend the applicability of Landau–Ginzburg theories to types of Calabi–Yau manifolds that were hitherto completely inaccessible by the standard formulation.

Further insight into the problem of redundancy in the construction of LG potentials can be gained by an LG theoretic analysis of this example. From the weights of the scaling variables in the LG configuration above it is clear that the spectrum of this LG configuration remains the same if the last three coordinates are set to zero. In the potential

$$W = \sum_{i=1}^{2} \left(u_i^2 v_i + v_i^2 w_i + w_i^2 x_i + x_i^2 \right) + (v_3^2 w_3 + w_3^2 x_3 + x_3^2) \tag{3.3.0.39}$$

described by the CICY polynomials (3.3.0.32), these variables cannot be set to zero because they are coupled to other fields; hence it seems impossible to reduce the number of fields. Consider however the following change of variables

$$y_i = x_i + \frac{1}{2}w_i^2, \quad i = 1, 2, 3. \tag{3.3.0.40}$$

It follows from these transformations that the potential defined by by (3.3.0.39) is equivalent to

$$W = \sum_{i=1}^{2} \left(u_i^2 v_i + v_i^2 w_i - \frac{1}{4} w_i^4 \right) + v_3^2 w_3 - \frac{1}{4} w_3^4. \tag{3.3.0.41}$$

Adding a trivial factor and splitting this potential into three separate polynomials

$$p_1 = u_1^2 v_1 + u_2^2 v_2, \quad p_2 = v_1^2 w_1 + v_2^2 w_2 + v_3^2 w_3, \quad p_3 = -\frac{1}{4}\left(w_1^4 + w_2^4 + w_3^4\right) + x^2 \tag{3.3.0.42}$$

we see that the original model is equivalent to a weighted configuration

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_{(1,1,1,2)} \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}, \tag{3.3.0.43}$$

which again describes manifolds with nine variables. Thus the original configuration (3.3.0.31) is in fact equivalent to two different (weighted) CICY representations

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \equiv \begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \equiv \begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_{(1,1,1,2)} \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}. \tag{3.3.0.44}$$

To summarize the last few paragraphs, we have shown two things: first that by adding trivial factors and coupling them to the remaining fields we can give a Landau–Ginzburg description of a larger class of Calabi–Yau manifolds than previously thought possible. Furthermore we can use topological identities to obtain an LG formulation of CY manifolds, which do not admit a canonical LG potential at all. Incidentally we have also shown that it is possible to relate complete intersection manifolds embedded in products of projective spaces to weighted complete intersection manifolds embedded in products of weighted projective space.

Similarly the number of fields can grow without bound if we not only allow fields that do not contribute to the central charge but also fields with a negative contribution. Again such fields provide redundant descriptions of simpler LG theories; they are nevertheless important for the LG/CY relation and occur in the constructions of splitting and contraction introduced in ref. [31]. Even though these constructions were discussed in [31] only in the context of Calabi–Yau manifolds embedded in products of ordinary projective spaces they readily generalize to the more general framework of weighted projective spaces.

In special circumstances, the splitting or contraction process does not change the spectrum of the theory and hence it provides another tool to relate LG potentials with at most nine variables manifolds with more than nine homogeneous coordinates. Consider e.g. the manifold embedded in

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix}^{(2,128)}_{-252} \tag{3.3.0.45}$$

which is described by the zero locus of the two polynomials

$$\begin{aligned} p_1 &= x_1^2 y_1 + x_2^2 y_2 \\ p_2 &= y_1^6 + y_2^6 + y_3^6 + y_4^6 + y_5^2 \end{aligned} \tag{3.3.0.46}$$

that can be described by a superpotential $W = p_1 + p_2$ defining a Landau–Ginzburg theory in

$$\mathbb{C}^{\star}_{(5,5,2,2,2,2,6)}[12]^{(2,128)}_{-252}. \tag{3.3.0.47}$$

This manifold can be rewritten via an ineffective split in an infinite sequence of different representations as

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 6 \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix} \longrightarrow \cdots \tag{3.3.0.48}$$

which are described by LG potentials in the alternating classes

$$\mathbb{C}^{\star 7}_{(5,5,2,2,2,2,6)}[12] \longrightarrow \mathbb{C}^{\star 9}_{(1,1,10,10,2,2,2,2,6)}[12] \longrightarrow \mathbb{C}^{\star 11}_{(5,5,2,2,10,10,2,2,2,2,6)}[12] \longrightarrow \cdots \tag{3.3.0.49}$$

i.e. the infinite sequence belongs to the configurations

$$\mathbb{C}^{\star 7+4k}_{(5,5,2,2,10,10,2,2,10,10,\ldots,2,2,10,10,2,2,6)}[12] \tag{3.3.0.50}$$

where the part $(2,2,10,10)$ occurs k times, and

$$\mathbb{C}^{\star 5+4k}_{(1,1,10,10,2,2,10,10,2,2,\ldots 10,10,2,2,2,2,6)}[12]. \tag{3.3.0.51}$$

where $(10,10,2,2)$ occurs k times.

The construction above easily generalizes to a number of examples which all belong to a class of spaces discussed in ref. [159]. Consider manifolds embedded in

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 \\ k_1 & k \end{bmatrix}, \tag{3.3.0.52}$$

where $k = k_1 + k_3 + k_4 + k_5$. These spaces can be split into the infinite sequences

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 \\ k_1 & k \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & k_1 & k \end{bmatrix} \longrightarrow \begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k_1 & k \end{bmatrix} \longrightarrow \cdots \tag{3.3.0.53}$$

If the weights are such that k/k_i is an integer, then it is easy to write down the tensor model that corresponds to it (but this is not essential). In such models the levels l_i of the tensor model $l_1^2 \cdot l_3 \cdot l_4 \cdot l_5$ in terms of the weights are given by

$$l_1 = l_2 = \frac{2k}{k_1} - 2, \quad l_i = \frac{k}{k_i} - 2, \ i = 3, 4, 5 \tag{3.3.0.54}$$

and the corresponding LG potentials live in

$$\mathbb{C}^{\star 7}_{(k-k_1,k-k_1,2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \longrightarrow \mathbb{C}^{\star 9}_{(k_1,k_1,2(k-k_1),2(k-k_1),2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \longrightarrow \cdots \tag{3.3.0.55}$$

i.e. they belong to the sequences

$$\mathbb{C}^{\star 7+4p}_{((k-k_1),(k-k_1),2k_1,2k_1,2(k-k_1),2(k-k_1),....,2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \tag{3.3.0.56}$$

where the part $(2k_1, 2k_1, 2(k-k_1), 2(k-k_1))$ occurs p times, and

$$\mathbb{C}^{\star 9+4k}_{(k_1,k_1,2(k-k_1),2(k-k_1),2k_1,2k_1,...,2k_3,2k_4,2k_5)}[2k]. \tag{3.3.0.57}$$

where $(2(k-k_1), 2(k-k_1), 2k_1, 2k_1)$ occurs p times.

All these models are constructed in such a way that they have central charge nine, but in contrast to the example discussed previously, there now appear fields with negative central charge. In the case at hand, however, these dangerous fields only occur in a *coupled* subpart of the theory; the smallest subsystem which involves these fields and which one can isolate is in fact a theory with positive central charge. In the series of splits just described e.g., the fields with negative central charge that occur in the first split always appear in the subsystem described by the configurations

$$\mathbb{C}^{\star}_{(k_1,2(k-k_1),2k_1)}[2k] \tag{3.3.0.58}$$

with potentials of the form

$$x^2y + yz + z^{2k/k_1}. \tag{3.3.0.59}$$

Thus the contribution to the central charge of this sector becomes

$$c = 3\left[\left(1 - \frac{k_1}{k}\right) + \left(1 - \frac{2(k-k_1)}{k}\right) + \left(1 - \frac{2k_1}{k}\right)\right], \tag{3.3.0.60}$$

which is always positive. This formula suggests that it ought to be possible to dispense with the variables y, z altogether, as their total contribution to the central charge adds up to zero, and that this theory is equivalent to that of a single monomial of degree k/k_1.

More generally one may consider the Landau–Ginzburg theory defined by the potential

$$x_1^a x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n^b. \tag{3.3.0.61}$$

From the central charge

$$c = \left\{ \begin{array}{ll} 6\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right), & n \text{ even} \\ 3\left(1-\frac{2}{ab}\right), & n \text{ odd} \end{array} \right\}, \tag{3.3.0.62}$$

as well as from the dimension of the chiral ring

$$\dim R_n = \prod \left(\frac{1}{q_i} - 1\right) = \left\{ \begin{array}{ll} ab-b-1, & n \text{ even} \\ ab-1, & n \text{ odd} \end{array} \right\}, \tag{3.3.0.63}$$

we expect this theory to be equivalent to

$$x_1^a x_2 + x_2 x_3 + \cdots x_{n-1} x_n + x_n^b = \left\{ \begin{array}{ll} x_1^a x_n + x_n^b, & n \text{ even} \\ x_1^{ab}, & n \text{ odd} \end{array} \right\}. \tag{3.3.0.64}$$

Such relations are supported by the identification of rather different LG configurations such as

$$\mathbb{C}^\star_{(1,1,1,6,6,6,3,3,3,3,3,3)}[9] \equiv \mathbb{C}^\star_{(1,1,1,3,3)}[9] \tag{3.3.0.65}$$

as well as many other identifications which are rather nontrivial in the context of the associated manifolds.

It follows from the above considerations that we have to assume, in order to avoid redundant reconstructions of LG theories, that the central charge of all scaling fields of the potential should be positive. In order to relate the potentials to manifolds, we may then add one or several trivial factors or more complicated theories with zero central charge.

Using the above results we derive in the following more detailed finiteness conditions. Observe first that from eq. (3.3.0.30) written as

$$\sum_{i=1}^{r} q_i = \left(\frac{r}{2} - \frac{c}{6}\right) := \hat{c} \tag{3.3.0.66}$$

we obtain $r > c/3$.

Now let p be a polynomial of degree d in r variables. For the index set $\mathcal{I}_1$ the conditions for transversality imply the existence of $n_i \in \mathbb{N}^+$, $i = 1, \ldots, r$ and of a map $j : \mathcal{I}_r \to \mathcal{I}_r$ such that for all i there exists $j(i)$ such that

$$q_i = \frac{1 - q_{j(i)}}{n_i}. \tag{3.3.0.67}$$

Let us first see how many non-trivial fields can occur at most. Fields which have charge $q_i \le 1/3$ contribute $c_i \ge 1$ to the conformal anomaly. Now consider fields with larger charge. Since we assume $c_i > 0$, they are in the range $1/3 < q_i < 1/2$. Among these fields the transversality condition (1.) cannot hold, because two of them are not enough and three of them are too many fields in order to form a monomial of charge one. Transversality condition (2.) implies that each of them has to occur together with a partner field $z_{j(i)}$. These pairs contribute to the conformal anomaly according to (3.3.0.67,3.3.0.30) $c_i + c_{j(i)} > 2$, so we can conclude that $r \le c$.

In order to construct all transversal LG potentials for a given c, we choose a specific r in the range obtained above and consider all possible maps j of which there are r^r. Without restricting on the generality, we may *then* assume the n_i to be ordered $n_1 \le \cdots \le n_r$. Starting with (3.3.0.66) we obtain via eq. (3.3.0.67) and the positivity of the charges a bound $n_1 < r/c$. Now we choose n_1 in the allowed range and use (3.3.0.67) in order to eliminate the q_1, if necessary in favour of the $q_2, \ldots, q_r$. This yields an equation of the general form ($p = 2$)

$$\sum_{i=p}^{r} w_i^{(p)} q_i = \hat{c}^{(p)}. \tag{3.3.0.68}$$

If $\hat{c}^{(p)} \ne 0$, eq. (3.3.0.68) either is in contradiction with the positivity of the charges or one can derive a finite bound for n_p. Assume $\hat{c}^{(p)} > 0$ and let $\mathcal{I}_+$ be the indices of the positive $w_i^{(p)}$; then one has $n_p < (\sum_{i\in\mathcal{I}_+} w_i^{(p)})/\hat{c}^{(p)}$; likewise if $\hat{c}^{(p)} < 0$ we have $n_p < (\sum_{i\in\mathcal{I}_-} w_i^{(p)})/\hat{c}^{(p)}$.

Consider the case $\hat{c}^{(p)} = 0$. If the $w_i^{(p)}$ are indefinite we get no bound from (3.3.0.68). However we will show that the existence of certain monomials, which are required by the transversality conditions, implies a bound for n_p. Let $\mathcal{I}_a$ denote the indices of the already bounded n_i and $\mathcal{I}_b$ the others. The charge of the field z_a with $a \in \mathcal{I}_a$ will depend on the unknown charge of a field $z_{b(a)}$ with $b \in \mathcal{I}_b$ if there is a chain of indices $a_0 = a, a_1 = j(a), \ldots, a_l = j(\ldots j(a)\ldots) =: b(a)$ linked by the map j. The charge of z_a is given by

$$q_a = \frac{1}{n_a} - \frac{1}{n_a n_{a_1}} + \cdots - \frac{(-1)^l}{n_a \ldots n_{l-1}} + \frac{(-1)^l q_{b(a)}}{n_a \ldots n_{l-1}}. \tag{3.3.0.69}$$

Indefiniteness of the $w_i^{(p)}$ can only occur if there are fields z_a, $a \in \mathcal{I}_a$, with odd l, i.e. the last term in (3.3.0.69) $s_a q_{b(a)} := (-1)^l/(n_a \ldots n_{a_{l-1}} q_{b(a)})$ is negative. Call the index set of these fields $\mathcal{I}_a^-$. Assume first that the transversality condition (1.) holds. This implies the existence of $m_i \in \mathbb{N}^+$ ($m_i < 2n_i$) such that $\sum_{i\in\mathcal{I}_a^-} m_i q_i = 1$. For the unknown q_i, $i \in \mathcal{I}_b$, we get an equation of the form $\sum w_i q_i = \epsilon$, which yields a bound for the lowest n_i, $i \in \mathcal{I}_b$, since $w_i > 0$. The lowest possible value for $\epsilon > 0$ can be readily calculated from the denominators occurring in (3.3.0.69). If transversality condition (2.) applies, we have $|\mathcal{I}_a^-|$ equations of the form $\sum_{i\in\mathcal{I}_a^-} m_i^{(j)} q_i = 1 - q_{e_j}$ which can be rewritten as $\sum_{i\in\mathcal{I}_b} w_i^{(j)} q_i = \epsilon^{(j)}$. Only if all $w_i^{(j)}$ happen to be indefinite and all $\epsilon^{(j)}$ are zero we get *no bound* from this condition. Assuming this to be true we have

$$\sum_{i\in\mathcal{I}_b} m^{(j)} s_i q_i - s_{e_j} q_{b(e_j)} = 0, \tag{3.3.0.70}$$

where $s_{e_j} := 1$ and $b(e_j) := b$ if $e_j \in \mathcal{I}_b$. Note that $\sum_i m_i^{(j)} \geq 2$ in order to avoid quadratic mass terms. Now we can rewrite (3.3.0.68) in the form

$$\sum s_i q_{b(i)} = 0. \tag{3.3.0.71}$$

Using now (3.3.0.70) and $\sum_i m_i^{(j)} \geq 2$ in order to eliminate the negative s_i, one finds $\sum_i w_i q_i \leq 0$ with $w_i > 0$, which is in contradiction with the positivity of the charges, hence we get a bound in any case.

This procedure of restricting the bound for n_p, given $n_i, \ldots, n_{p-1}$, was implemented in a computer program. It allows all configurations to be found without testing unnecessarily many combinations of the n_i. The actual upper bounds for the $(n_i, \ldots, n_r)$ in the four–variable case are $(7, 17, 83, 1805)$, and we have found $2,390$ configurations which allow for transversal polynomials. In the five–variable case the bounds are $(4, 6, 14, 62, 923)$ and $5,165$ configurations exist. By adding a trivial mass term z_5^2 in the four–variable case, these configurations lead to the total of 7,555 three–dimensional Calabi–Yau manifolds described by a one polynomial constraint in a four–dimensional weighted projective space.

The same figures for the six–variable case and seven–variable case are $(3, 3, 5, 11, 41, 482)$, $2,567$ and $(2, 2, 3, 4, 8, 26, 242)$, 669 respectively.

Likewise for eight– and nine–variable potentials, we have bounds $(2, 2, 2, 2, 2, 3, 3, 5, 14)$ with 47 examples and $(2, 2, 2, 2, 2, 2, 2, 2, 2)$ with 1 example respectively.

3.4 Results and Comparisons

In [122] we have constructed all 10,839 Landau–Ginzburg theories which lead to 2997 different spectra, i.e. pairs of generations and antigenerations. The massless spectrum is very rough information about a theory and it is likely that the degeneracy is lifted to a large degree when additional information, such as the number of singlets and/or the Yukawa couplings, becomes available. We expect the situation to be very similar to the class of CICYs [31], which only leads to some 250 different spectra [94], but for which a detailed analysis of the Yukawa couplings [104] shows that it contains several thousand distinct theories.

It is clear however that there is in fact some redundancy in this class of Landau–Ginzburg theories even beyond the one discussed in the previous sections. In the list there appear, for instance, two theories with spectrum $(h^{(1,1)}, h^{(2,1)}, \chi) = (2, 122, -240)$ involving five variables

$$\mathbb{C}^*_{(2,2,2,1,7)}[14] \ni \{z_1^7 + z_2^7 + z_3^7 + z_4^{14} + z_5^2 = 0\} \tag{3.4.0.72}$$

and

$$\mathbb{C}^*_{(1,1,1,1,3)}[7] \ni \{y_1^7 + y_2^7 + y_3^7 + y_4^7 + y_4 y_5^2 = 0\}. \tag{3.4.0.73}$$

Using the fractional transformations introduced in ref. [141] it is easy to show that these two models are equivalent, even though this is not obvious by just looking at the potentials. To see this consider first the orbifold

$$\mathbb{C}^*_{(2,2,2,1,7)}[14]/\mathbb{Z}_2 : [\ 0\ 0\ 0\ 1\ 1] \tag{3.4.0.74}$$

of the first model where the action indicated by $\mathbb{Z}_2 :\ [\ 0\ 0\ 0\ 1\ 1]$ means that the first three coordinates remain invariant, whereas the latter two variables are to be multiplied with the generator of the cyclic $\mathbb{Z}_2$, $\alpha = -1$. Since the action of this $\mathbb{Z}_2$ on the weighted projective space is part of the projective equivalence, the orbifold is of course isomorphic to the original model. On the other hand the fractional transformation

$$z_i = y_i,\ i = 1, 2, 3; \quad z_4 = y_4^{1/2}, \quad z_5 = y_4^{1/2} y_5 \tag{3.4.0.75}$$

associated with this symmetry [140] defines a 1–1 coordinate transformation on the orbifold, which transforms the first theory into the second; these are therefore equivalent as well.

Similarly the equivalences

$$\begin{aligned} \mathbb{C}^*_{(2,2,2,3,9)}[18]^{(4,112)}_{-216} &= \mathbb{C}^*_{(1,1,1,3,3)}[9] \\ \mathbb{C}^*_{(2,6,6,7,21)}[42]^{(17,65)}_{-96} &= \mathbb{C}^*_{(1,3,3,7,7)}[21] \\ \mathbb{C}^*_{(2,5,14,14,35)}[70]^{(27,59)}_{-64} &= \mathbb{C}^*_{(1,5,7,7,15)}[35] \end{aligned} \tag{3.4.0.76}$$

can be shown, as well as a number of others.

It should be noted that even though we now have constructed Landau–Ginzburg potentials with an arbitrary number of scaling fields, the basic range of the spectrum has not changed as compared with the results of [41] where it was found that the spectra of all 6000 odd theories constructed there lead to Euler numbers which fall in the range [5]

$$-960 \le \chi \le 960. \tag{3.4.0.77}$$

[5]This should be compared with the result for the complete intersection Calabi–Yau manifolds where $-200 \le \chi \le 0$ [31].

In fact not only do all the LG spectra fall into this range, all known Calabi–Yau spectra and all the spectra from exactly solvable tensor models are contained in this range as well! This suggests that perhaps the spectra of all string vacua based on $c = 9$ will be found within this range. To put it differently, we conjecture that the Euler numbers of all Calabi–Yau manifolds are contained in the range $-960 \leq \chi \leq 960$.

Similarly to the results in [41], the Hodge pairs do not pair up completely. In fact the mirror symmetry of the space of Landau–Ginzburg vacua is just about 77%. As an illustration of this fact, we have listed in table 1 the Euler numbers that occur in our list and in Figs. 3.1 and 3.2 we have plotted all the spectra and only those without mirror partners respectively. It thus appears that orbifolding is an essential ingredient in the construction of a mirror–symmetric slice of the configuration space of the heterotic string. It is in fact easy to produce examples of orbifolds whose spectrum does not appear in our list of LG vacua. An example of a mirror pair is furnished by the orbifold

$$\mathbb{C}^*_{(1,1,1,1,2)}[6]^{(1,103)}_{-204}/\mathbb{Z}_6 : [\,3 \quad 2 \quad 1 \quad 0 \quad 0\,] \tag{3.4.0.78}$$

which has the spectrum $(11, 23, -24)$ and the space

$$\mathbb{C}^*_{(1,1,1,1,2)}[6]^{(1,103)}_{-204}/\mathbb{Z}_6 \times \mathbb{Z}_3 : \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \tag{3.4.0.79}$$

which has the mirror–flipped spectrum $(23, 11, 24)$. The space of LG orbifolds that has been constructed so far [128] is indeed much closer to being mirror–symmetric than the space of LG theories itself. Even though the construction in [128] is incomplete, about 94% of the Hodge numbers pair up.

By now there exist many different prescriptions to construct string vacua and it is useful to put the LG framework into the context of other left–right–symmetric constructions. Among the more prominent ones other than Calabi–Yau manifolds, which are obviously closely related to LG theories, there are constructions which have traditionally been called, somewhat misleadingly, orbifolds (what is meant are orbifolds of tori) [63], free–fermion constructions [118, 2], lattice constructions [133], and interacting exactly solvable models [86][120].

None of these classes are known completely, even though much effort has gone into the exploration of some of them. Because powerful computational tools are available, toroidal orbifolds have been analysed in some detail [64],[103] and much attention has focused on the explicit construction of exactly solvable theories in the context of tensor models via $N = 2$ superconformal minimal theories [138, 139, 140, 137, 82, 83, 84] and Kazama–Suzuki models [77, 135, 89, 142, 155, 82, 1] as well as their in depth analysis [135, 89, 160, 46, 47, 48]. In the context of (2,2) vacua, orbifolds lead only to some tens of distinct models, whereas the known classes of exactly solvable theories lead only to a few hundred models with distinct spectra. Similar results have been obtained so far via the covariant–lattice approach [134] and hence it is obvious that these constructions do not exhaust the configuration space of heterotic string by far. The class of Landau–Ginzburg string vacua thus appears as a rather extensive source of (2,2)–symmetric models.

It should also be remarked that the (2,2) Landau–Ginzburg theories we have constructed here can be used to build a probably much larger class of (2,0) models along the lines described in [55], using an appropriate adaptation of the work of [33] [123][75] in order to determine the instantons on which the existence of a certain split of a vector bundle has to be checked.

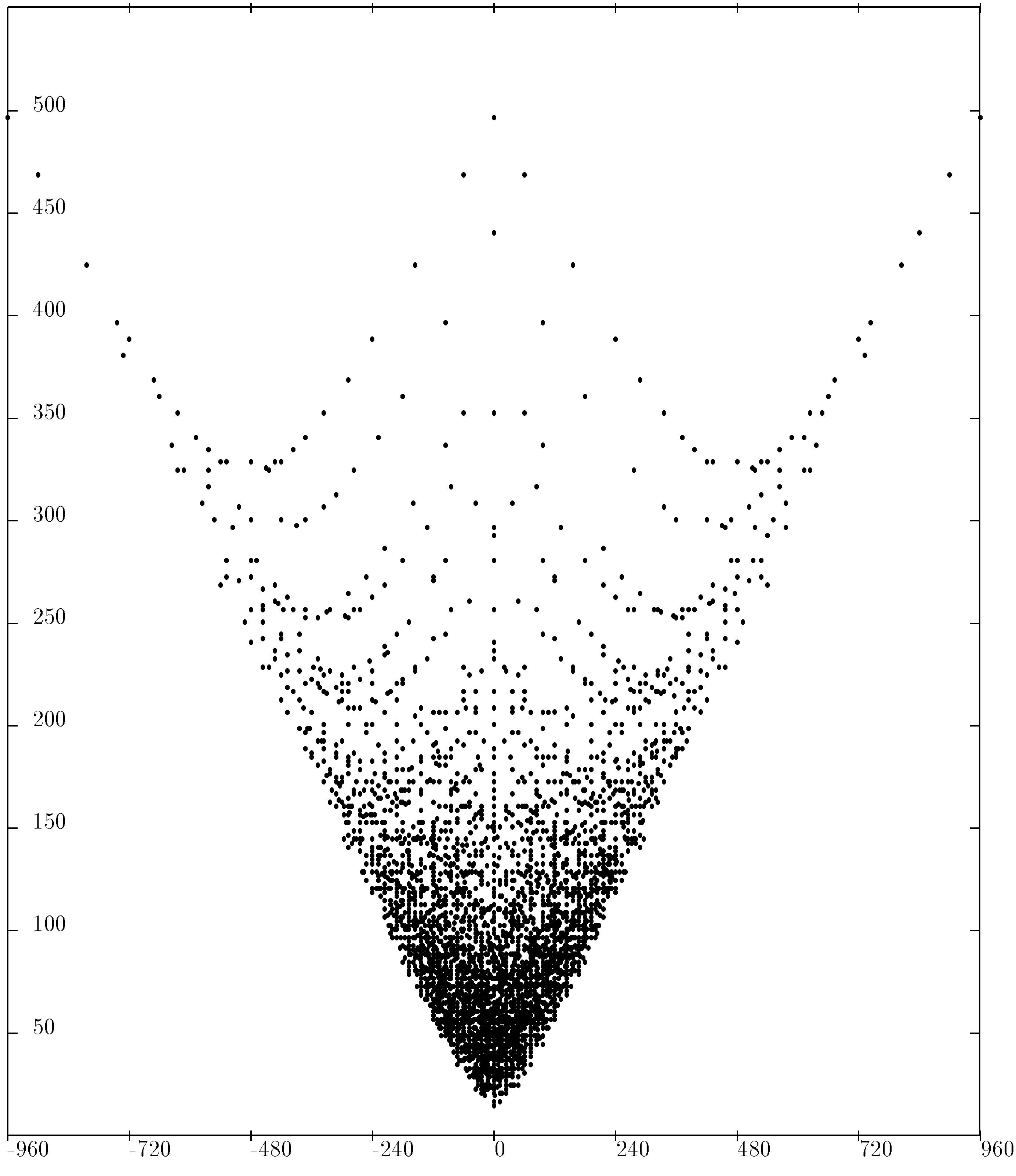


Fig. 3.1 *A plot of Euler numbers against $\bar{n}_g + n_g$ for the 2997 spectra of all the LG potentials.*

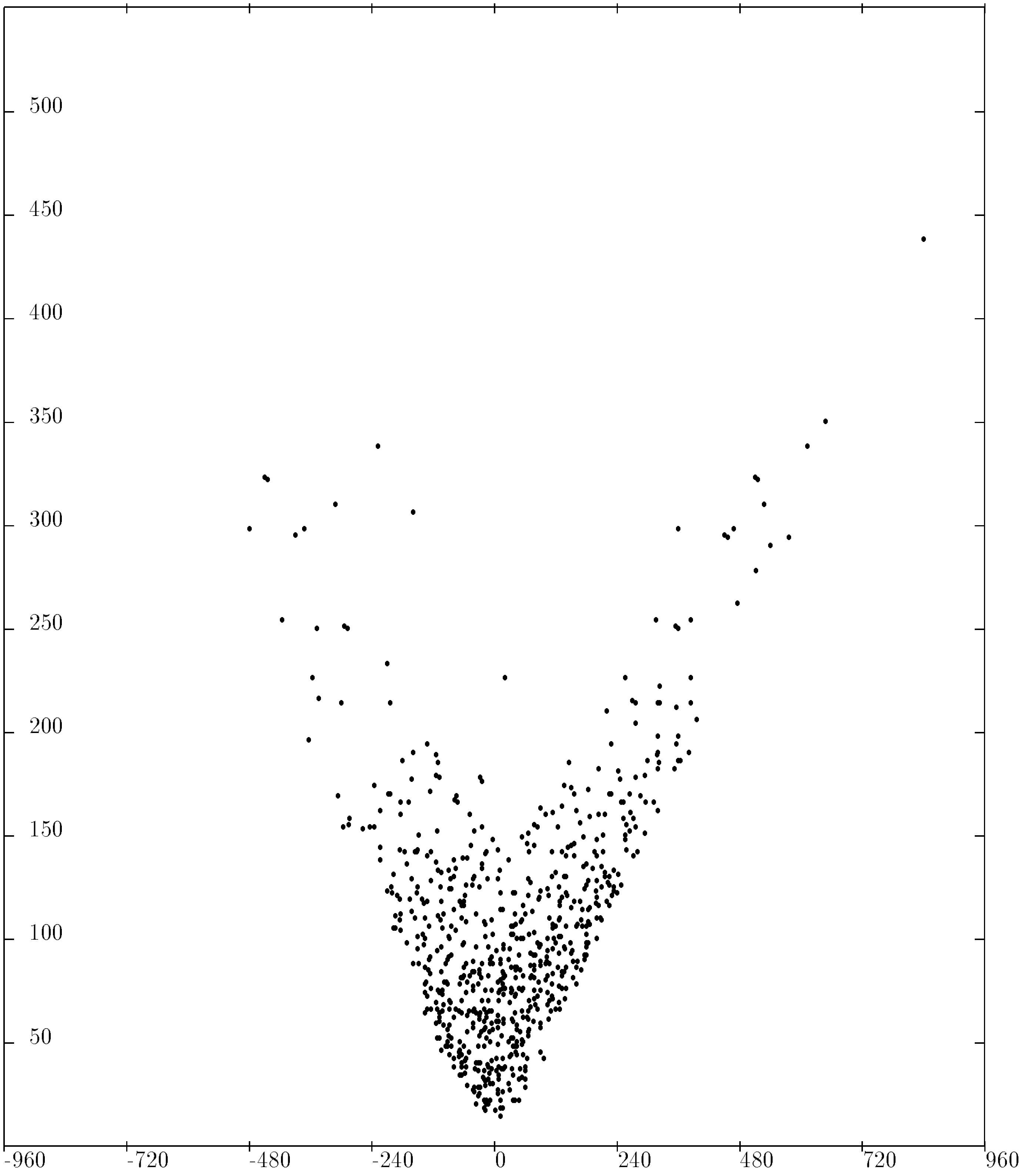


Fig. 3.2 *A plot of Euler numbers against $\bar{n}_g + n_g$ for the 695 spectra of the LG potentials without a mirror candidate in the list.*

Table 3.1 *A list of all the Euler numbers that arise via Landau–Ginzburg vacua*

0		− 64	64	− 134	$\star$	− 208	208	− 296	296	− 432	432
$\star$	2	− 66	66	− 136	136	− 210	210	− 300	300	− 444	444
− 4	4	− 68	68	− 138	138	$\star$	212	− 304	304	− 450	450
− 6	6	− 70	70	− 140	140	− 216	216	− 306	306	− 456	456
− 8	8	− 72	72	− 142	142	− 220	220	− 312	312	− 468	468
− 10	10	− 74	74	− 144	144	− 222	222	$\star$	316	$\star$	476
− 12	12	− 76	76	$\star$	146	− 224	224	$\star$	318	− 480	480
− 14	14	− 78	78	− 148	148	− 228	228	− 320	320	− 492	492
− 16	16	− 80	80	− 150	150	$\star$	230	$\star$	322	− 504	504
− 18	18	− 84	84	− 152	152	− 232	232	− 324	324	$\star$	510
− 20	20	− 86	86	$\star$	154	− 234	234	− 330	330	$\star$	512
− 22	$\star$	− 88	88	− 156	156	− 236	236	− 336	336	− 516	516
− 24	24	− 90	90	− 160	160	− 240	240	− 340	340	− 528	528
− 26	$\star$	− 92	92	− 162	162	$\star$	242	− 342	342	− 540	540
− 28	28	− 94	$\star$	− 164	164	− 244	$\star$	− 344	$\star$	− 552	552
− 30	30	− 96	96	− 166	$\star$	− 246	246	− 348	348	− 564	564
− 32	32	− 98	$\star$	− 168	168	$\star$	248	$\star$	352	− 576	576
$\star$	34	− 100	100	$\star$	170	− 252	252	$\star$	354	− 588	588
− 36	36	− 102	102	− 172	172	− 256	256	− 356	356	− 612	612
− 38	$\star$	− 104	104	− 174	174	− 258	258	− 360	360	− 624	624
− 40	40	− 106	106	− 176	176	− 260	260	− 364	364	− 636	636
− 42	42	− 108	108	− 180	180	− 264	264	− 368	368	$\star$	648
− 44	44	− 110	110	− 184	184	$\star$	266	− 372	372	− 660	660
− 46	$\star$	− 112	112	− 186	186	$\star$	270	− 376	376	− 672	672
− 48	48	− 114	114	− 190	$\star$	− 272	272	$\star$	380	− 720	720
− 50	50	− 116	116	− 192	192	− 276	276	− 384	384	− 732	732
− 52	52	− 120	120	− 194	$\star$	− 280	280	− 390	$\star$	− 744	744
− 54	54	− 124	124	− 196	196	− 284	$\star$	− 396	396	− 804	804
− 56	56	− 126	126	− 198	198	− 286	286	− 408	408	$\star$	840
− 58	$\star$	− 128	128	− 200	200	− 288	288	− 416	$\star$	− 900	900
− 60	60	− 130	130	− 202	$\star$	− 292	292	− 420	420	− 960	960
− 62	$\star$	− 132	132	− 204	204	− 294	294	− 426	426		

Table 3.2 *Three–generation models that arise in the set of Landau–Ginzburg vacua*

N^0	Manifold	Polynomial	χ	$h^{(1,1)}$	$h^{(1,2)}$
1	$\mathbb{C}(3,8,30,79,117)[237]$	$z_1^{79} + z_2^{15} z_5 + z_3^4 z_5 + z_4^3 + z_5^2 z_1 + z_3^3 z_2^{18} z_1$	6	77	74
2	$\mathbb{C}(4,5,26,65,95)[195]$	$z_1^{25} z_5 + z_2^{39} + z_3^5 z_4 + z_4^3 + z_5^2 z_2$	6	70	67
3	$\mathbb{C}(1,21,30,38,45)[135]$	$z_1^{135} + z_2^5 z_3 + z_3^3 z_5 + z_4^3 z_2 + z_5^3$	6	50	47
4	$\mathbb{C}^*(1,16,23,29,34)[103]$	$z_1^{103} + z_2^5 z_3 + z_3^3 z_5 + z_4^3 z_2 + z_5^3 z_1$	6	50	47
5	$\mathbb{C}^*(5,6,14,45,65)[135]$	$z_1^{27} + z_2^{15} z_4 + z_3^5 z_5 + z_4^3 + z_5^2 z_1$	6	45	42
6	$\mathbb{C}^*(2,9,12,23,37)[83]$	$z_1^{37} z_2 + z_3^5 z_4 + z_4^2 z_5 + z_5^2 z_2 + z_4 z_1^{30} + z_5 z_1^{23}$	6	40	37
7	$\mathbb{C}^*(4,7,13,41,58)[123]$	$z_1^{29} z_2 + z_3^5 z_5 + z_4^3 + z_5^2 z_2 + z_3^3 z_1^{21} + z_5 z_1^{13} z_3 + z_3^3 z_2^{12}$	6	40	37
8	$\mathbb{C}^*(4,6,15,35,45)[105]$	$z_1^{15} z_5 + z_2^{15} z_3 + z_3^7 + z_4^3 + z_5^2 z_3 + z_5 z_2^{10}$	6	37	34
9	$\mathbb{C}^*(2,6,9,17,17)[51]$	$z_1^{21} z_3 + z_2^7 z_3 + z_4^3 + z_5^3 + z_1^{17} z_4$	6	34	31
10	$\mathbb{C}^*(2,9,19,24,27)[81]$	$z_1^{36} z_2 + z_2^9 + z_3^3 z_4 + z_4^3 z_2 + z_5^3 + z_3 z_1^{31}$	6	32	29
11	$\mathbb{C}^*(5,8,12,15,35)[75]$	$z_1^{15} + z_2^5 z_5 + z_3^5 z_4 + z_4^5 + z_5^2 z_1$	6	30	27
12	$\mathbb{C}^*(3,4,6,13,13)[39]$	$z_1^{13} + z_2^9 z_1 + z_3^6 z_1 + z_4^3 + z_5^3 + z_3 z_2^5 z_4$	6	29	26
13	$\mathbb{C}^*(4,5,10,11,19)[49]$	$z_1^{11} z_2 + z_3^3 z_5 + z_4^4 z_2 + z_5^2 z_4 + z_4^3 z_1^4 + z_5 z_2^6$	6	23	20
14	$\mathbb{C}^*(5,6,9,10,21)[51]$	$z_1^9 z_2 + z_2^7 z_3 + z_4^3 z_5 + z_5^2 z_3 + z_3^4 z_1^3 + z_5 z_1^6 + z_5 z_2^5$	6	23	20
15	$\mathbb{C}^*(10,12,13,15,25)[75]$	$z_1^6 z_4 + z_2^5 z_4 + z_3^5 z_1 + z_4^5 + z_5^3 + z_2 z_1^5 z_3 + z_3^3 z_2^3$	6	20	17
16	$\mathbb{C}^*(3,4,6,6,7,7,9)[21]$	$z_1^7 + z_2^3 z_7 + z_3^3 z_1 + z_4^3 z_1 + z_5^3 + z_6^3 + z_7^2 z_1 + z_3^2 z_7 + z_7 z_4^2 + z_4 z_2^2 z_5$	6	19	16
17	$\mathbb{C}^*(5,6,9,14,17)[51]$	$z_1^9 z_2 + z_2^7 z_3 + z_4^3 z_3 + z_5^3 + z_3^4 z_1^3 + z_4^2 z_2^3 z_1$	6	18	15
18	$\mathbb{C}^*(5,8,9,11,12)[45]$	$z_1^9 + z_2^5 z_1 + z_3^5 + z_4^3 z_5 + z_5^3 z_3$	6	16	13
19	$\mathbb{C}^*(3,10,18,59,87)[177]$	$z_1^{59} + z_2^9 z_5 + z_3^5 z_5 + z_4^3 + z_5^2 z_1 + z_3^3 z_2^{12} z_1$	−6	57	60
20	$\mathbb{C}^*(2,8,29,49,59)[147]$	$z_1^{59} z_3 + z_2^{11} z_5 + z_4^3 + z_5^2 z_3 + z_5 z_1^{44}$	−6	48	51
21	$\mathbb{C}^*(1,18,32,39,45)[135]$	$z_1^{135} + z_2^5 z_5 + z_3^3 z_4 + z_4^3 z_2 + z_5^3$	−6	47	50
22	$\mathbb{C}^*(1,13,23,28,32)[97]$	$z_1^{97} + z_2^5 z_5 + z_3^3 z_4 + z_4^3 z_2 + z_5^3 z_1$	−6	47	50
23	$\mathbb{C}^*(3,5,8,24,35)[75]$	$z_1^{25} + z_2^{15} + z_3^9 z_1 + z_4^3 z_1 + z_5^2 z_2 + z_3^5 z_5$	−6	40	43
24	$\mathbb{C}^*(2,6,15,23,31)[77]$	$z_1^{31} z_3 + z_2^9 z_4 + z_4^2 z_5 + z_5^2 z_3 + z_4 z_1^{27} + z_5 z_1^{23}$	−6	37	40
25	$\mathbb{C}^*(3,6,8,23,37)[77]$	$z_1^{23} z_3 + z_2^9 z_4 + z_3^5 z_5 + z_4^3 z_3 + z_5^2 z_1 + z_4 z_1^{18}$	−6	37	40
26	$\mathbb{C}^*(3,9,17,22,24)[75]$	$z_1^{25} + z_2^8 z_1 + z_3^3 z_5 + z_4^3 z_2 + z_5^3 z_1 + z_5^2 z_2^3$	−6	35	38
27	$\mathbb{C}^*(3,4,14,21,21)[63]$	$z_1^{21} + z_2^{15} z_1 + z_3^3 z_4 + z_4^3 + z_5^3$	−6	32	35
28	$\mathbb{C}^*(3,7,18,26,27)[81]$	$z_1^{27} + z_2^9 z_3 + z_3^3 z_5 + z_4^3 z_1 + z_5^3$	−6	29	32
29	$\mathbb{C}^*(3,8,21,30,31)[93]$	$z_1^{31} + z_2^9 z_3 + z_3^3 z_4 + z_4^3 z_1 + z_5^3$	−6	29	32
30	$\mathbb{C}^*(3,8,13,15,36)[75]$	$z_1^{25} + z_2^9 z_1 + z_3^3 z_5 + z_4^5 + z_5^2 z_1 + z_5 z_2^3 z_4$	−6	29	32
31	$\mathbb{C}^*(3,12,15,16,17)[63]$	$z_1^{21} + z_2^5 z_1 + z_3^4 z_1 + z_4^3 z_3 + z_5^3 z_2 + z_3 z_2^4$	−6	29	32
32	$\mathbb{C}^*(3,4,12,17,19)[55]$	$z_1^{17} z_2 + z_3^3 z_5 + z_4^3 z_2 + z_5^2 z_4 + z_4^2 z_1^7 + z_5 z_1^{12} + z_5 z_2^9$	−6	26	29
33	$\mathbb{C}^*(3,5,8,9,20)[45]$	$z_1^{15} + z_2^9 + z_3^5 z_2 + z_4^5 + z_5^2 z_2 + z_5 z_3^2 z_4$	−6	23	26
34	$\mathbb{C}^*(5,12,13,15,20)[65]$	$z_1^{13} + z_2^5 z_1 + z_3^5 + z_4^4 z_1 + z_5^3 z_1 + z_4^3 z_5 + z_5^2 z_2 z_3$	−6	23	26
35	$\mathbb{C}^*(4,4,11,17,19)[55]$	$z_1^{11} z_3 + z_2^{11} z_3 + z_3^5 + z_4^3 z_1 + z_5^2 z_4 + z_1^9 z_5 + z_4^3 z_2 + z_5 z_2^9$	−6	21	24
36	$\mathbb{C}^*(4,4,7,13,15)[43]$	$z_1^9 z_3 + z_2^9 z_3 + z_3^4 z_5 + z_4^3 z_1 + z_5^2 z_4 + z_1^7 z_5 + z_4^3 z_2 + z_5 z_2^7$	−6	20	23
37	$\mathbb{C}^*(3,5,8,14,15)[45]$	$z_1^{15} + z_2^9 + z_3^5 z_2 + z_4^3 z_1 + z_5^3$	−6	20	23
38	$\mathbb{C}^*(4,5,7,8,19)[43]$	$z_1^9 z_3 + z_2^7 z_4 + z_3^5 z_4 + z_4^3 z_5 + z_5^2 z_2 + z_2^3 z_1^7 + z_5 z_1^6 + z_3^4 z_2^3$	−6	20	23
39	$\mathbb{C}^*(4,4,5,5,7)[25]$	$z_1^5 z_3 + z_2^5 z_3 + z_3^5 + z_4^5 + z_5^3 z_1 + z_1^5 z_4 + z_5^3 z_2$	−6	17	20
40	$\mathbb{C}^*(4,7,9,10,15)[45]$	$z_1^9 z_3 + z_2^5 z_4 + z_3^5 + z_4^3 z_5 + z_5^3 + z_2^3 z_1^6$	−6	13	16

Chapter 4

Calabi–Yau Manifolds

In this chapter the focus is on the third supporting column of the relation

$$(1 \cdot 16^3)_{A_2 \otimes E_7^3} \sim \mathbb{C}_{(2,2,2,3,3,3,3)}[9] \sim \begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_3 \end{array} \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}, \tag{4.0.0.1}$$

i.e. we wish to analyze weighted complete intersection manifold on their own terms with particular focus on the massless 4D spectrum encoded in their cohomology.

4.1 Weighted Complete Intersection Calabi–Yau Manifolds

The configurations which are the focus of this chapter can be succinctly summarized by the following object, which will be called a configuration

$$\begin{array}{c} \mathbb{P}_{(k_1^1,\ldots,k_{n_1+1}^1)} \\ \mathbb{P}_{(k_1^2,\ldots,k_{n_2+1}^2)} \\ \vdots \\ \mathbb{P}_{(k_1^F,\ldots,k_{n_F+1}^F)} \end{array} \begin{bmatrix} d_1^1 & d_2^1 & \ldots & d_N^1 \\ d_1^2 & d_2^2 & \ldots & d_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_1^F & d_2^F & \ldots & d_N^F \end{bmatrix} \tag{4.1.0.2}$$

where $N = \left(\sum_{i=1}^F n_i - 3\right)$ is the number of polynomials of F–degree $(d_i^1, \ldots, d_i^F)$.

The following paragraphs will elucidate some of the salient features of these manifolds, which describe intersections of hypersurfaces defined by the zero loci cut out by the polynomials $p_1, ..., p_N$ in the ambient space

$$\mathbb{P}_{(k_1^1,\ldots,k_{n_1+1}^1)} \times \cdots \times \mathbb{P}_{(k_1^F,\ldots,k_{n_F+1}^F)} \tag{4.1.0.3}$$

of F weighted projective $\mathbb{P}_n$s. A weighted $\mathbb{P}_n$ with weights $(k_1, \ldots, k_{n+1})$, which is denoted by $\mathbb{P}_{(k_1,\ldots,k_{n+1})}$, is most easily described in terms of $(n+1)$ complex 'homogeneous coordinates' $(z_1, z_2, ..., z_{n+1})$, not all zero, which are subject to the identification

$$(z_1, ..., z_{n+1}) \simeq (\lambda^{k_1} z_1, ..., \lambda^{k_{n+1}} z_{n+1}) \tag{4.1.0.4}$$

for all nonzero λ. Thus a weighted projective space is a generalization of ordinary projective space and $\mathbb{P}_n = \mathbb{P}_{(1,\ldots,1)}$ in this notation. In the following, when referring to a generic weighted $\mathbb{P}_n$, we shall frequently consider the weights to be understood and write $\mathbb{P}_n$ for $\mathbb{P}_{(k_1,k_2,\ldots,k_{n+1})}$.

An important property of these spaces is that weighted projective spaces have orbifold singularities owing to the identification (4.1.0.4), except for the case that the weights are all unity. This is most easily seen by setting $z_j = (\zeta_j)^{k_j}$ so that (4.1.0.4) becomes $(\zeta_1, ..., \zeta_5) \simeq \lambda(\zeta_1, ..., \zeta_5)$. However in virtue of the definition of ζ_i we must also identify $\zeta_j \simeq e^{2\pi i/k_j}\zeta_j$. So we see that

$$\mathbb{P}_{(k_1,k_2,\ldots,k_{n+1})} = \frac{\mathbb{P}_{n+1}}{\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_{n+1}}} \tag{4.1.0.5}$$

These identifications lead to singular sets.

A simple example is $\mathbb{P}_{(1,1,1,2,5)}$ which has weights that are mutually prime. If we take in turn $\lambda = -1$ and $\lambda = \alpha$ with α a fifth root of unity in (4.1.0.4) then we see that

$$\begin{aligned} (z_1, z_2, z_3, z_4, z_5) &\simeq (-z_1, -z_2, -z_3, z_4, -z_5) \\ &\simeq (\alpha z_1, \alpha z_2, \alpha z_3, \alpha^2 z_4, z_5). \end{aligned} \tag{4.1.0.6}$$

Consider now a neighborhood of the point $(0,0,0,1,0)$. We can take coordinates on the neighborhood by setting $\lambda = z_4^{-1/2}$ and writing $u_j = z_j/z_4^{1/2}$ for $j = 1,2,3$ and $u_5 = z_5/z_4^{5/2}$. We can therefore think of points in the neighborhood as corresponding to $(u_1, u_2, u_3, 1, u_5)$. However by the first of identifications (4.1.0.6) we have

$$(u_1, u_2, u_3, 1, u_5) \simeq (-u_1, -u_2, -u_3, 1, -u_5) \tag{4.1.0.7}$$

so that there is a $\mathbb{Z}_2$ identification on the space and hence on a neighborhood of the point and the action fixes $(0,0,0,1,0)$. In the same way there is a $\mathbb{Z}_5$ action which fixes $(0,0,0,0,1)$. In this case the singular set consists of points owing to the fact that the weights are mutually prime.

Consider now $\mathbb{P}_{(1,1,1,2,4)}$ then the identification is

$$(z_1, z_2, z_3, z_4, z_5) \simeq (\lambda z_1, \lambda z_2, \lambda z_3, \lambda^2 z_4, \lambda^4 z_5). \tag{4.1.0.8}$$

If we set $\lambda = z_5^{-1/4}$ and choose coordinates $w_j = z_j/z_5^{1/4}$, $w_4 = z_4/z_5^{1/2}$ then due to the freedom to set $\lambda = -1$ in (4.1.0.8) we have

$$(w_1, w_2, w_3, w_4, 1) \simeq (-w_1, -w_2, -w_3, w_4, 1) \tag{4.1.0.9}$$

and we see that we have a $\mathbb{Z}_2$ action which fixes the curve $(0,0,0,z_4,z_5)$. There is in addition a $\mathbb{Z}_4$ action with fixed point $(0,0,0,0,1)$ that lies within this curve. In general there is a fixed point for each weight that is greater than unity, a fixed curve for every pair of weights k_i, k_j whose greatest common factor, which we denote by (k_i,k_j), is greater than unity, a fixed surface for each triple with $(k_i,k_j,k_l) > 1$ and so on.

An assumption that we will make is to require the polynomials to be transverse, that is $p = 0$ and $dp_1 \wedge \cdots \wedge dp_N = 0$ have no common solution. Keeping this requirement but allowing for orbifold singularities provides at present for the most general framework that aside from technical difficulties can be dealt with in principle. The technical difficulties of implementing the transversality condition however are much more complicated than in the case of mere hypersurfaces, which has been solved in the previous chapter. They have, so far, prevented the complete construction of this class, even though a number of subclasses have been analyzed.

To illustrate a way of proceeding alternative to the one chosen in the previous chapter for the complete construction of the Landau–Ginzburg class we briefly present the explicit analysis started in [41] for hypersurfaces. The crucial point is that the weights of a given configuration may prevent all polynomials of the required degree from being transverse. This is in contradistinction to CICYs where every configuration admits a smooth representative (in fact almost all representatives are smooth [92]). To see this let

$$d_j := d - k_j \ , \ \ j = 1, \ldots, n+1 \tag{4.1.0.10}$$

and expand p in powers of z_1, say,

$$p = \sum_{r=0}^{n+1} C_r(z_m)\, z_1^r \ , \quad m \neq 1 \ . \tag{4.1.0.11}$$

It follows from Euler's Theorem (4.2.0.21) that $p = 0$ and $dp = 0$ if and only if there is a simultaneous solution of the equations

$$\frac{\partial p}{z_j} = 0 \ , \ \ j = 1, \ldots, n+1 \ . \tag{4.1.0.12}$$

Applying this to (4.1.0.11) we have

$$\begin{aligned} \frac{\partial p}{z_1} &= \sum_{r=1}^{n+1} r C_r(z_m)\, z_1^{r-1} \\ \frac{\partial p}{z_m} &= \sum_{r=0}^{n+1} \frac{\partial C_r}{z_m}\, z_1^r \end{aligned} \tag{4.1.0.13}$$

Now the degrees of the C_r are fixed by (4.1.0.11)

$$\begin{aligned} \deg\,(C_r) &= d - r k_1 = d_1 - (r-1)k_1 \ , \ \text{for } r \geq 1 \ , \\ \deg\left(\frac{\partial C_r}{z^m}\right) &= d_m - r k_1 \ . \end{aligned} \tag{4.1.0.14}$$

and a polynomial of negative degree is understood to vanish identically. Unless at least one of the coefficients in (4.1.0.13) has degree precisely zero then equations (4.1.0.13) will be satisfied for $z_1 = 1$ and $z_m = 0$ and p will not be transverse. This leads to the following neccessary condition on the weights if p is to be transverse:

For each i there must exist a j such that $k_i|d_j$ (k_i divides d_j).

This condition is quite restrictive. For example it is immediate that the only manifolds of the form $\mathbb{P}_{(1,1,1,1,k)}[k+4]$ that it allows are those for the four values $k = 1, 2, 3, 4$, and the assiduous reader may care to check that apart from these there only eleven other allowed cases of the form $\mathbb{P}_{(1,1,1,k,l)}[k+l+3]$.

This criterion however is not a sufficient condition. A counterexample being furnished by the configuration $\mathbb{P}_{(1,2,2,2,2)}[9]$ whose most general polynomial is of the form

$$p(z_1, z_2, z_3, z_4, z_5) = z_1 \tilde{p}(z_1, z_2, z_3, z_4, z_5). \tag{4.1.0.15}$$

Transversality for hypersurfaces requires that the equations $p = 0$ and $dp = 0$ have no common solution. Consider then a neighborhood U_5 say, on which we can take $z_5 = 1$. Taking the differential gives $dp = \tilde{p} dz_1 + z_1 d\tilde{p}$ which has a zero for $z_1 = 0$ and $\tilde{p} = 0$. These equations give two constraints in 4 variables and therefore always have a solution. Since the polynomial constraint is identically satisfied it follows that this configuration cannot admit a transverse realization.

Thus there are further criteria which must be satisfied in order for a polynomial to be transverse. A little thought shows that polynomials of Fermat type for which $k_i|d_i$ for each i *are* transverse. These are of the form

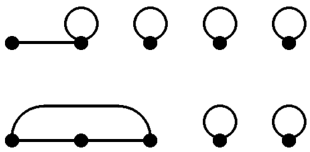
$$z_1^a + z_2^b + z_3^c + z_4^d + z_5^e \tag{4.1.0.16}$$

to which we have appended a diagrammatic shorthand.

There are other types which are also always transverse such as

$$z_1^a z_2 + z_2^b + z_3^c + z_4^d + z_5^e$$

$$z_1^a z_2 + z_2^b z_3 + z_3^c z_1 + z_4^d + z_5^e \tag{4.1.0.17}$$



Parts of these expressions corresponding to connected subdiagrams can also be combined together to produce yet other transverse polynomials such as

 or  (4.1.0.18)

Following in this constructive manner of explicitly producing polynomials that define transverse hypersurfaces 30 different types of five–dimensional catastrophes were found in [41] and all possible concrete realization were enumerated. A classification of all types was later achieved in [129].

4.2 Geometry of Weighted CICYs

Given the weights of the ambient space the requirement of a vanishing first Chern class puts constraints on the degree of the polynomial as in the case of CICYs. To derive the explicit condition we need to digress briefly on the Chern classes of the submanifold $\mathcal{M}$ defined by the equations $p_1 = \cdots = p_N = 0$.

Denote by $\mathcal{T}_{\mathbf{P}_n}$ and $\mathcal{T}_{\mathcal{M}}$ the tangent spaces to the $\mathbb{P}_n$s and $\mathcal{M}$ and by $\mathcal{N}$ the normal bundle of a hypersurface $\mathcal{M}$ in a weighted $\mathbb{P}_n$. Proceeding in a standard manner one finds

$$\mathcal{T}_{\mathbf{P}_n} = \mathcal{T}_{\mathcal{M}} \oplus \mathcal{N} \tag{4.2.0.19}$$

which allows us to calculate the Chern polynomial of $\mathcal{M}$ once those for $\mathbb{P}_n$ and $\mathcal{N}$ are known. The tangent space of $\mathbb{P}_n$ may be thought of as the set of vectors

$$\mathcal{V} = \mathcal{V}^j \frac{\partial}{\partial z^j} \tag{4.2.0.20}$$

which act on bona fide functions of the homogeneous coordinates, i.e. functions of degree zero. From Euler's theorem for homogeneous functions of degree m we have that

$$\sum_j k_j z^j \frac{\partial}{\partial z^j} f = mf. \tag{4.2.0.21}$$

So when acting on functions of degree zero we see that we may regard the $\mathcal{V}^j$ as independent apart from the identification $\mathcal{V}^j \simeq \mathcal{V}^j + k_j z^j$. This reduces the dimension of the space of $\mathcal{V}^j$ to n as is necessary. It follows that

$$\mathcal{T}_{\mathbf{P}_n} = \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_{n+1})/\mathcal{O} \tag{4.2.0.22}$$

where $\mathcal{O}(k)$ is a line bundle with $c_1 = kh$. It is also the line bundle whose fibre coordinates transform like a polynomial of degree k. $\mathcal{O}$ is the trivial bundle. Since $\mathcal{O}(k)$ is one dimensional we have $c(\mathcal{O}(k)) = 1 + kh$ and hence

$$c(\mathcal{T}_{\mathbf{P}_n}) = \prod_{j=1}^{n+1} (1 + k_j h) . \tag{4.2.0.23}$$

We can regard the defining polynomial p as a fibre coordinate on $\mathcal{N}$ so if p is of degree d we have $\mathcal{N} = \mathcal{O}(d)$ and $c(\mathcal{N}) = 1 + dh$. Hence

$$c(\mathcal{T}_{\mathcal{M}}) = \frac{\prod_{j=1}^{n+1} (1 + k_j h)}{(1 + dh)} \tag{4.2.0.24}$$

Applying these results to the configurations (4.1.0.2) one finds from

$$\oplus_{i=1}^{F} \mathcal{T}_{\mathbf{P}_{n_i}} = \mathcal{T}_{\mathcal{M}} \oplus \left(\oplus_{j=1}^{N} \mathcal{N}_i\right) \tag{4.2.0.25}$$

and therefore

$$c(\mathcal{T}_{\mathcal{M}}) = \frac{\prod_{i=1}^{F} \prod_{j=1}^{n_i+1} \left(1 + k_j^i h_i\right)}{\prod_{i=1}^{N} \left(1 + \sum_{j=1}^{F} d_i^j h_j\right)} . \tag{4.2.0.26}$$

Since the first Chern class of the i'th weighted projective space is given by $c_1 = (\sum_{j=1}^{n_i+1} k_j^i) h_i$ where $h_i \in H^2(\mathbb{P}_{(k_1^i, \ldots, k_{n_i+1}^i)}, \mathbb{Z})$ is the generator of the second cohomology group of the i'th weighted projective space it follows that the condition for the configurations (4.1.0.2) to describe Calabi–Yau manifolds the weights and the degree matrix elements have to satisfy

$$\sum_{l=1}^{n_i+1} k_l^i \quad = \quad \sum_{\alpha=1}^{N} d_\alpha^i, \qquad i = 1, \ldots, F. \tag{4.2.0.27}$$

For later reference we record some very useful expressions for the second and third Chern classes of weighted CICYs which can be easily derived from the total Chern class expression (4.2.0.26):

$$c_2 \quad = \quad \frac{1}{2} \left[\sum_{\alpha=1}^{N} \left(\sum_{i=1}^{F} d_\alpha^i h_i \right)^2 - \sum_{i=1}^{F} \sum_{j=1}^{n_i+1} \left(k_j^i h_i\right)^2 \right] \tag{4.2.0.28}$$

$$c_3 \quad = \quad -\frac{1}{3} \left[\sum_{\alpha=1}^{N} \left(\sum_{i=1}^{F} d_\alpha^i h_i \right)^3 - \sum_{i=1}^{F} \sum_{j=1}^{n_i+1} \left(k_j^i h_i\right)^3 \right] \tag{4.2.0.29}$$

These formulas can be easily derived by applying the theory of Chern characters and their relation to Chern classes [31] to the case of weighted CICYs.

The Euler number of weighted CICYs can be calculated by using Poincaré duality to lift the computation to the ambient space. The only further ingredient necessary to complete the computation is the normalization of the generators H_i:

$$\int_{\mathbf{P}_{(k_1^i,\ldots,k_{n_i+1}^i)}} H_i^{n_i} = \frac{1}{k_1^i k_2^i \cdots k_{n_i+1}^i} \tag{4.2.0.30}$$

By multiplying the third Chern class with the first Chern class of the normal bundles it is then easy to read off the coefficient of the monomial with the highest power in the generators. Dividing through with the product of the weights therefore gives the (singular) Euler characteristic χ_s.

To obtain the Euler number relevant for the string spectrum we have to take into accoung the contributions from the singularities and correct the singular Euler number by terms coming from the resolution of such

$$\chi = \chi_s - \sum_i \frac{\chi(S_i)}{n_i} + \sum_i n_i \chi(S_i) \tag{4.2.0.31}$$

where $\chi(S_i)$ is the Euler number of the singular set S_i.

In fact one can do better than just comparing the Euler number of the models. It is easy in principle if not in practice to compute for these models the Hodge diamond in a canonical manner. The way to proceed is to first compute the number of (1,1)–forms of the Calabi–Yau manifold and then use the Euler number to determine the only other independent Hodge number $h^{(2,1)} = \dim\, H^{(2,1)}(M, \mathbb{Z})$.

4.3 Spectrum of Weighted Calabi–Yau Hypersurfaces

Before discussing the more general case we illustrate the method in the simpler situation of Calabi–Yau manifolds embedded in one ambient factor. The first order business is to show that the singularities of the variety M corresponding to the locus $p = 0$ in a weighted $\mathbb{P}_4$ of a transverse polynomial p can always be resolved. It is known that such resolution is always possible when the singular subsets of M are of dimension 0 or 1. We shall show that these are in fact the only types of singularity that can arise if p is transverse. In other words that singular subsets of M cannot have dimension 2 or 3.

First we show that the embedding $\mathbb{P}_4$ cannot have singular sets of dimension greater than or equal to 3. Recall that the $\mathbb{P}_4$ has a singular point for each $k_i > 1$, a singular curve for each pair with $(k_i, k_j) > 1$, a singular subset of dimension 2 for each triple with $(k_i, k_j, k_l) > 1$ *etc.* In the definition of $\mathbb{P}_4$ we have assumed that the weights have no common factor so there are no singular sets of dimension 4. Consider the possibility of a singular set of dimension 3. Such a set would correspond to weights such that

$$(k_2, k_3, k_4, k_5) = m > 1 \quad \text{but} \quad m \not| k_1 \ .$$

Since m does not divide k_1 but does divide the other k's it cannot divide the degree $d = \sum_{j=1}^{5} k_j$. Every monomial of degree d must therefore contain at least one factor of z_1. Thus $p(z) = z_1 \tilde{p}(z)$ and so is not transverse. On the other hand singular sets of dimension 2 can occur and these will generically intersect M in subsets of dimension 1. It remains to show that a singular subset of dimension 2 cannot lie within M. To this end suppose there is a fixed point set of dimension 2 for the identification (4.1.0.4) and that this subset lies within M. We may choose coordinates (x, y, z) such that, locally, the fixed point set is $(x, y, 0)$. Suppose the identification is represented by a matrix A. Since A fixes $(x, y, 0)$ it has the form

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix} \ .$$

The three–form $dx \wedge dy \wedge dz$ transforms as

$$dx \wedge dy \wedge dz \longrightarrow \det A \, dx \wedge dy \wedge dz$$

we must have $\det A = 1$ and hence $c = 1$. Since A is contained in a finite group $A^n = 1$ for some n, however

$$A^n = \begin{pmatrix} 1 & 0 & na \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{pmatrix}$$

so a and b must, in fact, vanish. Thus the only identification that fixes a two–dimensional subset is the identity.

Turning to the spectrum, one of the (1,1)–forms clearly is present in all these manifolds: this is the pullback of the Kähler form of the weighted projective space. Whenever the CY is smooth this is the only one and therefore one finds $h^{(1,1)} = 1$ in those cases. If there are singularities however the blow–up procedure introduces additional (1,1)–forms. The situation for isolated singular points has been discussed in great detail by Roan and Yau [154]. Their discussion will not be duplicated here.

We now briefly describe the technique for blowing up curves [159]. The resolution of general singularities in three–dimensional Calabi–Yau manifolds is an unsolved difficult problem. In the case at hand however we are just looking at $\mathbb{Z}_n$–singularities which are much simpler. Furthermore it is possible to reduce the problem of blowing up curves in a three–dimensional manifold to a two–dimensional problem and make use of beautiful techniques that have been developed to deal with surface singularities.

Consider the action of a $\mathbb{Z}_n$ on a weighted CICY

$$\mathbb{Z}_n \ni g : \mathbb{P}_{(l_1,\cdots,l_5)}[d] \longrightarrow \mathbb{P}_{(l_1,\cdots,l_5)}[d] \tag{4.3.0.32}$$

which leaves a curve invariant. In the three–dimensional CY the normal bundle of this curve has fibres $\mathbb{C}_2$ and therefore this discrete group induces an action on the fibres that can be written as

$$\mathbb{Z}_n \ni g : \mathbb{C}_2 \longrightarrow \mathbb{C}_2 \quad : g = \begin{pmatrix} \alpha^{mq} & 0 \\ 0 & \alpha^m \end{pmatrix} \tag{4.3.0.33}$$

for some $0 \leq m < n$ where $\alpha^n = 1, \alpha \neq 1$ and n and q are coprime. The singularity is then defined to be of type (n, q). The action has an isolated singularity which needs to be resolved. To proceed we use some techniques pioneered by F.Hirzebruch in his dissertation [107] (for first applications of these techniques in string theory see [158, 159]). The essential point is that the singularity of $\mathbb{C}_2/\mathbb{Z}_n$ can be described as the singular set of the surface

$$\mathbb{C}_3 \supset S = \{(z_1, z_2, z_3) \in \mathbb{C}_3 | z_3^n = z_1 z_2^{n-q}\} \tag{4.3.0.34}$$

and therefore can be resolved by a Hirzebruch–Jung tree with a valuation that is determined by the type (n, q) through the method of continued fractions. The expansion of $\frac{n}{q}$ in a continued fraction

$$\begin{aligned} \frac{n}{q} &= [b_1, ..., b_s] \\ &= b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_s}}} \end{aligned} \tag{4.3.0.35}$$

determines the numbers $b_i > 1$ which specify uniquely the plumbing process which replaces the singularity. This process consists of patching together $s+1$ copies of $\mathbb{C}_2$ in a way that is determined by the numbers b_i. These numbers determine the maps that glue together the various patches. The magic of the construction is that if we just want to know the change of the spectrum that the blow–up procedure induces we don't need to go through the details of the glueing process every time. Instead the construction shows that the glueing of these $s + 1$ patches introduces s spheres, each of which can of course support an additional (1,1)–form. The Hirzebruch–Jung tree is a shorthand notation for the geometry of the blow–up in that it encodes not only the number of spheres but also how these spheres intersect and what their intersection numbers are (basically the b_i's). In our case we only have to consider $\mathbb{Z}_n$–blow ups for which the Hirzebruch–Jung tree is just an SU$(s + 1)$ Dynkin diagram with the values of the b_i's attached to the nodes. Each node of the diagram corresponds to a sphere and the diagram encodes that in the case of $\mathbb{Z}_n$ blow–ups only neighboring spheres intersect.

Some examples should make it clear how the computation of the cohomology proceeds in general. Consider e.g. one of the manifolds whose exact model is known:

$$\left(2^3 \cdot 3 \cdot 18\right)_{A_3^3 \otimes A_4 \otimes A_{19}} \sim \mathbb{P}_{(1,4,5,5,5)}[20] \tag{4.3.0.36}$$

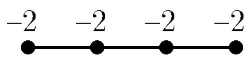
whose singularity set consists only of a $\mathbb{Z}_5$–curve $C = \mathbb{P}_2[4]$ with Euler number $\chi_C = -4$. With $\chi_s = -100\frac{4}{5}$ one finds for the Euler number of the smoothed out CY

$$\chi = \chi_s + \sum \left(\chi(C_i)n_i - \frac{\chi(C_i)}{n_i}\right) = -120. \tag{4.3.0.37}$$

In this formula χ_s is the Euler number of the singular space and $\chi(C_i)$ is the Euler numbers of the fixed point set C_i of the discrete group $\mathbb{Z}_{n_i}$. The action of the $\mathbb{Z}_5$ on the normal bundle is of type $(n,q) = (5,4)$ and therefore the singularity of $\mathbb{C}_2/\mathbb{Z}_4$ is equivalent to the singular set of the surface

$$\mathbb{C}_3 \supset S = \{(z_1, z_2, z_3) \in \mathbb{C}_3 | z_3^5 = z_1 z_2\}. \tag{4.3.0.38}$$

The resolution is specified by the Hirzebruch–Jung tree



which in turn is completely specified by the continued fraction

$$\frac{5}{4} = [2,2,2,2] = 2 - \frac{1}{2 - \frac{1}{2-\frac{1}{2}}}. \tag{4.3.0.39}$$

The blow–up therefore is of type $S^2 \vee S^2 \vee S^2 \vee S^2$ and therefore contributes four generators to H^2. Together with the Kähler form this results in a 5–dimensional second cohomology group H^2, in agreement with the exact result. Below we will discuss some other examples with non–diagonal invariants.

Putting the results of Roan and Yau for the blow–up of points together with the prescription just reviewed allows for the computation of the spectrum of general Calabi–Yau manifolds. The results of the calculations for the manifolds agree with those of the exact models (when these are known), lending further support for the identification proposed in [144, 176, 98] in terms of sheer numerical agreement.

It must be emphasized however that the above techniques to compute are utterly awkward as they entail a detailed analysis of the singular sets S_i for each manifold. These depend on the divisibility property of the weights as well as the type of the individual catastrophe involved and therefore need to be determined on a case by case basis which is not easily automated. This should be compared with the Landau–Ginzburg formulation of the same computation described in the previous chapter, which is purely combinatorical in character and therefore is much easier to use. Thus whenever possible to choose then the LG theories provide the framework in which it can actually be done without much pain.

4.4 Higher Codimension Weighted CICYs

Tensor models with equal or less than five scaling fields provide the majority of cases in the class of Gepner models. However it does happen that there are more than five superfields that define the Landau–Ginzburg action – namely whenever there are more than two non–diagonal invariants for the models with $r = 4$ or at least one non–diagonal invariant in those with $r = 5$. A subclass of these cases behave like the diagonal models with $r > 5$: these spaces are described by manifolds that are products of lower dimensional Calabi–Yau spaces (in some cases with an additional modding), even though there are non–diagonal invariants involved now.

Consider e.g. the model

$$(2^2 \cdot 3^2 \cdot 8)_{A_3^2 \otimes A_4^2 \otimes D_6}. \tag{4.4.0.40}$$

The computation of the U(1)–charge for this model gives $\frac{3}{2}$ and therefore we need to add an additional trivial factor. The Landau–Ginzburg potential then can be split into two pieces of integral charge

$$W = \left(\Phi_1^4 + \Phi_2^4 + \Phi_3^2\right) + \left(\Psi_1^5 + \Psi_2^5 + \Psi_3^5 + \Psi_3 \Psi_4^2\right). \tag{4.4.0.41}$$

By redefining the coordinates appropriately and integrating out some of the variables one sees that the manifold described is given by

$$M = \mathbb{P}_{(1,1,2)}[4] \times \mathbb{P}_{(1,1,1,2)}[5] \tag{4.4.0.42}$$

i.e. it is the product of a torus and a K3 surface. This manifold has clearly Euler number zero and $h^{(1,1)} = 21$ matching the exact result. In fact the exact analogs of many of the various representations of $\mathrm{T}_1 \times$ K3 as manifolds of the type

$$M = \mathbb{P}_{(k_1,k_2,k_3)}[K] \times \mathbb{P}_{(l_1,l_2,l_3,l_4)}[L] \tag{4.4.0.43}$$

with $K = \sum k_i$ and $L = \sum l_i$ can be found in the list. Of course not all possible representations can appear since the ADE–singularities do not describe the most general type of catastrophe in 4 variables, i.e. there are representations of K3 not covered by our models.

There are also models whose superpotential can be split in the above way but which need additonal moddings as described in [98]. Consider e.g. the model

$$(2^3 \cdot 3 \cdot 18)_{A_3^3 \otimes A_4 \otimes D_{11}} \tag{4.4.0.44}$$

Computation of the U(1)–charge for this model gives again $\frac{3}{2}$ and therefore we need to add an additional trivial factor in this case as well. The Landau–Ginzburg potential

$$W = \Phi_1^4 + \Phi_2^4 + \Phi_3^4 + \Phi_4^5 + \Phi_5\Phi_6^2 + \Phi_5^{10} + \Phi_7^2 \tag{4.4.0.45}$$

then decomposes into two polynomials with charge 1 and therefore the manifold naively is described by

$$M = \mathbb{P}_{(1,1,2)}[4] \times \mathbb{P}_{(2,4,5,9)}[20] \tag{4.4.0.46}$$

i.e. it is again the product of a torus and a K3 surface (cf. eq. (78)). This manifold however has Euler number zero whereas the exact model has Euler number –24, i.e. one needs some additonal moddings.

There exist also models with U(1)–charge greater than 1 where the LG–potential cannot be decomposed into two polynomials of unit charge. In those cases we should expect the manifold to be embedded in an ambient product of weighted projective spaces in such a way that it does not decompose into two factors.

An example is furnished by the model

$$(1 \cdot 2 \cdot 4 \cdot 6^2)_{A_2 \otimes A_3 \otimes A_5 \otimes D_5^2} \tag{4.4.0.47}$$

(which appears as $(4 \cdot 6^2 \cdot 10)_{AD^2E}$ in our list) which has 55 generations and 7 anti–generations. The U(1)–charge of this model is 2 and since we have seven coordinates we expect it to be described by two polynomials in a product space. The fields (Φ_i, Ψ_i) in the Landau–Ginzburg potential

$$W = \Phi_1\Psi_1^2 + \Phi_1^4 + \Phi_2\Psi_2^2 + \Phi_2^4 + \Phi_3^3 + \Phi_4^4 + \Phi_5^6 \tag{4.4.0.48}$$

have charges $(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{3}{8}, \frac{3}{8})$ if we consider the potential as a polynomial with degree 1. The LG–potential does not split into two integrally charged pieces. This means that the manifold will not be a simple product but an irreducible space. The idea then is simply to consider polynomials of multidegree according to monomials that consist of several variables. Consider e.g. the class of manifolds described by the transverse configurations in

$$\begin{matrix} \mathbb{P}_{(3,3,4,3,2)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} 3 & 12 \\ 2 & 0 \end{bmatrix} \tag{4.4.0.49}$$

Interpreting the Φ_i's as coordinates on the weighted $\mathbb{P}_4$ and the Ψ_i's as coordinates on the $\mathbb{P}_1$ it can be easily shown that the polynomials

$$\begin{aligned} p^1 &= \Phi_1\Psi_1^2 + \Phi_2\Psi_2^2 \\ p^2 &= \Phi_1^4 + \Phi_2^4 + \Phi_3^3 + \Phi_4^4 + \Phi_5^6 \end{aligned} \tag{4.4.0.50}$$

describe a transverse intersection in the ambient space (77). To check this we first compute the Euler number. The singular weighted CICY has Euler number $\chi_s = -62\frac{1}{3}$. The weighted projective space has three types of singularities: a $\mathbb{Z}_2$–singular curve, a $\mathbb{Z}_3$–singular surface and a $\mathbb{Z}_4$–singular point. The singular point does not descend down to the CY manifold. The $\mathbb{Z}_2$–singular curve however induces on the Calabi–Yau manifold three singular curves $C = \mathbb{P}_1[3] \times \mathbb{P}_1$ with Euler number $\chi_C = 6$. The $\mathbb{Z}_2$–singular surface induces the singular curve $C' = \begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ with Euler number $\chi_{C'} = -16$ and therefore the resolution of the singularities leads to a smooth manifold with Euler characteristic

$$\chi = -62\frac{1}{3} - \frac{\chi_C}{2} + 2\chi_C - \frac{\chi_{C'}}{3} + 3\chi_{C'} - = -96 \tag{4.4.0.51}$$

which agrees again with the exact result. Furthermore we can check the spectrum. Applying the techniques described above for the blow–up of these curves one finds that the $\mathbb{Z}_2$–curve introduces 3 new (1,1)–forms whereas the $\mathbb{Z}_3$–curve contributes 2 additional generators to H^2. Adding the two generators coming from the pullback of the Kähler forms of the ambient space gives the total dimension seven for the second cohomology group, as found in the exact model.

The manifold above is an example of a whole class of irreducible (non–product) weighted CICYs whose exact models can be found in a similar fashion [159]. The models are embedded in a product of two ambient spaces as

$$\begin{array}{c} \mathbb{P}_{(l_1,l_2,l_3,l_4,l_5)} \\ \mathbb{P}_{(1,1)} \end{array} \begin{bmatrix} l_1 & l \\ 2 & 0 \end{bmatrix} \tag{4.4.0.52}$$

where $l = l_2 + l_3 + l_4 + l_5$. In [159] it was assumed that the $\mathbb{N} \ni l_i > 0$ are such that $l/l_i \in \mathbb{N}$ for $i = 2, \ldots, 5$ and indeed in the example above we encountered such a situation. Obviously the manifolds defined in (80) have vanishing first Chern class and complex dimension three and therefore define CYs of the type we are interested in. Using again the techniques of section 3 one finds for the Euler numbers of these configurations

$$\chi_s = -\frac{2l}{l_1 \cdots l_5}\left[l_1^3 + \sum_{i=2}^{5} l_i^2 \hat{l}_i + \sum_{\substack{i<j<k \\ i,j,k \in \{2,..,5\}}} 2 l_i l_j l_k \right] \tag{4.4.0.53}$$

where $\hat{l}_i = l - l_i$. As in the case of one weighted projective space this formula does not take blow–ups into account. Now set $l_1 = l_2$. To each of these manifolds one can associate an exactly solvable model in the following way. Starting from the weights l_i of the weighted projective space we define levels as follows

$$k_1 = k_2 = \frac{2l}{l_1} - 2, \quad k_i = \frac{l}{l_i} - 2, \quad i = 3,4,5 \tag{4.4.0.54}$$

For the central charge of this model one finds

$$c = 2\frac{3k_1}{k_1+2} + \sum_{i=3}^{5} \frac{3k_i}{k_i+2} = 9 \tag{4.4.0.55}$$

as is necessary for an allowed configuration.The point in the moduli space of each of these configurations which we wish to identify with this exact model is defined by the following choice of polynomials

$$p_1 = y_1 z_1^2 + y_2 z_2^2 \tag{4.4.0.56}$$

$$p_2 = \sum_{i=1}^{5} y_i^{l/l_i} \tag{4.4.0.57}$$

where we denote the coordinates of the first factor by y_i and the coordinates of the second factor by z_j. From this it is clear that we should identify these weighted CICYs with the models $k_1^2 \cdot k_3 \cdot k_4 \cdot k_5$ in which the left and right sectors are glued together with the affine invariants $D_{\frac{l}{l_i}+1}$ in the first two factors and the diagonal affine invariant in the last factors. In these cases we understand the relation

$$\begin{matrix} \mathbb{P}_{(l_1,l_1,l_3,l_4,l_5)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} l_1 & l \\ 2 & 0 \end{bmatrix} \sim \left(2\left(\frac{l}{l_1} - 1\right)\right)^2 \cdot \prod_{i=3}^{5} \left(\frac{l}{l_i} - 2\right) \tag{4.4.0.58}$$

to hold as an equivalence between the exact model and the CY with all the singularities *resolved*, i.e. the blowing up procedure is implicitly assumed whenever necessary.

The assumption made in [159], restricting the l_i such that $l/l_i \in \mathbb{N}$ for $i = 2, ..., 5$ is however not essential for this construction but simply allows a more canonical formulation. It will not be satisfied whenever there are more than two non–diagonal invariants involved. An example that is a bit involved but rather instructive is furnished by the model

$$(46^2 \cdot 6 \cdot 1)_{D_{25}^2 \otimes D_5 \otimes A_2} \tag{4.4.0.59}$$

whose LG–potential

$$W = \Phi_1 \Psi_1^2 + \Phi_1^{24} + \Phi_2 \Psi_2^2 + \Phi_2^{24} + \Phi_3 \Phi_4^2 + \Phi_3^4 + \Phi_5^3 \tag{4.4.0.60}$$

suggests that we should split it up into two polynomials describing the Calabi–Yau space embedded in the ambient space

$$\begin{matrix} \mathbb{P}_{(1,1,6,8,9)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} 1 & 24 \\ 2 & 0 \end{bmatrix} \tag{4.4.0.61}$$

by the defining equations

$$\begin{aligned} p^1 &= x_1 y_1^2 + x_2 y_2^3 \\ p^2 &= x_1^{24} + x_2^{24} + x_3^4 + x_3 x_4^2 + x_5^3. \end{aligned} \qquad (4.4.0.62)$$

where the $\{y_i\}$ are the homogeneous coordinates of the $\mathbb{P}_1$ and the $\{x_i\}$ are the coordinates of the weighted $\mathbb{P}_4$. This space is transverse but has again orbifold singularities. The singular space has Euler number $\chi_s = -458\frac{1}{9}$. The singularities on the ambient space are associated with the discrete groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_8$, and $\mathbb{Z}_9$. The $\mathbb{Z}_2$ fixed point set is just a sphere, with Euler number 2. The $\mathbb{Z}_3$ fixed point set is given by $\mathbb{P}_{(2,3)}[8] \times \mathbb{P}_1$ and therefore seems to consist of two spheres. Note however that one of the points of $\mathbb{P}_{(2,3)}[8]$ is not a $\mathbb{Z}_3$ fixed point at all but a $\mathbb{Z}_9$ fixed point and therefore should not be counted at this stage, else we would overcount. The $\mathbb{Z}_8$ does not induce any fixed points on the Calabi–Yau manifold, so we are left the $\mathbb{Z}_9$ fixed point set which contributes, as just mentioned, another sphere. Plugging all this into the Euler number formula gives $\chi = -432$ for the blown–up manifold. This agrees with the exact result. Furthermore we can determine the spectrum by adding up all the various contributions to the second cohomology: aside from the two Kähler forms there are eight contributions of the $\mathbb{Z}_9$ blow–up, two from the $\mathbb{Z}_3$ resolution and finally one last one coming from the $\mathbb{Z}_2$ fixed points making up a grand total of 13, in agreement of course with the exact result.

As a comment on the significance of this class we remark that as explained in [159], these manifolds actually describe generalized splits of Calabi–Yau manifolds embedded in one ambient factor. Originally *splitting* was introduced in [31] as a technique to construct all complete intersection Calabi–Yau manifolds embedded in products of complex projective spaces. It involves deforming a given smooth Calabi–Yau manifold to a singular limit and then smoothing it out again in a different way. In the process the Euler number and the number of generations and anti–generations will be changed, i.e. the two manifolds related by this process are topologically distinct.

It was remarked in ref. [38] that the numerology of this construction is very reminiscent to phase transition and it has been speculated by these authors that splitting and contraction (its inverse process) do in fact describe possible phase transition. (A possible objection to this interpretation was removed in [32]).

In order to shed some light on this interesting suggestion it would be very helpful to be able to recast this construction in terms of the Landau–Ginzburg formulation or the language of exactly solvable models. Unfortunately the Calabi–Yau manifold examples that have been discussed so far do not lend themselves to such a formulation as neither their LG–formulation nor their exactly solvable counterparts are known. The models introduced in [159] however do not suffer from this problem: their LG–potentials are of standard type and the structure of their exactly solvable counterparts is known in great detail. More precisely these relations can be summarized in the following diagram

$$\begin{array}{ccc}
\mathbb{P}_{(\frac{l_1}{2},\frac{l_1}{2},l_3,l_4,l_5)}[l] & \longrightarrow & \left(2\left(\frac{l}{l_1}-1\right)\right)^2_{A^2_{\frac{2l}{l_1}-1}} \cdot \prod_{i=3}^5 \left(\frac{l}{l_i}-2\right)_{A_{\frac{l}{l_i}-1}} \\
\downarrow & & \downarrow \\
\begin{matrix}\mathbb{P}_{(l_1,l_1,l_2,l_3,l_4)} \\ \mathbb{P}_{(1,1)}\end{matrix} \begin{bmatrix} l_1 & l \\ 2 & 0 \end{bmatrix} & \longrightarrow & \left(2\left(\frac{l}{l_1}-1\right)\right)^2_{D^2_{\frac{l}{l_1}+1}} \cdot \prod_{i=3}^5 \left(\frac{l}{l_i}-2\right)_{A_{\frac{l}{l_i}-1}}
\end{array}$$

where the analogous step taken in the conformal field theory has been indicated as well. The horizontal arrows describe the LG–type identifications of exact models and weighted CICYs. The vertical arrows describe the replacement of the diagonal invariants by the D–invariants in the first two factors.

In fact there is a deeper geometrical link between these two models. Consider the zero–set of the two polynomials p_1, p_2 in more detail. For a generic point in $\mathbb{P}_{(l_1,l_1,l_3,l_4,l_5)}$ the polynomial constraint $p_1 = 0$ has two solutions on $\mathbb{P}_1$, i.e. generically the manifold is a double cover of $\mathbb{P}_{(l_1,l_1,l_3,l_4,l_5)}[l]$. If however either $y_1 = 0$ or $y_2 = 0$ then there is just one solution. Furthermore if $y_1 = 0 = y_2$ there is one full $\mathbb{P}_1$ for each point on the curve $\mathbb{P}_{(l_3,l_4,l_5)}$. The curve described by setting $y_1 = 0 = y_2$ however is precisely the curve that can be found in the diagonal manifold by setting the first two coordinates equal to zero.

Consider for example the manifold $\mathbb{P}_{(1,1,2,2,2)}[8]$ whose split is $\begin{matrix}\mathbb{P}_4 \\ \mathbb{P}_1\end{matrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$. These two manifolds have the same cohomology $(h^{(1,1)}, h^{(2,1)}) = (2, 86)$. The split manifold obviously has two (1,1)–forms that are restrictions of the two Kähler forms of the ambient factor. A more detailed picture however is revealed by repeating the analysis above. The split manifold is a double cover over $\mathbb{P}_4[4]$ with branch locus $\mathbb{P}_4[4,2]$ which contains the singular curve $\mathbb{P}_2[4]$ over each point of which there sits a full $\mathbb{P}_1$. It is precisely this $\mathbb{P}_1$ which supports the second Kähler form. In the diagonal manifold the origin of the second (1,1)–form is of course related to the blow–up. The singular set here is a $\mathbb{Z}_2$–curve which is precisely the curve $\mathbb{P}_2[4]$ found in the split manifold. The blow–up of this curve introduces one additional generator of the second cohomology.

Thus these manifolds are clearly related and it would be very interesting if a rationale for this relation on the level of conformal field theory or LG theory could be found in terms, say, of an RG flow from a diagonal SU(2) model to a model with a D–invariant. A conformal field theoretic investigation of the splitting and contraction process in this setting would very likely help to elucidate the physical interpretion of this geometrical construction.

To end the discussion of the relation between exactly solvable models and CY manifolds we must mention the very few models with U(1)–charge $Q = 3$ which lead to manifold defined by three equations. Most of these models in fact can be reduced to the previous discussion as the LG–potential splits into integrally charged sub–polynomials, i.e. these manifols are defined as products of lower dimensional Calabi–Yau manifolds (modulo moddings). There are only two models $((6^4)_{D^4}$ and $(4 \cdot 6^2 \cdot 10)_{D^4})$ which cannot described this way.

Chapter 5

Mirror Symmetry

As yet there exists no proof of mirror symmetry. But there exist constructions and computations which demonstrate beyond doubt that mirror symmetry exists. The focus in this chapter will be precisely this: to construct mirror pairs. The idea is to find such a construction which can be applied to the class of Landau–Ginzburg theories (and their geometrical counterpart when those exist) *and* which specializes to conformal field theory isomorphism for those theories for which the exactly solvable counterpart is known as well.

A string vacuum for which all three formulations, discussed in the first chapters, are known *and* for which mirror spectra appear in the the list of exactly solvable models, is furnished by

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A_2 \otimes A_7 \otimes A_{32} \otimes A_{87}} \sim \mathbb{C}_{(3,8,33,88,132)}[264] \sim \mathbb{P}_{(3,8,33,88,132)}[264], \tag{5.0.0.1}$$

leading to the spectrum $(\chi, h^{(1,1)}, \gamma) = (-48, 57, 81)$. The mirror spectrum for this theory does appear in the list of ADE models [140, 83] and is associated, among others, to the two models

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A_2 \otimes D_5 \otimes A_{32} \otimes A_{87}} \sim \mathbb{C}_{(3,8,66,88,99)}[264] \sim \mathbb{P}_{(3,8,66,88,99)}[264], \tag{5.0.0.2}$$

and

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A_2 \otimes A_7 \otimes A_{32} \otimes D_{45}} \sim \mathbb{C}_{(6,8,33,88,129)}[264] \sim \mathbb{P}_{(6,8,33,88,129)}[264]. \tag{5.0.0.3}$$

The *only* difference in these theories is that at some level k (here 6 or 86) the diagonal affine invariant is replaced by the D–invariant. The goal in this chapter is to understand and generalize whatever it is that has happened here.

5.1 The Basic Isomorphism

Proceeding in a somewhat Gaussian way, we simply remark that the essential ingredient in our mirror construction is the generalization of a well known low temperature – high temperature duality of the Ising model. At the critical point this duality translates into an isomorphism between the representation of the Ising model in terms of its order parameters and the representation in terms of the disorder parameters.

The generalization of this relation to the class of Landau–Ginzburg theories containing and extending the ADE theories is what we call the basic isomorphism

$$\begin{aligned} &\mathbb{C}_{\left(\frac{b}{g_{ab}},\frac{a}{g_{ab}}\right)}\left[\frac{ab}{g_{ab}}\right] \ni \left\{z_1^a + z_2^b = 0\right\} \Big/ \mathbb{Z}_b : [(b-1) \quad 1] \\ \sim\ &\mathbb{C}_{\left(\frac{b^2}{h_{ab}},\frac{a(b-1)-b}{h_{ab}}\right)}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \left\{y_1^{a(b-1)/b} + y_1 y_2^b = 0\right\} \Big/ \mathbb{Z}_{b-1} : [1 \quad (b-2)] \end{aligned} \tag{5.1.0.4}$$

induced by the fractional transformations

$$\begin{aligned} z_1 &= y_1^{1-\frac{1}{b}}, & y_1 &= z_1^{\frac{b}{b-1}} \\ z_2 &= y_1^{\frac{1}{b}} y_2, & y_2 &= \frac{z_2}{z_1^{\frac{1}{b-1}}} \end{aligned}$$

in the path integral of the theory. Here g_{ab} is the greatest common divisor of a and b and h_{ab} is the greatest common divisor of b^2 and $(ab - a - b)$. The action of a cyclic group $\mathbb{Z}_b$ of order b denoted by $[m \quad n]$ indicates that the symmetry acts like $(z_1, z_2) \mapsto (\alpha^m z_1, \alpha^n z_2)$ where α is the b^{th} root of unity.

For $a = 2k$ and $b = 2$ this isomorphism reduces to the well known isomorphism of the $\mathcal{A}_k/\mathbb{Z}_2$ orbifold and the $\mathcal{D}_{\frac{k}{2}+2}$ theory. It is important to note however that *only* for the values $a = 2k$ and $b = 2$ does this isomorphism lead to a complete intersection description of the orbifold theory. In general the identity relates different orbifolds. Nevertheless the isomorphism (5.1.0.4) is extremely useful for the analysis of the complete intersections constructed in [41, 122, 130], the reason being that in many examples the required modding on the rhs becomes part of the projective equivalence of the transformed space and hence trivial and can be ignored.

5.2 From A–Models to D–Models via Orbit Construction

For the minimal models it is easy to provide an explicit counting of the states provided by the two 'mirror theories'. It is well known from the ADE–classification of partition functions of ($N=2$)minimal models that the D–type models are equivalent to A–type models modded out by a $\mathbb{Z}_2$ discrete symmetry. As a consequence subsets of vacua of the Heterotic String involving the $N=2$ minimal series are also related via orbifold constructions In terms of the corresponding Landau-Ginzburg potentials of the exact models this can be seen as follows. The potential corresponding to a model at level k with a diagonal affine invariant is [115]

$$\mathrm{A}_{k+1} \quad \sim \quad \Phi_1^{k+2}+\Phi_2^2 \tag{5.2.0.5}$$

where we have added, with hindsight, a trivial factor. The LG potential corresponding to the exact model at the same level [4] but with the D–invariant is described by [144, 176]

$$\mathrm{D}_{\frac{k}{2}+2} \quad \sim \quad \tilde{\Phi}_1^{(k+2)/2}+\tilde{\Phi}_1\tilde{\Phi}_2^2. \tag{5.2.0.6}$$

By defining the transformation of the scaling variables

$$\Phi_1=\tilde{\Phi}_1^{1/2} \quad \text{and} \quad \Phi_2=\tilde{\Phi}_1^{1/2}\tilde{\Phi}_2 \tag{5.2.0.7}$$

one can map the diagonal theory of (5.2.0.5) into the nondiagonal model of (5.2.0.6). Moreover this transformation has a constant Jacobian and therefore one might naively expect that they are in fact equivalent descriptions of a given theory since their path integrals are equivalent. It is clear however that this is not the case as the two theories have a different spectrum. The local ring of the diagonal theory consists of states

$$\mathcal{R}_{\mathrm{A}_{k+1}}=\{1,\Phi_1,\Phi_1^2,\cdots,\Phi_1^k\} \tag{5.2.0.8}$$

i.e. the theory has $k+1$ states. The D–theory however has only $\frac{k}{2}+2$ states in its spectrum

$$\mathcal{R}_{\mathrm{D}_{\frac{k}{2}+2}}=\{1,\tilde{\Phi}_2,\tilde{\Phi}_1,\cdots,\tilde{\Phi}_1^{\frac{k}{2}}\}. \tag{5.2.0.9}$$

Hence these two theories are not the same. The resolution of this puzzle comes from the fact that the transformation of (5.2.0.7 is not a coordinate transformation since it is not a bijection but is 2–1 as it stands. To make it 1–1 we should identify

$$\Phi_i \sim -\Phi_i, \quad i=1,2. \tag{5.2.0.10}$$

Of the $k+1$ states of A_{k+1}–models $\frac{k}{2}+1$ are invariant with respect to the action of this $\mathbb{Z}_2$. By including the one twisted state we find precisely the states of the non–diagonal D–theory.

[4] The D–models only exist for even level $k=2n$.

This construction can immediately be applied to string compactification proper. It may seem unlikely at first that the simple modding of a $\mathbb{Z}_2$ should account for the different behaviour that the various tensor models exhibit under the exchange of a diagonal invariant by a D–invariant. A quick look at the results of refs. [140, 83] shows that this exchange generically changes the spectrum in some arbitrary way without any obvious systematics. In some special cases the spectrum is flipped, exchanging generations and anti–generations while in other models the spectrum does not change at all.

The reason for this erratic behaviour is not difficult to understand and can be traced to the fact that the other coordinates involved in the construction determine the fixed point structure of the discrete group in an essential way. Consider the following example where the exchange of the affine invariant does not change the spectrum

$$\mathbb{P}_{(1,7,2,2,2)}[14] \quad \ni \quad (12\cdot 5^3)_{A_{13}\otimes A_6^3} \quad \longrightarrow \quad (12\cdot 5^3)_{D_8\otimes A_6^3} \quad \in \quad \mathbb{P}_{(1,3,1,1,1)}[7]. \tag{5.2.0.11}$$

where a trivial factor has been added with hindsight. The reason that the spectrum does not change after the replacement of the diagonal affine invariant by the D–invariant is rather obscure from the point of view of the conformal field theory described by the tensor model $(12\cdot 5^3)$ with different affine invariants but can be understood easily from the orbifold point of view. In the manifold picture it simply follows from the fact that the $\mathbb{Z}_2$–action $[1,1,0,0,0]$, generated by

$$(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5) \longrightarrow (\alpha\Phi_1,\alpha\Phi_2,\Phi_3,\Phi_4,\Phi_5), \tag{5.2.0.12}$$

(where $\alpha^2=1, \alpha\neq 1$) is not an action at all on the Calabi–Yau manifold embedded in the ambient weighted projective space but it is part of the projective equivalence transformation because we have

$$\mathbb{P}_{(1,7,2,2,2)} = \mathbb{C}_5/\sim \tag{5.2.0.13}$$

with $(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5) \sim (\lambda\Phi_1,\lambda^7\Phi_2,\lambda^2\Phi_3,\lambda^2\Phi_4,\lambda^2\Phi_5)$, where $\lambda\in\mathbb{C}^*$. Therefore the $\mathbb{Z}_2$ does not transform the projective coordinates at all but is trivial. In the conformal field theory picture this translates into the fact that the action is actually part of the modding that has to be done to implement the GSO projection. This example then shows that not all our models are independent but that some models appear in different representations.

A more complicated example is furnished by the pair of minimal models

$$\mathbb{P}_{(1,5,1,1,2)}[10]^1_{-288} \quad = \quad (8^3\cdot 3)_{A_9^3\otimes A_4} \longrightarrow (8^3\cdot 3)_{D_6\otimes A_9^2\otimes A_4} \quad = \quad \mathbb{P}_{(2,4,1,1,2)}[10]^3_{-192}. \tag{5.2.0.14}$$

The superscript denotes the dimension of the second cohomology group $h^{(1,1)} = \dim\, \mathrm{H}^{(1,1)}(\mathcal{M})$ whereas the subscript denotes the Euler number. It is again not obvious from the point of view of the affine invariants involved why the spectrum changes the way it does but the result is clear from the orbifold construction. Defining the $\mathbb{Z}_2$–action as $[1,1,0,0,0]$ it follows that in this case the map is nontrivial. Its fixed point set consists of two curves $C_1 = \mathbb{P}_{(1,1,2)}[10]$ with Euler number $\chi_{C_1} = -30$ and $C_2 = \mathbb{P}_{(1,5,2)}[10]$ with $\chi_{C_2} = -2$. The Euler number of the resolved orbit manifold is therefore

$$\chi = -\frac{288}{2} - \frac{(-30-2)}{2} + 2(-30-2) = -192. \tag{5.2.0.15}$$

The cohomology can be computed as well. Given the Euler number we only need to compute either the second cohomology group or the number of (2,1)–forms. Using the results of [159] it is clear that each of the curves contributes one additional generator to the second cohomology, i.e. for the resolved manifold we have $h^{(1,1)} = 3$ and therefore the result of (5.2.0.15).

As an aside it should be remarked that these simple identifications of a class of Landau-Ginzburg potentials with orbifolds of other potentials can be considered as a generalization of the strange duality known from the exceptional singularities of modality one [3]. One of the 'strangely dual pairs' appearing in the classification of Landau–Ginzburg potentials in 3 variables with an isolated singularity at the origin is described by the two polynomials

$$K_{14} \quad : \quad \mathbb{P}_{(3,12,8)}[24] \ \ni \ \{\Phi_1^8 + \Phi_2^2 + \Phi_3^3 = 0\} \tag{5.2.0.16}$$
$$Q_{10} \quad : \quad \mathbb{P}_{(6,9,8)}[24] \ \ni \ \{\tilde{\Phi}_1^4 + \tilde{\Phi}_1\tilde{\Phi}_2^2 + \tilde{\Phi}_3^3 = 0\}. \tag{5.2.0.17}$$

The dimension of the chiral ring of these singularities is 10 and 14, respectively, as indicated by the subscript of Q and K. Each of these catastrophes is characterized by two triplets of numbers, the Dolgachev numbers and the Gabrielov numbers. For the above singularity Q_{10} the corresponding pair of triplets is $\mathcal{D}(Q_{10}) = (2,3,9)$ and $\mathcal{G}(Q_{10}) = (3,3,4)$ for the Dolgachev and Gabrielov numbers respectively whereas the singularity K_{14} leads to the triplets $(3,3,4)$ and $(2,3,9)$ For the above pair $\mathcal{D}(Q_{10}) = \mathcal{G}(K_{14})$ and $\mathcal{G}(Q_{10}) = \mathcal{D}(K_{14})$. It is in this sense that the two polynomials are called dual[7].

From our previous results it is clear that an alternative way to relate these two singularities flows from the description of affine D–invariants as orbifolds of diagonal invariants. The above polynomials can be viewed as the Landau–Ginzburg potentials of the tensor product $(1\cdot 6)$

$$Q_{10} \sim (1\cdot 6)_{A_2\otimes D_5} \quad \text{and} \quad K_{14} \sim (1\cdot 6)_{A_2\otimes A_7} \tag{5.2.0.18}$$

where we have added a trivial factor for the K_{14} singularity. Therefore it follows that

$$Q_{10} = K_{14}/\mathbb{Z}_2. \tag{5.2.0.19}$$

More explicitly consider the chiral ring of K_{14}

$$\mathcal{R}_K = \{1, \Phi_1, \Phi_1^2, ..., \Phi_1^6, \Phi_2, \Phi_1\Phi_2, ..., \Phi_1^6\Phi_2\}. \tag{5.2.0.20}$$

[7]The precise nature of these numbers, which are not of concern here, can be found in [3]

The action of the $\mathbb{Z}_2$ is defined as $[1, 0, 1]$ on the fields (Φ_1, Φ_2, Φ_3). The two sectors of the orbifold consist of the eight invariant states of $\mathcal{R}_K$ and of two twisted states giving the total of ten states necessary for Q.

This strangely dual pair can again be used as building block of string compactifications. In the models constructed in [139, 140, 82, 83] this involves vacua which have $(1 \cdot 6)_{A_1 \otimes A_7}$ or $(1 \cdot 6)_{A_1 \otimes D_5}$ as subfactors.

5.3 Mirror Application of the Basic Isomorphism for the A–D Transition

Coming back to our example at the beginning of this chapter

$$\mathbb{P}_{(3,8,33,88,132)}[264]^{57}_{-48} \ni \{z_1^{88} + z_2^{33} + z_3^8 + z_4^3 + z_5^2 = 0\} \tag{5.3.0.21}$$

we see that applying the analysis of the previous section with respect to the cyclic group

$$\mathbb{Z}_2 \ni [\,0 \quad 0 \quad 1 \quad 0 \quad 1\,] \tag{5.3.0.22}$$

leads to an orbifold which is isomorphic to the complete intersection manifold

$$\mathbb{P}_{(3,8,66,88,99)}[264]^{81}_{48} \ni \{z_1^{88} + z_2^{33} + z_3^4 + z_4^3 + z_3 z_5^2 = 0\}, \tag{5.3.0.23}$$

thus describing the mirror partner of (5.3.0.21).

There are other potential mirror partners in the ADE list and the question arises whether those can be matched as well. In order to pursue this it is natural to look at further symmetries that this manifold might have. At first sight, seems to be $\mathbb{Z}_{11} \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_2^2$. It can easily be seen however that the $\mathbb{Z}_{11}$ and $\mathbb{Z}_3$ groups are trivial, i.e. part of the projective equivalence.

The remaining factors act nontrivial, i.e. the symmetry group is generated by

$$\mathbb{Z}_8 \times \mathbb{Z}_2^2 : \quad \begin{bmatrix} 7 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.3.0.24}$$

It turns out however that the two $\mathbb{Z}_2$ actions are projectively equivalent since

$$(\alpha z_1, z_2, z_3, z_4, \alpha z_5) = \alpha(z_1, z_2, \alpha z_3, z_4, \alpha z_5) \tag{5.3.0.25}$$

and therefore the maximal supersymmetry preserving phase group is $\mathcal{D} = \mathbb{Z}_8 \times \mathbb{Z}_2$.

One would clearly hope that, as before, after modding out the $\mathbb{Z}_2$ the remaining symmetries are again part of the projective equivalence of the resulting orbifold. This is not the case. On the complete intersection mirror we have the equivalence relation

$$(z_1, z_2, z_3, z_4, z_5) \sim (\lambda^3 z_1, \lambda^8 z_2, \lambda^{66} z_3, \lambda^{88} z_4, \lambda^{99} z_5). \tag{5.3.0.26}$$

The $\mathbb{Z}_8$ obviously does not act trivially on the ambient space of the complete intersection. Surprisingly it is not an action on the mirror at all as can be seen from the polynomial . This represents our second qualification as regards to the selection criterion: part of the supersymmetry preserving discrete group does not act on the orbifolds.

Similar things happen for different moddings. Modding out the original Fermat by the $\mathbb{Z}_8$ leads to the manifold

$$\mathbb{P}_{(6,14,57,154,231)}[462]^{81}_{48} \ni \{z_1^{77} + z_2^{33} + z_1 z_3^8 + z_4^3 + z_5^2 = 0\} \tag{5.3.0.27}$$

on which the first $\mathbb{Z}_2$ acts trivially whereas the second $\mathbb{Z}_2$ is not a symmetry at all. Lastly, when modding out the second $\mathbb{Z}_2$ first one finds the mirror

$$\mathbb{P}_{(6,8,33,88,129)}[264]^{81}_{48} \ni \{z_1^{44} + z_2^{33} + z_3^8 + z_4^3 + z_1 z_5^2 = 0\}. \tag{5.3.0.28}$$

on which the first $\mathbb{Z}_2$ acts trivial and the $\mathbb{Z}_8$ does not act on the manifold.

Thus we have made explicit the two effects that generally modify the selection rule which says that is the maximal discrete group needs to be modded out to produce the mirror or a Fermat type manifold: it can happen that a subgroup of the full symmetry group becomes part of the projective equivalence of an intermediate orbifold or that it is not an action on the orbifold at all. There are many examples in which these effects make their appearance.

5.4 Iterating the Basic Isomorphism

It is easy to generalize this analysis to general transformations from Fermat–type polynomials to tadpole–type polynomials with N points attached. Consider the transformation

$$\sum_{i=1}^{N} \Phi_i^{l_i} \longrightarrow \left(\sum_{j=1}^{N-1} \tilde{\Phi}_j^{n_j} \tilde{\Phi}_{j+1}\right) + \tilde{\Phi}_N^{n_N} \tag{5.4.0.29}$$

defined by

$$\begin{aligned} \Phi_i &= \tilde{\Phi}_i^{n_i/l_i} \tilde{\Phi}_{i+1}^{1/l_i} \qquad i = 1, .., N-1; \\ \Phi_N &= \tilde{\Phi}_N^{n_N/l_n}. \end{aligned} \tag{5.4.0.30}$$

The condition that the Jacobian of this transformation is constant again leads to constraints on the exponents:

$$n_1 = l_1; \tag{5.4.0.31}$$
$$n_i = l_i\left[1-\frac{1}{l_{i-1}}\right], \; i=2,...,N. \tag{5.4.0.32}$$

Transformation (5.4.0.30) then becomes

$$\Phi_1 = \tilde{\Phi}_1\tilde{\Phi}_2^{1/l_1}; \tag{5.4.0.33}$$
$$\Phi_i = \tilde{\Phi}_i^{(l_{i-1}-1)/l_{i-1}}\tilde{\Phi}_{i+1}^{1/l_i} \qquad i=2,..,N-1; \tag{5.4.0.34}$$
$$\Phi_N = \tilde{\Phi}_N^{(l_{N-1}-1)/l_{N-1}}. \tag{5.4.0.35}$$

As before, this transformation is not well defined, but one can find a discrete group under which it is invariant. Let

$$\Phi_i \longrightarrow \alpha_i\Phi_i. \tag{5.4.0.36}$$

From eq. (5.4.0.35) it follows that under this transformation

$$\tilde{\Phi}_N \longrightarrow \alpha_N^{l_{N-1}/(l_{N-1}-1)}\tilde{\Phi}_N, \tag{5.4.0.37}$$

i.e. we find the constraint

$$\alpha_N^{l_{N-1}/(l_{N-1}-1)} = 1 \tag{5.4.0.38}$$

for the generator α_N. Plugging the solutions of this constraint iteratively into eqs. (5.4.0.35) one finds that all generators α_i have to satisfy $\alpha_i^{l_i} = 1$, $i = 1,..,N-1$ for the transformation of variables to be invariant under the identification (5.4.0.36). The group by which one has to mod out is $\prod_{i=1}^{N-1}\mathbb{Z}_{l_i}$. Similar considerations apply to the other direction of the transformation.

Consider e.g. the manifold

$$\mathbb{P}_{(1,3,3,3,5)}[15]^3_{-144} = \{\Phi_1^{15}+\Phi_2^5+\Phi_3^5+\Phi_4^5+\Phi_5^3=0\}. \tag{5.4.0.39}$$

We will show now that the orbifold of this model with respect to two cyclic groups $\mathbb{Z}_5$, the first generated by $[0,4,1,0,0]$ the second one by $\mathbb{Z}_5 : [0,0,4,1,0]$, is isomorphic to the manifold

$$\mathbb{P}_{(16,60,45,39,80)}[240]^{75}_{144} = \{\tilde{\Phi}_1^{15}+\tilde{\Phi}_2^4+\tilde{\Phi}_2\tilde{\Phi}_3^4+\tilde{\Phi}_3\tilde{\Phi}_4^5+\tilde{\Phi}_5^3=0\}. \tag{5.4.0.40}$$

Consider the first $\mathbb{Z}_5$. The fixed points of this action are determined by the requirement that

$$(\Phi_1,\alpha^4\Phi_2,\alpha\Phi_3,\Phi_4,\Phi_5) = c(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5) \tag{5.4.0.41}$$

where $c \in \mathbb{C}^*$. The fixed point set consists of one curve $\mathbb{P}_{(1,3,5)}[15]$ with Euler number $\chi = -6$ and one further fixed point $\mathbb{P}_{(3,5)}[15]$. The Euler number of the resolved orbifold therefore is

$\chi = \frac{-144}{5} - \frac{1}{5}(-6+1) + 5(-6+1) - \frac{1}{5} + 5 = -48$. The corresponding weighted CY–manifold of this space is described by

$$\mathbb{P}_{(4,15,9,12,20)}[60] \;=\; \{\phi_1^{15} + \phi_2^4 + \phi_2\phi_3^5 + \phi_4^5 + \phi_5^3 = 0\}. \tag{5.4.0.42}$$

and the coordinate transformation is defined as follows

$$(\Phi_1, \Phi_2, ..., \Phi_5) = (\phi_1, \phi_2^{4/5}, \phi_2^{1/5}\phi_3, \phi_4, \phi_5). \tag{5.4.0.43}$$

Modding out by the other $\mathbb{Z}_5$ leads to a fixed point set of three curves

$$\mathbb{P}_{(4,15,20)}[60]_{15}, \quad \mathbb{P}_{(15,9,20)}[60]_{16}, \quad \mathbb{P}_{(15,12,20)}[60]_{11}. \tag{5.4.0.44}$$

These three curves intersect in the point $\mathbb{P}_{(15,20)}[60] = \mathbb{P}_{(3,4)}[12]$ and the Euler number of the resolved manifold is given by

$$\chi = -\frac{48}{5} - \frac{1}{5}(15+11+16-2\times 5) + 5(15+11+16-2\times 5) = 144. \tag{5.4.0.45}$$

The term -2×5 comes from the fact that in all three curves we have blown up the $\mathbb{Z}_5$ fixed point when in fact we only should have done so once.

5.5 Mirror Mapping Moduli Spaces

It is important to realize that via fractional transformations it is possible to identify not only individual points in the moduli space as mirror pairs but instead to mirror map moduli spaces.

As a simple example which illustrates this phenomenon consider the quintic which at the exactly solvable point takes the form

$$(3^5)_{A_4^5} \sim \mathbb{C}^*(1,1,1,1,1)[5] \sim \mathbb{P}_4[5] \ni \left\{\sum_{i=1}^{5} z_i^5 = 0\right\}. \tag{5.5.0.46}$$

Using the iteration of the basic isomorphism as described in section 3 one finds that orbifolding the LG theory with respect to the cyclic group

$$\mathbb{Z}_5^3: \quad \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix} \tag{5.5.0.47}$$

leads to the mirror spectrum and using the corresponding fractional transformations shows that this orbifold is isomorphic to the complete intersection

$$\mathbb{P}_{(80,60,65,51,64)}[320]_{200}^{101} \ni \{z_1^4 + z_1 z_2^4 + z_2 z_3^4 + z_3 z_4^5 + z_5^5 = 0\}. \tag{5.5.0.48}$$

where again no orbifolding is necessary on the mirror side of the isomorphism. The manifold (5.5.0.48) is indeed one of two spaces in the list of [41] which has the appropriate mirror spectrum of the quintic.

As mentioned already the relation between mirror candidates is not 1–1; to a given manifold there usually exist many manifolds in the list featuring the mirror spectrum. For the quintic there appear two manifolds in the list with the mirror spectrum and an obvious question is whether the remaining one is a quintic mirror as well or whether this is perhaps a genuinely different space *. Since actions of groups with order smaller than 125 all lead to orbifolds with spectra different from the one needed the only possibility left is to consider larger groups.

Naively the maximal supersymmetry preserving discrete phase group appears to be $\mathcal{D} = (\mathbb{Z}_5)^4$, generated by the first three factors of the action 5.5.0.47 and an additional factor defined by

$$\mathbb{Z}_5: \ [0 \quad 0 \quad 0 \quad 4 \quad 1]. \tag{5.5.0.49}$$

The orbifold with respect to this $\mathbb{Z}_5^4$ has the mirror spectrum as well and using again the isomorphism 5.1.0.4 one can show that this orbifold is isomorphic to the complete intersection

$$\mathbb{P}_{(64,48,52,51,41)}[256]^{101}_{200} \ni \{z_1^4 + z_1 z_2^4 + z_2 z_3^4 + z_3 z_4^4 + z_4 z_5^5 = 0\}. \tag{5.5.0.50}$$

This is indeed the second mirror candidate that appears in the list of [41].

The group $(\mathbb{Z}_5)^4$ however is *not* the maximal supersymmetry phase group since any one its individual factors is projectively equivalent to a diagonal subgroup of the three others. Hence it should be expected that the two manifolds are in fact only different representations of one and the same manifold. To get a better insight into these two spaces recall the construction of the two mirrors of the quintic as orbifolds. Modding out the original quintic with the first two $\mathbb{Z}_5$s leads to the complete intersection

$$\mathbb{P}_{(20,15,13,16,16)}[80]^{49}_{88} \ni \{z_1^4 + z_1 z_2^4 + z_2 z_3^5 + z_4^5 + z_5^5 = 0\} \tag{5.5.0.51}$$

where the $\mathbb{Z}_4$ is again part of the projective equivalence of $\mathbb{P}_{(20,15,13,16,16)}$. Modding out the third $\mathbb{Z}_5$ then leads to the manifold (5.5.0.48) whereas modding out the last $\mathbb{Z}_5$ leads to manifold (5.5.0.50). The $\mathbb{Z}_4$ is trivial in both steps.

The fact that the two mirrors thus constructed are indeed isomorphic can easily be seen on the level of the complete intersections as well. Consider the action of the last $\mathbb{Z}_5$ on the first mirror in more detail. The generator of the group is described by

$$(z_1, z_2, z_3, z_4, z_5) \longrightarrow (z_1, z_2, z_3, \alpha^4 z_4, \alpha z_5). \tag{5.5.0.52}$$

*The Hodge diamond does not classify manifolds up to diffeomorphisms.

The weights of the space it acts on are $(80, 60, 65, 51, 64)$, i.e. the projective equivalence corresponds to the identification of the following vectors

$$(z_1, z_2, z_3, z_4, z_5) \sim (\lambda^{80} z_1, \lambda^{60} z_2, \lambda^{65} z_3, \lambda^{51} z_4, \lambda^{64} z_5). \tag{5.5.0.53}$$

it is clear from this that the action described in eq.(5.5.0.52) is in fact trivial since it is part of the projective equivalence. Hence these two models are isomorphic.

What is important to emphasize is the fact that indeed the mirror construction just described goes through for the whole part of the moduli space that is relevant: since the complex sector of the mirror of the quintic is complex 1–dimensional we need to establish a map that relates a one–dimensional subspace of the 101–dimensional space of complex deformations to the 1–dimensional subspace of the mirror. This 1–dimensional subspace is parametrized by

$$\mathbb{P}_4[5] \ni \left\{ \sum_{i=1}^{5} z_i^5 - 5\psi \prod_{i=1}^{5} z_i = 0 \right\}. \tag{5.5.0.54}$$

The crucial point here is that for each value of ψ the configurations possess the mirror discrete group (5.5.0.47) as a symmetry group and we can mod out this action. Furthermore for each value of ψ we can apply our fractional transformation and map this configuration to

$$\mathbb{P}_{(80,60,65,51,64)}[320]_{200}^{101} \ni \{ z_1^4 + z_1 z_2^4 + z_2 z_3^4 + z_3 z_4^5 + z_5^5 - 5\psi \prod_{i=1}^{5} z_i = 0 \}. \tag{5.5.0.55}$$

The same holds for the second representation of the mirror.

The same type of construction also applies to mirror pairs involving moduli spaces whose dimensions is larger than one: an example that has received attention recently [34, 37] is furnished by theory

$$(6^2 \cdot 2^3)_{A_7^2 \otimes A_3^3} \sim \mathbb{C}^*_{(1,1,2,2,2)}[8] \sim \mathbb{P}_{(1,1,2,2,2)}[8] \tag{5.5.0.56}$$

described by the polynomial

$$p = z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4 - 8\psi \prod_{i=1}^{5} z_i - 2\lambda z_1^4 z_2^4 \tag{5.5.0.57}$$

the Gepner point being characterized by $\psi = 0 = \lambda$. This theory has the spectrum $(\chi, h^{(1,1)}, \gamma) = (-168, 2, 86)$ and the mirror can be obtained by using the action

$$\mathbb{Z}_8 \times \mathbb{Z}_3 : \quad \begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix} \tag{5.5.0.58}$$

and using the fractional transformation

$$z_1 = y_1^{7/8}, \quad z_2 = y_1^{1/8} y_2^{3/4}, \quad z_3 = y_2^{1/4} y_3^{3/4}, \quad z_4 = y_4^{1/4} y_5, \quad z_5 = y_5 \tag{5.5.0.59}$$

we obtain the configuration

$$\mathbb{P}_{(4,4,8,5,7)}[28] \tag{5.5.0.60}$$

with the mirror spectrum $(\chi, h^{(1,1)}, \gamma) = (168, 86, 2)$, parametrized by the polynomial

$$p = y_1^7 + y_1 y_2^6 + y_2 y_3^3 + y_4 y_5^4 + y_5^4 - 8\psi \prod_{i=1}^{5} y_i - 2\lambda y_1^4 y_2^3. \tag{5.5.0.61}$$

5.6 Mirrors of Potentials which are not of Brieskorn–Pham Type

The representation of orbifolds in terms of Landau–Ginzburg theories or weighted complete intersection manifolds also makes it possible to analyze non–Fermat type theories as well. Consider e.g. the manifold embedded in

$$\mathbb{P}_{(3,6,6,4,5)}[24]^{10}_{-48} \tag{5.6.0.62}$$

as the zero locus of the polynomial

$$p = z_1^8 + z_2^4 + z_3^4 + z_4^6 + z_4 z_5^4. \tag{5.6.0.63}$$

Orbifolding this space with respect to a $\mathbb{Z}_4$ symmetry with the action

$$\mathbb{Z}_4 \;:\; [\ 0\ 0\ 1\ 0\ 3\ ] \tag{5.6.0.64}$$

produces a theory with the mirror flipped spectrum. Using the appropriate fractional transformations as discussed above leads to the mirror manifold embedded in

$$\mathbb{P}_{(9,12,13,18,20)}[72]^{34}_{48} \tag{5.6.0.65}$$

defined by the polynomial

$$p = z_1^8 + z_2^6 + z_2 z_3^3 + z_5 z_3^4 + z_4^4. \tag{5.6.0.66}$$

One particularly simple class are manifolds built from Fermat parts and tadpole potentials

$$\Phi_1^c \Phi_2 + \Phi_2^d \tag{5.6.0.67}$$

that can be described as the orbifold of two Fermat monomials. Consider the non-Fermat manifold

$$\mathbb{P}_{(1,1,1,2,3)}[8]^2_{-208} \ni \{z_1^8 + z_2^8 + z_3^8 + z_4^4 + z_4 z_5^2 = 0\}. \tag{5.6.0.68}$$

From (5.1.0.4) it follows that this manifold is the $\mathbb{Z}_7$ orbifold of

$$\mathbb{P}_{(1,1,1,1,4)}[8]^1_{-296} \ni \{z_1^8 + z_2^8 + z_3^8 + z_4^8 + z_5^2 = 0\}. \tag{5.6.0.69}$$

where the action is defined by $\mathbb{Z}_2 \ni [0,0,0,1,1]$. A manifold with the mirror spectrum of the non-Fermat manifold above can be constructed by orbifolding the original Fermat with respect to two $\mathbb{Z}_8$ groups generated by

$$\mathbb{Z}_8^2 : \begin{bmatrix} 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 \end{bmatrix}. \tag{5.6.0.70}$$

Its complete intersection representation is defined by the manifold

$$\mathbb{P}_{(56,48,43,49,196)}[392]^{106}_{208} \ni \{z_1 z_2^7 + z_2 z_3^8 + z_3^8 + z_4^8 + z_5^2 = 0\}. \tag{5.6.0.71}$$

Since the maximal symmetry group of the original Fermat manifold space is generated by

$$\mathcal{D} = \mathbb{Z}_8^2 \times \mathbb{Z}_2 : \quad \begin{bmatrix} 7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{5.6.0.72}$$

Thus we have found in this example that the mirror manifold of this non-Fermat type space can be obtained via the following steps: first one finds the Fermat manifold whose orbifold with respect to a subgroup $\mathcal{D}_t$ of maximal supersymmetry preserving phase group $\mathcal{D}$ this weighted CICY corresponds to. The mirror is then obtained by modding the original Fermat manifold with respect to the complement of the group $\mathcal{D}_t$.

5.7 Mirrors of LG/CY vacua of codimension larger than one.

One of the important questions clearly is to what extent the mirror symmetry found for the special classes of LG potentials and orbifolds discussed so far extends to other, more general, constructions. One important generalization of CY manifolds embedded in one ambient weighted projective space are spaces embedded in products of several factors. A preliminary exploration of such spaces shows that mirror pairs do exist in this more general framework as well. A small list of such spaces and their mirrors are collected in Table 6.1.

Manifold	LG Potential	Group	Mirror Action
$\begin{matrix}\mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,4,4,3)}\end{matrix}\begin{bmatrix}2 & 0\\ 1 & 12\end{bmatrix}$	$\mathbb{C}_{(8,8,6,2,11,2,11)}[24]^{(11,131)}$ $x_1^3+x_2^3+x_3^4+\sum_{i=1}^2(y_i^{12}+y_iz_i^2$	$\mathbb{Z}_3^2\times\mathbb{Z}_4$	$\begin{bmatrix}1&0&0&1&1&0&0\\1&0&0&0&0&1&1\\0&0&1&2&1&0&0\end{bmatrix}$
$\begin{matrix}\mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,2,2,1)}\end{matrix}\begin{bmatrix}2 & 0\\ 1 & 6\end{bmatrix}$	$\mathbb{C}_{(4,4,2,2,5,2,5)}[12]^{(5,101)}$ $x_1^3+x_2^3+x_3^6+\sum_{i=1}^2(y_i^6+y_iz_i^2)$	$\mathbb{Z}_6^2$	$\begin{bmatrix}0&0&1&4&1&0&0\\0&0&1&0&0&4&1\end{bmatrix}$
$\begin{matrix}\mathbb{P}_{(1,1)} \\ \mathbb{P}_{(3,3,4,4,1)}\end{matrix}\begin{bmatrix}2 & 0\\ 3 & 12\end{bmatrix}$	$\mathbb{C}_{(8,8,2,6,9,6,9)}[24]^{(19,43)}$ $x_1^3+x_2^3+x_3^{12}+\sum_{i=1}^2(y_i^4+y_iz_i^2)$	$\mathbb{Z}_4$	$\begin{bmatrix}0&0&1&2&1&0&0\end{bmatrix}$
$\begin{matrix}\mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,4,6)}\end{matrix}\begin{bmatrix}2 & 0\\ 1 & 12\end{bmatrix}$	$\mathbb{C}_{(2,11,2,11,2,8,12)}[24]^{(3,243)}$ $\sum_{i=1}^2(x_i^{12}+x_iy_i^2)+x_3^{12}+x_4^3+x_5^2$	$\mathbb{Z}_{12}^2\times\mathbb{Z}_2$	$\begin{bmatrix}10&1&0&0&1&0&1\\0&0&10&1&1&0&0\\0&0&0&0&1&0&1\end{bmatrix}$
$\begin{matrix}\mathbb{P}_2 \\ \mathbb{P}_3\end{matrix}\begin{bmatrix}3 & 0\\ 1 & 3\end{bmatrix}$	$\mathbb{C}_{(3,2,3,2,3,2,3)}[9]^{(8,35)}$ $\sum_i(x_i^3+x_iy_i^3)+y_4^3$	$\mathbb{Z}_9\times\mathbb{Z}_3$	$\begin{bmatrix}6&1&3&2&0&0&6\\0&0&0&0&0&2&1\end{bmatrix}$
$\begin{matrix}\mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)}\end{matrix}\begin{bmatrix}2 & 0\\ 1 & 6\end{bmatrix}$	$\mathbb{C}_{(2,5,2,5,2,2,6)}[12]^{(2,128)}$ $\sum_i(x_i^6+x_iy_i^2)+\sum_j x_j^6+x_5^2$	$\mathbb{Z}_6^2\times\mathbb{Z}_2$	$\begin{bmatrix}4&1&0&0&1&0&0\\0&0&0&0&5&1&0\\0&0&0&0&0&1&1\end{bmatrix}$

Table 6.1 *Some examples of codimension 2 Calabi-Yau/LG vacua and their mirror actions.*

5.8 Mirrors of Rigid Vacua

It should be noted that in the context of Landau–Ginzburg vacua the construction of mirror of rigid vacua poses no difficulty at all. Applying the constructions of [141] it is easy to find the corresponding Landau–Ginzburg potentials of the mirror vacua as well. In the list of Landau–Ginzburg theories with isolated singularities there appear two distinct rigid models, both of which were mentioned in section 3

$$\begin{aligned}(1^9)^{(0,84)}_{A^9} &\leftrightarrow \mathbb{C}^*_{(1,1,1,1,1,1,1,1,1)}[3] \quad \ni \quad \left\{W=\sum_{i=1}^3 z_i^3=0\right\}\\ (2^6)^{(0,90)}_{A^6} &\leftrightarrow \mathbb{C}^*_{(1,1,1,1,1,1,2)}[4] \quad \ni \quad \left\{W=\sum_i z_i^4+z_7^2=0\right\}\end{aligned} \tag{5.8.0.73}$$

Consider e.g. the orbifold

$$\mathbb{C}_{(1,1,1,1,1,1,1,1,1)}[3]/\mathbb{Z}_3^6 : \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix}. \qquad (5.8.0.74)$$

of the model $\mathbb{C}_{(1,1,1,1,1,1,1,1,1)}[3]$, $(0,84,-168)$. This orbifold has the spectrum $(84,0,168)$ and leads, via nonlinear changes, to the LG–potential

$$\mathbb{C}_{(96,48,72,60,66,63,43,64,64)}[192] \ni \left\{z_1^2 + z_1z_2^2 + z_2z_3^2 + z_3z_4^2 + z_4z_5^2 + z_5z_6^2 + z_6z_7^3 + z_8^3 + z_9^3 = 0\right\} \qquad (5.8.0.75)$$

which has the same spectrum $(84,0,168)$ (hence no modding is necessary).

Even though the above mirror is certainly a valid representation there is another form which is to be much preferred. Consider the orbifold

$$\mathbb{C}_{(1,1,1,1,1,1,1,1,1)}[3]/\mathbb{Z}_3^6 : \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}. \qquad (5.8.0.76)$$

which also has the spectrum $(84,0,168)$. Applying the rational transformations this becomes the complete intersection

$$\mathbb{C}_{(192,96,144,120,132,126,129,85,128)}[384] \ni \left\{z_1^2 + z_1z_2^2 + z_2z_3^2 + z_3z_4^2 + z_4z_5^2 + z_5z_6^2 + z_6z_7^2 + z_7z_8^3 + z_9^3 = 0\right\}. \qquad (5.8.0.77)$$

For the second theory the orbifolding

$$\mathbb{C}^*_{(1,1,1,1,1,1,2)}[4]/\mathbb{Z}_4^3 : \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \end{bmatrix} \qquad (5.8.0.78)$$

leads to the mirror spectrum and fractional transformations lead to the mirror potential

$$\mathbb{C}_{(108,72,84,80,61,81,162)}[324] \ni \{z_1^3 + z_1z_2^3 + z_2z_3^3 + z_3z_4^3 + z_4z_5^4 + z_6^4 + z_7^2. \qquad (5.8.0.79)$$

All representations of the two rigid vacua and their mirrors that appear in the list of all Landau–Ginzburg vacua can in fact be mapped into each other via fractional transformations. The precise relations are collected in Table 6.2 *.

*This table contains the distinct theories. The other theories in the list

LG–Theory	Spectrum	Group	Action	Mirror LG–Theory
$\mathbb{C}_{(1,1,1,1,1,1,1,1,1)}[3]$	(0,84)	$\mathbb{Z}_3^6$:	$\begin{bmatrix} 2&1&0&0&0&0&0&0&0\\ 0&2&1&0&0&0&0&0&0\\ 0&0&2&1&0&0&0&0&0\\ 0&0&0&2&1&0&0&0&0\\ 0&0&0&0&2&1&0&0&0\\ 0&0&0&0&0&2&1&0&0 \end{bmatrix}$	$\mathbb{C}_{(43,48,60,63,64,64,66,72,98)}[192]$
		$\mathbb{Z}_3^7$:	$\begin{bmatrix} 2&1&0&0&0&0&0&0&0\\ 0&2&1&0&0&0&0&0&0\\ 0&0&2&1&0&0&0&0&0\\ 0&0&0&2&1&0&0&0&0\\ 0&0&0&0&2&1&0&0&0\\ 0&0&0&0&0&2&1&0&0\\ 0&0&0&0&0&0&2&1&0 \end{bmatrix}$	$\mathbb{C}_{(57,64,80,84,85,86,88,96,128)}[256]$
		$\mathbb{Z}_3^8$:	$\begin{bmatrix} 2&1&0&0&0&0&0&0&0\\ 0&2&1&0&0&0&0&0&0\\ 0&0&2&1&0&0&0&0&0\\ 0&0&0&2&1&0&0&0&0\\ 0&0&0&0&2&1&0&0&0\\ 0&0&0&0&0&2&1&0&0\\ 0&0&0&0&0&0&2&1&0\\ 0&0&0&0&0&0&0&2&1 \end{bmatrix}$	$\mathbb{C}_{(85,96,120,126,128,129,132,144,192)}[384]$
$\mathbb{C}_{(1,1,1,1,1,1,2)}[4]$	(0,90)	$\mathbb{Z}_4^4$:	$\begin{bmatrix} 3&1&0&0&0&0&0\\ 0&3&1&0&0&0&0\\ 0&0&3&1&0&0&0\\ 0&0&0&3&1&0&0 \end{bmatrix}$	$\mathbb{C}_{(61,72,80,81,84,108,162)}[324]$
		$\mathbb{Z}_4^4 \times \mathbb{Z}_2$:	$\begin{bmatrix} 3&1&0&0&0&0&0\\ 0&3&1&0&0&0&0\\ 0&0&3&1&0&0&0\\ 0&0&0&3&1&0&0\\ 0&0&0&0&1&0&1 \end{bmatrix}$	$\mathbb{C}_{(54,60,61,63,76,81,91)}[243]$
		$\mathbb{Z}_4^5$:	$\begin{bmatrix} 3&1&0&0&0&0&0\\ 0&3&1&0&0&0&0\\ 0&0&3&1&0&0&0\\ 0&0&0&3&1&0&0\\ 0&0&0&0&3&1&0 \end{bmatrix}$	$\mathbb{C}_{(91,108,120,122,126,162,243)}[486]$
		$\mathbb{Z}_4^5 \times \mathbb{Z}_2$:	$\begin{bmatrix} 3&1&0&0&0&0&0\\ 0&3&1&0&0&0&0\\ 0&0&3&1&0&0&0\\ 0&0&0&3&1&0&0\\ 0&0&0&0&3&1&0\\ 0&0&0&0&0&1&1 \end{bmatrix}$	$\mathbb{C}_{(72,80,81,84,101,108,122)}[324]$

Table 6.2 *Rigid Landau–Ginzburg vacua and their mirrors.*

$\mathbb{C}_{(1,1,1,2,2,2,3)}[6]$	(0,84)	$\mathbb{Z}_6^2 \times \mathbb{Z}_3$	$\begin{bmatrix} 5&1&0&0&0&0&0\\ 0&5&1&0&0&0&0\\ 0&0&2&1&0&0&0 \end{bmatrix}$	$\mathbb{C}_{(48,60,63,79,100,100,150)}[300]$
		$\mathbb{Z}_6^2 \times \mathbb{Z}_3$	$\begin{bmatrix} 5&1&0&0&0&0&0\\ 0&5&1&0&0&0&0\\ 0&0&2&1&0&0&0\\ 0&0&0&2&1&0&0 \end{bmatrix}$	$\mathbb{C}_{(96,120,121,126,200,237,300)}[600]$
		$\mathbb{Z}_6^2 \times \mathbb{Z}_3^3$	$\begin{bmatrix} 5&1&0&0&0&0&0\\ 0&5&1&0&0&0&0\\ 0&0&2&1&0&0&0\\ 0&0&0&2&1&0&0\\ 0&0&0&0&2&1&0 \end{bmatrix}$	$\mathbb{C}_{(64,80,84,93,121,158,200)}[400]$
		$\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\begin{bmatrix} 5&1&0&0&0&0&0\\ 0&1&0&0&0&0&1\\ 0&0&2&1&0&0&0\\ 0&0&0&2&1&0&0\\ 0&0&0&0&2&1&0 \end{bmatrix}$	$\mathbb{C}_{(48,55,60,64,75,88,90)}[240]$

are isomorphic to the ones listed in Table 6.1.

Aside from the rigid vacua that arise in the Landau–Ginzburg formulation of string vacua there exist of course also rigid vacua that are obtained by orbifolding Landau–Ginzburg theories with respect to discrete groups. More importantly however there exist further rigid Calabi–Yau manifolds whose mirrors cannot be described in the Landau–Ginzburg framework. Examples are furnished by the resolution of particular nontransverse quintics constructed by Schoen [164] and van Straten [177].

5.9 Loop Potentials

The previous sections of this chapter have established that the discrete groups of the LG–CY theories play a role comparable to the discrete symmetries of the fusion rules in conformal field theories even though for most of the models at hand the corresponding exact theory is not known. We have seen that in general there are constraints on the weights of the potentials that can be identified with with orbifolds of other models. Nevertheless, in the types of potentials discussed above there was enough freedom to have nontrivial identifications. This is not true in general as we will show now.

Consider potentials of the type

$$\tilde{\Phi}_1^c\tilde{\Phi}_2 + \tilde{\Phi}_2^d\tilde{\Phi}_1 \tag{5.9.0.80}$$

and more general polynomials of this type with an arbitrary number of fields. For these polynomials it is possible to find a 1-1 coordinate transformation from Fermat–type polynomials, but the condition that the Jacobian is constant leads to the conclusion that only deformations of the original space can be obtained in this way.

The transformation of a Fermat type polynomial into a loop–type polynomial

$$\sum_{i=1}^{N}\Phi_i^{l_i} \longrightarrow \left(\sum_{i=1}^{N-1}\tilde{\Phi}_i^{n_i}\tilde{\Phi}_{i+1}\right) + \tilde{\Phi}_N^{n_N}\tilde{\Phi}_1. \tag{5.9.0.81}$$

leads to the transformation of variables

$$\begin{aligned} \Phi_i &= \tilde{\Phi}_i^{n_i/l_i}\tilde{\Phi}_{i+1}^{1/l_i} \qquad i = 1,..,N-1; && (5.9.0.82)\\ \Phi_N &= \tilde{\Phi}_N^{n_N/l_N}\tilde{\Phi}_1^{1/l_N}. && (5.9.0.83)\end{aligned}$$

The condition that the Jacobian be constant leads to the constraints

$$\begin{aligned} n_1 &= l_1\left[1-\frac{1}{l_N}\right]; && (5.9.0.84)\\ n_i &= l_i\left[1-\frac{1}{l_{i-1}}\right], \qquad i = 2,..,N; && (5.9.0.85)\end{aligned}$$

so that the coordinate transformation (41,42) becomes

$$\Phi_1 = \tilde{\Phi}_1^{(l_N-1)/l_N} \tilde{\Phi}_2^{1/l_1} \tag{5.9.0.86}$$
$$\Phi_i = \tilde{\Phi}_i^{(l_{i-1}-1)/l_{i-1}} \tilde{\Phi}_{i+1}^{1/l_i}, \qquad i = 2, .., N. \tag{5.9.0.87}$$

From this it follows however that the charges of the loop fields are precisely those of the fields of the Fermat type polynomial, i.e. the loop–type model is a trivial deformation of the original one.

Chapter 6

Mirror Symmetry and Spectra from Fano Manifolds

Even though we have constructed many mirror pairs in the previous chapter, it is not easy, unfortunately, to say, in full generality, what mirror symmetry is without reference to any particular framework for string compactification. What we have done so far, as a first step, is to essentially consider the conventional framework in which left–right symmetric compactifications without torsion are described in a number of related ways

- By internal spaces which are compact complex Kähler manifolds and admit a covariantly constant spinor, i.e. have vanishing first Chern class, so–called Calabi–Yau manifolds.
- By (2,2)–supersmmetric Landau–Ginzburg theories.
- By (2,2)–superconformal exactly solvable field theories.

It has become clear that no matter in which framework one starts, when all is said and done the GSO projection leads to a theory which represents a geometrical object. Now, one would like to argue that in the case of (2,2)–theories this underlying construct is a Calabi–Yau manifold. However, this cannot be the case when the theory is one without modes that correspond to Kähler deformations. This is the problem which we are turning to presently.

In this chapter we analyze a class of Fano type manifolds, described in [161] (see also [36]), which generalizes Calabi–Yau spaces of complex dimension D_{crit} in a natural way. These manifolds are of complex dimension $(D_{crit} + 2(Q-1))$, $Q > 1$, and, being Fano manifolds they have a positive first Chern class. However they are Fano varieties of a special type: their first Chern class is quantized in in a particular way. Thus they do not describe, a priori, consistent string groundstates. Surprisingly however, it is possible to derive from these higher dimensional manifolds the spectrum of critical string vacua. This can be done not only for the generations but also for the antigenerations. For particular types of these new manifolds it is furthermore possible to construct the corresponding D_{crit}–dimensional Calabi–Yau manifold directly from the $(D_{crit} + 2(Q-1))$–dimensional space.

The new class of manifolds of [161], however, not in one to one correspondence with the class of Calabi–Yau manifolds as it contains manifolds which describe string vacua that do not contain massless modes corresponding to antigenerations. It is precisely this new type of manifold that is needed in order to construct mirrors of rigid Calabi–Yau manifolds without generations. A toric description of the constructions of [161] has been given by Batyrev and Borisov and can be found [14] (see also [9]).

6.1 Higher Dimensional Manifolds with Quantized Positive First Chern Class

The construction of noncritical manifolds proceeds via the following prescription [161]:

- Fix the central charge c of the (2,2)–vacuum states and its critical dimension

$$D_{crit} = c/3. \tag{6.1.0.1}$$

- Choose a positive integer $Q \in \mathbb{N}$.

- Introduce $(D_{crit} + 2Q)$ complex coordinates $(z_1, ..., z_{D_{crit}+2Q})$, $z_i \in \mathbb{C}$.

- Define an equivalence relation

$$(z_1, ..., z_{D_{crit}+2Q}) \sim (\lambda^{k_1} z_1, ..., \lambda^{D_{crit}+2Q} z_{D_{crit}+2Q}) \tag{6.1.0.2}$$

  where $\lambda \in \mathbb{C}^*$ is a nonzero complex number and the positive integers $k_i \in \mathbb{N}$ are the weights of these coordinates. The set of these equivalence classes defines so–called weighted projective spaces, compact manifolds which are denoted by $\mathbb{P}_{(k_1,...,k_{D_{crit}+2Q})}$.

- Define hypersurfaces in the ambient weighted projective space by imposing a constraint defined by polynomials p of degree
$$d = \frac{1}{Q}\sum_i k_i \tag{6.1.0.3}$$
i.e. $p(\lambda^i z_i) = \lambda^d p(z_i)$.

The family of hypersurfaces embedded in the ambient space as the zero locus of p will be denoted by

$$\begin{aligned} M_{D_{crit}+2(Q-1)} &= \{p(z_1,\ldots,z_{D_{crit}+2Q})=0\} \cap \mathbb{P}_{(k_1,\ldots,k_{D_{crit}+2Q})} \\ &= \mathbb{P}_{(k_1,\ldots,k_{D_{crit}+2Q})}\left[\frac{1}{Q}\sum_{i=1}^{D_{crit}+2Q} k_i\right] \end{aligned} \tag{6.1.0.4}$$

and is called a configuration.

Relation (6.1.0.3) is the defining property of the class of spaces to be considered below. It is a rather restrictive condition in that it excludes many types of varieties which are transverse and even smooth but are not of physical relevance [1]. A simple example is the Fermat hypersurface

$$\mathbb{P}_{(420,280,210,168,140,120,105)}[840] \ni \{p = \sum_{i=1}^{7} z_i^{i+1} = 0\} \tag{6.1.0.5}$$

which is a rather nice, transverse, i.e. quasismooth manifold. It is also interesting from a different point of view. A curious aspect of Calabi–Yau hypersurfaces is that they are automatically what is called *well formed*, i.e. they do not contain orbifold singularities that are surfaces (in the case of threefolds). More generally this fact translates into the statement that the only resolutions that have to be performed are so–called small resolutions, i.e. the singular set are of codimension larger than one. The same is true for the higher dimensional manifolds defined above whereas the manifold (6.1.0.5) contains the singular 4–fold $S = \mathbb{P}_{(210,140,105,84,70,60)}[420]$.

Alternatively, manifolds of the type above may be characterized via a curvature constraint. Because of (6.1.0.3) the first Chern class is given by

$$c_1(M_{D_{crit}+2(Q-1)}) = (Q-1)\ c_1(\mathcal{N}) \tag{6.1.0.6}$$

where $c_1(\mathcal{N}) = dh$ is the first Chern class of the normal bundle $\mathcal{N}$ of the hypersurface $M_{D_{crit}+2(Q-1)}$ and h is the pullback of the Kähler form $\mathrm{H} \in \mathrm{H}^{(1,1)}\left(\mathbb{P}_{(k_1,\ldots,k_{D_{crit}+2Q})}\right)$ of the ambient space. Hence

[1] It will become clear below that this definition is rather natural in the context of the theory of Landau–Ginzburg string vacua with an arbitrary number of scaling fields. A particular simple manifold in this class, the cubic sevenfold $\mathbb{P}_8[3]$, has been the subject of refs. [174][36].

the first Chern class is quantized in multiples of the degree of the hypersurface $M_{D_{crit}+2(Q-1)}$. For $Q = 1$ the first Chern class vanishes and the manifolds for which (6.1.0.3) holds are Calabi–Yau manifolds, defining consistent groundstates of the supersymmetric closed string. For $Q > 1$ the first Chern class is nonvanishing and therefore these manifolds cannot possibly describe vacua of the critical string, or so it seems.

It has been shown in [161] however that it is possible to derive from these higher dimensional manifolds the massless spectrum of critical vacua. It is furthermore possible, for certain subclasses of hypersurfaces of type (6.1.0.4), to construct Calabi–Yau manifolds M_{CY} of dimension D_{crit} and complex codimension

$$codim_{\mathbb{C}}(M_{CY}) = Q \tag{6.1.0.7}$$

directly from these manifolds. The integer Q thus plays a central rôle: the critical dimension is the dimension of the noncritical manifolds offset by twice the coefficient of the first Chern class of the normal bundle of the hypersurface, which involves Q. The physical interpretation of the integer Q is that of a total charge associated to the corresponding Landau–Ginzburg theory which determines the codimension of the Calabi–Yau manifold which it describes.

As mentioned already in the introduction the class of spaces defined by (6.1.0.4) contains manifolds with no antigenerations and hence it is necessary to have some way other than Calabi–Yau manifolds to represent string groundstates if one wants to compare them with the higher dimensional manifolds. One possible way to do this is to relate them to Landau–Ginzburg theories: any manifold of type (6.1.0.4) can be viewed as a projectivization of an affine noncompact hypersurface

$$\mathbb{C}_{(k_1,\ldots,k_{N+2})}[d] \ni \{p(z_1, \ldots, z_{N+2}) = 0\} \tag{6.1.0.8}$$

via a weighted equivalence relation

$$(z_1, \ldots, z_{N+2}) \equiv (\lambda^{k_1} z_1, \ldots, \lambda^{k_{N+2}} z_{N+2}). \tag{6.1.0.9}$$

Because the polynomial p is assumed to be transverse in the projective ambient space the affine variety has a very mild singularity: it has an isolated singularity at the origin defining what is called a catastrophe in the mathematics literature. The polynomial ring of such a catastrophe is isomorphic to the chiral ring of a N=2 supersymmetric Ginzburg–Landau theory and hence allows us to compute physical quantities via methods in this framework.

Even though the manifolds (6.1.0.4) therefore correspond to LG theories of central charge $c = 3D_{crit}$ they can, however, not be identical to such theories: Consider the case when the critical dimension of the internal space corresponds to our world, i.e. $D_{crit} = 3$ and $Q = 2$. The cohomology of M_5 leads to the Hodge diamond

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & (1,1) & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & (2,2) & & 0 & & 0 & \\
0 & & (4,1) & & (3,2) & & (3,2) & & (4,1) & & 0 \\
 & 0 & & 0 & & (2,2) & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & (1,1) & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

where (p,q) denotes the dimension $h^{(p,q)}$ of the cohomology group $\mathrm{H}^{(p,q)}(M_5)$.

It is clear from this Hodge diamond that the higher dimensional manifolds will contain more modes than the critical vacuum and hence the relation of the spectrum of the critical vacuum and the cohomology of the noncritical manifolds will be a nontrivial one.

6.2 Noncritical Manifolds and Critical Vacua

Even though for $Q > 1$ the spaces (6.1.0.4) clearly are not of Calabi–Yau type and hence do not, a priori, lead to consistent string vacua, they do in fact encode important information about these string vacua. Firstly it is possible to derive from these higher dimensional manifolds the massless spectrum of critical vacua. Furthermore we will see that for certain subclasses of hypersurfaces of type (6.1.0.3) it is possible to construct Calabi–Yau manifolds M_{CY} of dimension D_{crit} and complex codimension $codim_{\mathbb{C}}(M_{CY}) = Q$ directly from these manifolds.

A first step in the analysis of any theory is to determine its physical spectrum; hence the obvious question is whether it is possible to derive the massless spectrum of the corresponding string vacua from these higher dimensional manifolds. In order to be able to formulate our geometrical framework in a concise way it is useful to recall the salient features of hypersurfaces in weighted projective spaces. The most important feature for the following comes from the fact that weighted projective spaces are singular objects, the singularities being of orbifold type. Hypersurfaces will, in general intersect these singularities and thus be singular themselves. Such singular spaces, called V–manifolds in the older literature. In string physics these objects themselves have not been of interest, rather it was the where the smoothed out, 'resolved' space is the physical object. Such is the case for critical situation $Q = 1$.

For $Q > 1$ it is clear that a projection is needed which selects from the Fano fold cohomology the relevant part. In the following I will present abundant evidence that the following paradigm provides for a geometrical projection of the Hodge diamond.

The generation and antigeneration spectrum of a $c = D_{crit}$ vacuum of the N=1 space-time supersymmetric string is parametrized by the cohomology carried by the singular subvarieties of the Fano fold together with the monomials of charge unity.

It must be emphasized that this prescription is rather different from the $Q = 1$ case where not only the cohomology carried by the singular sets is important but also the cohomology induced by the fibers of the bundles that are introduced in the process of resolving the singularities.

At first sight it might appear hopeless to find a construction corresponding to the analysis of (2,1)–forms because of the following example. Consider the orbifold $T_1^3/\mathbb{Z}_3^2$ where the two actions are defined as $(z_1, z_4) \longrightarrow (\alpha z_1, \alpha^2 z_4)$, all other coordinates invariant and $(z_1, z_7) \longrightarrow (\alpha z_1, \alpha^2 z_7)$, all other invariant. Here α is the third root of unity. The resolution of the singular orbifold leads to a Calabi–Yau manifold with 84 antigenerations and no generations [98]. This is precisely the mirror flipped spectrum of the exactly solvable tensor model 1^9 of 9 copies of $N = 2$ superconformal minimal models at level $k = 1$ [86] which can be described in terms of the Landau–Ginzburg potential $W = \sum z_i^3$ which belongs to the configuration $\mathbb{C}_{(1,1,1,1,1,1,1,1,1)}[3]$. The generations of this theory are obtained by counting the number of monomials of charge unity, of which there are 84. Imposing the GSO projection by modding out a $\mathbb{Z}_3$ symmetry this Landau–Ginzburg theory does not lead to new generations and furthermore does not create antigenerations. Thus this LG theory leads to the same spectrum as the 1^9 model [2]. [3].

This Landau–Ginzburg theory clearly is a mirror candidate for the resolved torus orbifold just mentioned [174][36] and the question arises whether a manifold corresponding to this LG potential can be found. Since the theory does not contain modes corresponding to (1,1)–forms it appears that the manifold cannot be Kähler and hence not projective. Thus it appears that the 7–dimensional manifold $\mathbb{P}_8[3]$ whose polynomial ring is identical to the chiral ring of the LG

[2] It would seem that an obvious generalization of this 7–dimensional smooth manifold is the infinite class of models $\mathbb{C}_{(1,1,1,1,1,1,1,1,1+3p)}[3+p]$, but since the manifolds (6.1.0.4) are required to be transverse the only possibility is $p = 0$.

[3] It might appear that the idea to extract the antigenerations in a heterotic string vacuum from the singularity structure of higher dimensional manifolds with integral first Chern class is obviously doomed because of apparent counterexamples, such as $\mathbb{P}_{(1,3,2,2,2,2,2,2,2)}[6]$ and $\mathbb{P}_{(1,1,1,2,2,2,3)}[6]$ which both are certainly singular but describe string vacua without antigenerations. More precisely they correspond to ground states with the spectrum $(h^{(1,1)}, h^{(1,2)}, \chi) = (0, 84, -168)$, i.e. the same spectrum featured by the model $\mathbb{P}_8[3]$. In fact these two theories are equivalent to the latter as can be seen easily by using the fractional transformation introduced in [2].

theory is merely useful as an auxiliary device in order to describe one sector of the critical LG string vacuum.

There exists one further theory in the space of all LG vacua with no antigenerations which is distinct from the one just discussed. It is described in the CFT/LG framework by

$$(2^6)^{(0,90)}_{A^6} \equiv \mathbb{C}^*_{(1,1,1,1,1,1,2)}[4] \ni \{\sum_i z_i^4 + z_7^2 = 0\} \qquad (6.2.0.10)$$

and leads to an obviously smooth manifold $\mathbb{P}_{(1,1,1,1,1,1,2)}[4]$ as well.

Thus these two rigid theories fit in very well in our way of thinking about the critical spectrum of the corresponding string vacuum.

Vacua without antigenerations are rather exceptional however; the generic groundstate will have both sectors, generations and antigenerations. The idea described above to derive the antigenerations works for higher dimensional manifolds corresponding to different types of critical vacua as will shown now.

To be concrete consider the exactly solvable tensor theory $(1{\cdot}16^3)_{A_2\otimes E_7^3}$ with 35 generations and 8 antigenerations which corresponds to a Landau–Ginzburg theory belonging to the configuration

$$\mathbb{C}^*_{(2,3,2,3,2,3,3)}[9]^{(8,35)} \qquad (6.2.0.11)$$

and which induces, via projectivization, a 5–dimensional weighted hypersurface

$$\mathbb{P}_{(2,2,2,3,3,3,3)}[9] \ni \left\{p = \sum_{i=1}^{3}(y_i^3 x_i + x_i^3) + x_4^3 = 0\right\}. \qquad (6.2.0.12)$$

with orbifold singularities

$$\begin{aligned} \mathbb{Z}_3 \ &: \ \mathbb{P}_3[3] \ni \left\{p_1 = \sum_{i=1}^{4} x_i^3 = 0\right\} \\ \mathbb{Z}_2 \ &: \ \mathbb{P}_2. \end{aligned} \qquad (6.2.0.13)$$

The $\mathbb{Z}_3$–singular set is a smooth cubic surface which supports seven (1,1)–form as can be easily shown. The $\mathbb{Z}_2$ singular set is just the projective plane and therefore adds one further (1,1)–form. Hence the singularities induced on the hypersurface by the singularities of the ambient weighted projective space give rise to a total of eight (1,1)–forms. A simple count leads to the result that the subring of monomials of charge 1 is of dimension 35. Thus we have derived the spectrum of the critical theory from the noncritical manifold (6.2.0.12).

Looking at the singularities (6.2.0.13) it becomes clear that the idea introduced above of relating the spectrum of the string vacuum to the singularity structure of the noncritical manifold leads to an interesting byproduct: it also leads to a prescription of how to derive from these higher dimensional manifolds the Calabi–Yau manifold of critical dimension! Thus a canonical prescription is obtained which also allows to pass from the Landau–Ginzburg theory to its geometrical counterpart.

This works as follows: Recall that the structure of the singularities of the weighted hypersurface just involved part of the superpotential, namely the cubic polynomial p_1 which determined the $\mathbb{Z}_3$ singular set described by a surface. The superpotential thus splits naturally into the two parts

$$p = p_1 + p_2 \tag{6.2.0.14}$$

where p_2 is the remaining part of the polynomial. The idea now is to consider the product $\mathbb{P}_3[3] \times \mathbb{P}_2$ where the factors are determined by the singular sets of the higher dimensional space and to impose on this 4–dimensional space a constraint described by the remaining part of the polynomial which did not take part in constraining the singularities of this ambient space. In the case at hand this leaves a polynomial of bidegree $(3, 1)$ and hence we are lead to a manifold embedded in

$$\begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_3 \end{array} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \tag{6.2.0.15}$$

defined by polynomials

$$\begin{aligned} p_1 &= y_1^3 x_1 + y_2^3 x_2 + y_3^3 x_3 \\ p_2 &= \sum_{i=1}^{4} x_i^3 \end{aligned} \tag{6.2.0.16}$$

which is precisely the manifold constructed in [158], the exactly solvable model of which was later found in [87]. Thus we have found how to construct from the noncritical manifold (6.2.0.12) a critical Calabi–Yau manifold.

A class of manifolds of a different type which can be discussed in this framework rather naturally is defined by

$$\mathbb{P}_{(2k, K-k, 2k, K-k, 2k_3, 2k_4, 2k_5)}[2K] \tag{6.2.0.17}$$

where $K = k + k_3 + k_4 + k_5$. Assume, for the moment, that K/k and K/k_i are integers. The potentials then can be chosen to be of the form

$$W = \sum_{i=1}^{2} (x_i^{K/k} + x_i y_i^2) + x_3^{K/k_3} + x_4^{K/k_4} + x_5^{K/k_5}. \tag{6.2.0.18}$$

The singularities in these manifolds are

$$\begin{aligned} \mathbb{Z}_2 &: \quad \mathbb{P}_{(k,k,k_3,k_4,k_5)}[K] \ni \{\sum_i x_i^{K/k_i} = 0\} \\ \mathbb{Z}_{K-k} &: \quad \mathbb{P}_1. \end{aligned} \tag{6.2.0.19}$$

This suggests that there exists a canonical way to proceed to the corresponding Calabi–Yau manifold for these theories by considering manifolds embedded in

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_{(k,k,k_3,k_4,k_5)} \end{array} \begin{bmatrix} 2 & 0 \\ k & K \end{bmatrix} \tag{6.2.0.20}$$

defined by polynomials

$$\begin{aligned} p_1 &= y_1^2 x_1 + y_2^2 x_2 \\ p_2 &= x_1^{K/k} + x_2^{K/k} + x_3^{K/k_3} + x_4^{K/k_4} + x_5^{K/k_5}. \end{aligned} \tag{6.2.0.21}$$

That this correspondence is in fact correct can be inferred from the work of [9] where it was shown that these codimension–2 weighted CICYs correspond to $N = 2$ minimal exactly solvable tensor models of the type

$$\left[2\left(\frac{K}{k} - 1\right)\right]_D^2 \otimes \prod_{i=3}^{5} \left(\frac{K}{k_i} - 2\right)_A. \tag{6.2.0.22}$$

where the subscripts indicate the affine invariants chosen for the individual levels.

The constraints that k and the k_i divide K has been imposed purely for convenience since it allows for a nice coherent description of the resulting class. These restrictions however can be relaxed to more general types of potentials, the discussion of which will take up the remainder of this section.

• **Single tadpole** involving, say, the variables x_4, x_5

$$x_4^a + x_4 x_5^b \tag{6.2.0.23}$$

which leads to the modified divisibility criterion $\frac{K}{k_4}, \frac{1}{k_5}(K - k_4) \in \mathbb{N}$. An example is furnished by

$$\mathbb{P}_{(7,2,7,2,4,4,6)}[16] \ni \{p = \sum_{i=1}^{2} \left(x_i^2 y_i + y_i^8\right) + y_3^4 + y_4^4 + y_4 y_5^2 = 0\} \tag{6.2.0.24}$$

which leads to

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_{(1,1,2,2,3)} \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 8 \end{bmatrix} \ni \left\{ \begin{array}{ll} p_1 = & \sum_{i=1}^2 x_i^2 y_i = 0 \\ p_2 = & y_1^8 + y_2^8 + y_3^4 + y_4^4 + y_4 y_5^2 = 0 \end{array} \right\} \tag{6.2.0.25}$$

• **Isolated double tadpole**

$$x_3^a + x_3 x_4^b + x_4 x_5^c \tag{6.2.0.26}$$

involving three coordinates for which $\frac{K}{k_3}, \frac{1}{k_4}(K - k_3), \frac{1}{k_5}(K - k_4) \in \mathbb{N}$. An example of this type obtains via

$$\mathbb{P}_{(17,2,17,2,8,12,14)}[36] \ni \{p = \sum_{i=1}^{2} \left(x_i^2 y_i + y_i^{18}\right) + y_3^3 y_4 + y_4^3 + y_3 y_5^2 = 0\} \tag{6.2.0.27}$$

from which our construction derives the CY 3–fold

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_{(1,1,4,6,7)} \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 18 \end{bmatrix} \ni \left\{ \begin{array}{ll} p_1 = & \sum_{i=1}^2 x_i^2 y_i = 0 \\ p_2 = & y_1^{18} + y_2^{18} + y_3^3 y_4 + y_4^3 + y_3 y_5^2 = 0 \end{array} \right\} \tag{6.2.0.28}$$

• **Nonisolated tadpoles.** It is not necessary however to have no couplings between x_3, x_4, x_5 from the rest of the coordinates. Examples abound where they a re linked to in a nontrivial way to other variables as well. This is illustrated by the 5–fold

$$\mathbb{P}_{(3,4,2,3,4,2,2)}[10] \ni \{p = \sum_{i=1}^{2} \left(x_i^2 y_i + y_i^2 z_i + z_i^5\right) + z_3^5 = 0\} \tag{6.2.0.29}$$

shows from which with the above construction the CY 3–fold

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_{(2,2,1,1,1)} \end{array} \begin{bmatrix} 2 & 0 \\ 2 & 5 \end{bmatrix} \ni \left\{ \begin{array}{ll} p_1 = & \sum_{i=1}^2 x_i^2 y_i = 0 \\ p_2 = & \sum_{i=1}^2 (y_i^2 z_i + z_i^5) + z_3^5 = 0 \end{array} \right\} \tag{6.2.0.30}$$

can be derived.

• **Exceptional types.** Finally, it should be emphasized that our construction also works for polynomials which contain so–called exceptional terms, an example in point is

$$\mathbb{P}_{(2,4,5,2,4,5,6)}[14] \ni \{p = \sum_{i=1}^{2} \left(x_i^2 y_i + y_i^3 z_i + z_i^7\right) + z_1 u + u y_1^2 = 0\} \tag{6.2.0.31}$$

shows from which with the above construction the CY 3–fold

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_{(2,2,1,1,3)} \end{array} \begin{bmatrix} 2 & 0 \\ 2 & 7 \end{bmatrix} \ni \left\{ \begin{array}{ll} p_1 = & \sum_{i=1}^2 x_i^2 y_i = 0 \\ p_2 = & \sum_{i=1}^2 (y_i^3 z_i + z_i^7) + z_1 u + u y_1^2 = 0 \end{array} \right\} \tag{6.2.0.32}$$

It is clear from the previous discussion that the spaces described by (6.2.0.20) describes a much wider class of models than the known exactly solvable spaces which were addressed in [159].

The picture that emerges from these constructions then is the following: embedded in the higher dimensional manifold is a submanifold which is fibered where the base and the fibres are determined by the singular sets of the ambient manifold. The Calabi–Yau manifold itself is a hypersurface embedded in this fibered submanifold. A heuristic sketch of the geometry is shown in the Figure 1.

The examples above illustrate the simplest situation that can appear. In more complicated manifolds the singularity structure will consist of hypersurfaces whose fibers and/or base themselves are fibered, leading to an iterative procedure. The submanifold to be considered will, in those cases, be of codimension larger than one and the Calabi–Yau manifold will be described by a submanifold with codimension larger than one as well. To illustrate this point consider the 9–fold

$$\mathbb{P}_{(5,5,6,6,6,4,4,4,8,8,8)}[16] \ni \{\sum_{i=1}^{2}\left(u_i^2v_i + v_i^2w_i + w_i^2x_i + x_i^2\right) + v_3^2w_3 + w_3^2x_3 + x_3^2 = 0\}. \tag{6.2.0.33}$$

The $\mathbb{Z}_2$–fibering leads to the split

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_{(3,3,3,2,2,2,4,4,4)} \end{array} \begin{bmatrix} 2 & 0 \\ 3 & 16 \end{bmatrix} \tag{6.2.0.34}$$

which in turn leads to the $\mathbb{Z}_2$ split

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_{(1,1,1,2,2,2)} \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \tag{6.2.0.35}$$

which finally leads to

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \ni \left\{ \begin{array}{ll} p_1 & = \sum u_i^2 v_i \\ p_2 & = \sum v_i^2 w_i \\ p_3 & = \sum w_i^2 x_i \\ p_4 & = \sum x_i^2 \end{array} \right\}. \tag{6.2.0.36}$$

Thus the 9–fold fibers iteratively and the splits of the polynomial p are dictated by the fibering.

Another manifold leading to a codimension larger than 2 critical vacuum that will make another appearence in Section 9 is the 7–fold

$$\mathbb{P}_{(1,1,6,6,2,2,2,2,2)}[8] \ni \left\{\sum_{i=1}^{2}\left(x_i^2y_i + y_iz_i + z_i^4\right) + z_3^4 + z_4^4 + z_5^4 = 0\right\} \tag{6.2.0.37}$$

which leads to the $\mathbb{Z}_2$ fibering $\mathbb{P}_1 \times \mathbb{P}_{(3,3,1,1,1,1,1)}[4]$ which in turn leads to the $\mathbb{Z}_3$ fibering $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_4[4]$ and finally to the Calabi–Yau configuration

$$\begin{array}{l} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_4 \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \ni \left\{ \begin{array}{ll} p_1 = & \sum_{i=1}^{2} x_i^2 y_i = 0 \\ p_2 = & \sum_{i=1}^{2} y_i z_i = 0 \\ p_3 = & \sum_{j=1}^{5} z_i^4 = 0 \end{array} \right\} \tag{6.2.0.38}$$

which is of codimension 3. This example will be of use later in order to show that there are nontrivial relations between these higher dimensional manifolds. One way to see this is via the process of splitting and contraction of Calabi–Yau manifolds introduced in ref. [31] . We will see that in fact the Calabi–Yau manifold (6.2.0.38) is an ineffective split of a Calabi–Yau manifold in the class (6.2.0.17). Thus there also exists a corresponding relation between the higher dimensional manifolds.

6.3 Generalization to arbitrary critical dimensions

Even though the examples discussed in the previous section were all concerned with 6–dimensional Calabi–Yau manifolds and the way they are embedded, in a sense, in the new class of spaces, it should be clear that the ideas presented are not specific to this dimension. Instead of considering compactifications down to the physical dimension, 4, one might contemplate compactifying down to 2, 6 or 8 dimensions, or else, discuss the class of manifolds above not in the context of compactification at all.

To illustrate this point consider the doubly infinite class of $n+1$–dimensional complex compact manifolds

$$\mathbb{P}_{(m+1,n-1,m+1,n-1,\ldots,m+1,n-1,m+1,\cdots,m+1)}[(m+1)n] \qquad (6.3.0.39)$$

with $(m+1)$ pairs of coordinates with weights $(m+1, n-1)$ and $(n-m)$ coordinates of weight $(m+1)$, defined by polynomials

$$p = \sum_{i=1}^{m+1}(x_i^n + x_i y_i^{m+1}) + \sum_{j=m+2}^{n+1} x_j^n. \qquad (6.3.0.40)$$

According to the considerations above these $(n+1$–dimensional spaces lead to Calabi–Yau manifolds embedded in

$$\begin{matrix}\mathbb{P}_m \\ \mathbb{P}_n\end{matrix}\begin{bmatrix}(m+1) & 0 \\ 1 & n\end{bmatrix} \qquad (6.3.0.41)$$

via the equations

$$p_1 = \sum_i y_i^{m+1} x_i, \qquad p_2 = \sum_{i=1}^{n+1} x_i^n. \qquad (6.3.0.42)$$

The simplest example is, of course, the case $n = 2$ where the higher dimensional manifold is a 3–fold described by

$$\mathbb{P}_{(2,1,2,1,2)}[4] \ \ni \ \left\{\sum_{i=1}^{2}(z_i^2 + z_i y_i^2) + z_3^2 = 0\right\} \qquad (6.3.0.43)$$

with a $\mathbb{Z}_2$ singular set isomorphic to the sphere $\mathbb{P}_2[2] \sim \mathbb{P}_1$ which contributes one (1,1)–form, the remaining one being provided by the $\mathbb{P}_1$ defined by the remaining coordinates. The singularity structure of the 3–fold then relates this space to the complex torus described by the algebraic curve

$$\begin{matrix}\mathbb{P}_1 \\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2 & 0 \\ 1 & 2\end{bmatrix}. \qquad (6.3.0.44)$$

The Landau–Ginzburg theory corresponding to this theory derives from an exactly solvable tensor model $(2^2)_{D^2}$ described by two $N = 2$ superconformal minimal theories at level $k = 2$ equipped with the affine D–invariant.

It is of interest to consider the cohomology groups of the 3–fold itself. With the third Chern class $c_3 = 2h^3$ the Euler number of the singular space is

$$\chi_s = \int c_3 = 1 \tag{6.3.0.45}$$

and hence the Euler number of the resolved manifold is

$$\tilde{\chi} = 1 - (2/2) + 2 \cdot 2 = 4. \tag{6.3.0.46}$$

Since the singular set is a sphere its resolution contributes just one (1,1)–form and hence the second Betti number becomes $b_2 = 2$. With $\tilde{\chi} = 2(1 + h^{(1,1)}) - 2h^{(2,1)}$ it follows that $h^{(2,1)} = 1$ and therefore the Hodge diamond becomes

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 0 & & 1 & & 1 & & 0 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The case $n = 3$ is exceptional because it leads to weights that are all equal and hence involves a higher dimensional manifold $\mathbb{P}_5[3]$ that is smooth and hence the naive expectation is that the construction breaks down. This, however, is not the case, the observation being that the polynomial to be considered is not of Fermat type and indeed has a $\mathbb{Z}_2$ symmetry. Instead of considering the smooth manifold $\mathbb{P}_5[3]$ the space to use instead then is the orbifold $\mathbb{P}_5[3]/\mathbb{Z}_2$ and compare the cohomology groups of this space with the spectrum of K3

$$K3 = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_3 \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \tag{6.3.0.47}$$

which consists of $h^{(0,0)} = h^{(2,2)} = h^{(2,0)} = h^{(0,2)} = 1$ and $h^{(1,1)} = 20$, all other Hodge numbers are zero. Hence the Euler number becomes $\chi(K3) = 24$. The Euler number for the noncritical manifold

$$\mathbb{P}_5[3] \tag{6.3.0.48}$$

is easily computed to be $\chi = 27$. Since the manifold is Kähler $h^{(p,q)} = h^{(q,p)}$ and because of Poincaré duality $b_p = b_{8-p}$. Because the manifold has positive first Chern class it follows from Kodaira's vanishing theorem that $h^{(p,0)} = 0$ for $p \neq 0$ and via Lefshetz's hyperplane theorem it is known that below the middle dimension all the cohomology is inherited from the ambient space and therefore the only nonvanishing cohomology groups lead to $h^{(0,0)} = h^{(1,1)} = 1$. It can be shown that $h^{(3,1)} = h^{(1,2)} = 1$ and therefore the only remaining cohomology is in $H^{(2,2)}$. Since

$$\chi = 2(b_0 + b_2) + 2 + h^{(2,2)} = 6 + h^{(2,2)} \tag{6.3.0.49}$$

it follows that $h^{(2,2)} = 21 = 20+1$. The $\mathbb{Z}_2$ orbifold of this space introduces one further $(1,1)$–form and hence we obtain the spectrum of K3 plus one additional mode. Thus the Hodge diamond splits as

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 2 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 2 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
=
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
+
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

This example is also useful because it indicates a generalization of the considerations of the previous section. The surprising new feature of this manifold is that even though the higher dimensional manifold did not have any orbifold singularities it was nevertheless possible to split it in such a way as to construct a Calabi–Yau manifold from it. This was possible because the defining equation was not of Fermat type but involved coupling between the fields. Because of this the manifold featured a new $\mathbb{Z}_2$ symmetry not present in the Fermat hypersurface and it is this new symmetry that dictated how to perform the split. This indicates that even for smooth higher dimensional manifolds it is possible to relate them to Calabi–Yau manifolds once one moves away from the symmetric point.

Similar situations appear whenever

$$n = m + 2 \tag{6.3.0.50}$$

in which case the defining equations in the ordinary projective space are not of Fermat type and feature an additional $\mathbb{Z}_{m+1}$ symmetry that is not induced by the ambient space.

The series (6.3.0.40) can be generalized to weighted spaces as is illustrated by the following example leading to a 4–dimensional critical manifold:

$$p = \sum_{i=1}^{3} \left(x_i^3 + x_i y_i^3 \right) + \sum_{j=4}^{5} x_i^6 \tag{6.3.0.51}$$

corresponds to the tensor model $(16^3 \cdot 4)_{E^3 \otimes A^2}$ with central charge $c = 12$ and belongs to the configuration

$$\mathbb{P}_{(6,4,6,4,6,4,3,3)}[18]. \tag{6.3.0.52}$$

The critical manifold derived from this 6–fold belongs to the configuration class

$$\begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_{(2,2,2,1,1)} \end{array} \begin{bmatrix} 3 & 0 \\ 2 & 6 \end{bmatrix} \tag{6.3.0.53}$$

which is indeed a Calabi–Yau deformation class.

Another infinite class of Calabi–Yau manifolds which are contained in higher dimensional spaces of the type (6.1.0.4) is

$$\mathbb{P}_n[n+1], \qquad n \in \mathbb{N} \tag{6.3.0.54}$$

which are related to $(n+1)$–folds

$$\begin{aligned} &\mathbb{P}_{(1,1,\dots,1,\frac{n+1}{2},\frac{n+1}{2})}[n+1], && n+1 \quad \textit{even} \\ &\mathbb{P}_{(2,2,\dots,2,n+1,n+1)}[2(n+1)], && n+1 \quad \textit{odd}. \end{aligned} \tag{6.3.0.55}$$

For $(n+1)$ even the $\mathbb{Z}_{\frac{n+1}{2}}$–singularity set just consists of 2 points and therefore one obtains two copies of the Calabi–Yau $\mathbb{P}_n[n+1]$ whereas for odd $(n+1)$ the $\mathbb{Z}_{n+1}$ singular set is again two points and the $\mathbb{Z}_2$ singular set is the Calabi–Yau.

The simplest case is $n = 2$ for which the resolution of the orbifold singularities of the noncritical 3–fold

$$\mathbb{P}_{(2,2,2,3,3)}[6] \tag{6.3.0.56}$$

leads to two independent Hodge numbers $h^{(1,1)} = 4$, $h^{(2,1)} = 2$ and hence the Hodge diamond contains *twice* the Hodge diamond of the torus, as it must, according to the geometrical picture described above. Similarly $\mathbb{P}_{(2,2,2,2,2,5,5)}[10]$ leads to two copies of the critical quintic.

The construction is not restricted to the infinite series defined in (6.3.0.40) or its weighted generalization as the next example illustrates. A five–dimensional critical vacuum of higher codimension is obtained by considering the Landau–Ginzburg potential

$$W = \sum_{j=1}^{2} \left(u_i^3 + u_i v_i^2\right) + \sum_{i=3}^{5} \left(u_i^3 + u_i w_i^3\right) \tag{6.3.0.57}$$

which corresponds to the exactly solvable model $(16^3 \cdot 4^2)_{E_7^3 \otimes D^2}$. The nine–dimensional noncritical manifold

$$\mathbb{P}_{(3,2,3,2,3,2,3,3,3,3)}[9] \tag{6.3.0.58}$$

leads, via its singularity structure, to the five–dimensional critical manifold

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_4 \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 3 \end{bmatrix}. \tag{6.3.0.59}$$

It is crucial that a polynomial was chosen which is not of Brieskorn–Pham type for the last four coordinates in the noncritical manifold.

There are of course other types of weighted n–folds which solve the conditions (6.1.0.4). An example that does not belong the class above is

$$\mathbb{P}_{(2,1,2,1,1,1)}[4] \ni \left\{ \sum_{i=1}^{2} \left(x_i^2 + x_i y_i^2 \right) + x_5^4 + x_6^4 = 0 \right\} \tag{6.3.0.60}$$

which leads to the critical surface

$$\begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_{(2,2,1,1)} \end{matrix} \begin{bmatrix} 2 & 0 \\ 2 & 4 \end{bmatrix}. \tag{6.3.0.61}$$

The cohomology of the noncritical 4–fold is easy to determine: since the $\mathbb{Z}_2$–singular set consists of two points the resolution of which introduces two (1,1)–forms. Thus $h^{(1,1)} = 3$. With $c_4 = 25h^4$ one finds $\chi_s = 25$ and hence

$$\chi = 2(1 + h^{(1,1)}) - 2 = 25 - 1 + 4 = 28 \tag{6.3.0.62}$$

Thus there appears, in general, not just one additional Kähler mode in the spectrum of the higher dimensional manifold but more new fields whose (possible) interpretation remains obscure at this point.

An example involving a 4–dimensional critical manifold not of the type just considered starts from the Landau–Ginzburg potential

$$W = \sum_{i=1}^{3} \left(x_i^3 + x_i y_i^3 \right) + \sum_{j=4}^{5} x_i^6 \tag{6.3.0.63}$$

with central charge $c = 12$ which corresponds to the tensor model $16_E^3 4_A^2$ and belongs to the configuration

$$\mathbb{P}_{(6,4,6,4,6,4,3,3)}[18]. \tag{6.3.0.64}$$

The 4–dimensional manifold derived from this LG configuration belongs to the configuration class

$$\begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_{(2,2,2,1,1)} \end{matrix} \begin{bmatrix} 3 & 0 \\ 2 & 6 \end{bmatrix} \tag{6.3.0.65}$$

which is indeed a Calabi–Yau deformation class.

6.4 Cohomology and physical modes

In this section I will describe in a more coherent way the cohomology of the higher dimensional manifolds and its relation to the critical spectrum. The results described in the previous sections suggest that the cohomology of the critical vacuum (NOT described by a Calabi–Yau manifold in general) is embedded in the cohomology of the higher dimensional space as:

$$H^{(p,q)}(V_{D_{crit}}) \subset H^{(p+Q-1,q+Q-1)}(\mathcal{M}_{D_{crit}+2(Q-1)}) \tag{6.4.0.66}$$

or, put differently, there exists a projection π such that

$$\pi\left(H^{(p,q)}(\mathcal{M}_{D_{crit}+2(Q-1)})\right) = H^{(p-(Q-1),q-(Q-1))}(V_{D_{crit}}) \tag{6.4.0.67}$$

and what has been shown in this chapter is to describe a geometrical framework for this projection for a number of classes of manifolds.

There are two main results that follow from our geometrical construction. First:

- The critical spectrum is determined by the cohomology carried by the singular sets of the higher dimensional manifolds as well as the polynomial (chiral) ring.

This is in contrast to Calabi–Yau manifolds where the spectrum is described by the *resolution* of the singularities and the polynomial ring. From this an immediate consequence is that

- The cohomology of the higher dimensional ambient weighted projective space does not contribute to the critical spectrum.

By using a combination of different techniques it is possible to compute the Euler number and, more detailed, the cohomology groups of the higher dimensional spaces. First one may recall that one way to compute the Euler number of weighted CICYs is to consider them as coverings of the weighted spaces themselves.

Consider a Fermat configuration

$$\mathbb{P}_{(k_1,\ldots,k_{N+2})}[d] \ni p = \sum_i z_i^{a_i}. \tag{6.4.0.68}$$

Write this as

$$z_1^{a_1} = -z_2^{a_2} - \cdots - z_{N+2}^{a_{N+2}}. \tag{6.4.0.69}$$

Clearly this describes an a_1–fold covering of

$$\mathbb{P}_{(k_2,\ldots,k_{N+2})} \tag{6.4.0.70}$$

as long as p_1 does not vanish identically, the solution simply being

$$z_1 = \left(p_1(z_2,\ldots,z_{N+2})\right)^{1/a_1}, \qquad (z_2,\ldots,z_{N+2}) \in \mathbb{P}_{(k_2,\ldots,k_{N+2})}. \tag{6.4.0.71}$$

When $p_1 = 0$ then the only solution is $z_1 = 0$. Thus the space of solutions of the equation $p = 0$ becomes a a_1–fold covering of $\mathbb{P}_{(k_2,\ldots,k_{N+2})}$ branched over the hypersurface

$$\mathbb{P}_{(k_2,\ldots,k_{N+2})}[d]. \tag{6.4.0.72}$$

This is an example of a general situation: if M is an n–fold covering of the space N branched over a submanifold $B \in N$ then the Euler number of the spaces involved are related via

Riemann–Hurwitz: $\chi(M) = n\chi(N) - (n-1)\chi(B)$.

This is the result when M is a smooth manifold. In the case at hand however the manifold is singular and Riemann–Hurwitz computes the Euler number for the singular manifold which has to be resolved: if the manifold has orbifold singularities S_i with respect to cyclic groups $\mathbb{Z}_{n_i}$ of order n_i then the Euler number of the resolved manifold $\tilde{M}$ is given by

$$\begin{aligned} \chi(\tilde{M}) &= \chi(M) + \sum_i (n_i - 1)\chi(S_i) \\ &= n\chi(N) - (n-1)\chi(B) + \sum_i (n_i - 1)\chi(S_i). \end{aligned} \tag{6.4.0.73}$$

A simple but useful example because of matters of principle is the 3–fold

$$\mathbb{P}_{(2,2,2,3,3)}[6] \ni \left\{ \sum_i x_i^3 + \sum y_j^2 = 0 \right\} \tag{6.4.0.74}$$

with the singular curve

$$\mathbb{Z}_2 : \quad \mathbb{P}_2[3], \qquad \chi_3 = 0 \tag{6.4.0.75}$$

which describes the torus, and the singular points

$$\mathbb{Z}_3 : \quad \mathbb{P}_1[2] = 2 \text{ pts}. \tag{6.4.0.76}$$

Thus the fiber collapses to two points in this example and the critical manifold becomes

$$\begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \tag{6.4.0.77}$$

i.e. it consists of two copies of the torus.

The noncritical spectrum can be obtained as follows: first one computes via RH the Euler number, which turns out to be

$$\chi_s = 2 \cdot 4 - \chi(\mathbb{P}_{2,2,2,3)}[6]) = 8 - 2 \cdot 3 + \chi(\mathbb{P}_2[3]) = 2 \tag{6.4.0.78}$$

which, when resolved, leads to

$$\chi = -\chi_s + \sum_i (n_i - 1)\chi(S_i) = 2 + 2 \cdot 2 = 6. \qquad (6.4.0.79)$$

Checking this result with the standard resolution formula for the Euler number, using $c_3 = 8h^3$, it follows that

$$\chi = \frac{8 \cdot 6}{8 \cdot 9} - \frac{2}{3} + 3 \cdot 2 = 6 \qquad (6.4.0.80)$$

which agrees with the Riemann–Hurwitz result.

Resolving these singularities leads to the surface with the local fibering

$$\mathbb{P}_2[3] \times \mathbb{P}_1, \qquad (6.4.0.81)$$

which, via Künneth, leads to one additional (1,1)–form wheras the resolution of the two $\mathbb{Z}_3$ points introduces 2 additional (1,1)–forms. Using the result for the Euler number then leads to

$$\chi = 6 = 2(1 + 4) - 2h^{(2,1)} \qquad (6.4.0.82)$$

and hence to $h^{(4,1)} = 4$.

Thus the Hodge diamond of the 3–fold is completely determined and leads to (including the 1 monomial of charge 1)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 4 & & 0 & \\ 0 & & 2 & & 2 & & 0 \\ & 0 & & 4 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

This example is important because it shows that the degeneration phenomenon also occurs for critical vacua in the physical dimension, i.e. for special Fano folds that lead to Calabi–Yau manifolds of dimension $dim_{\mathbb{C}} M = 3$. Thus it follows that also in this context, the higher dimensional analog of the critical holomorphic $(D_{crit}, 0)$ is not always 1–dimensional, the simple reason being, as mentioned before, that multiple copies of the critical manifold can be embedded in the higher dimensional space.

An example that leads to a phenomenological interesting string vacuum is furnished by

$$\mathbb{P}_{(2,2,2,3,3,3,3)}[9] \ni \left\{ \sum_i \left(x_i^3 y_i + y_i^3 \right) + x_4^3 = 0 \right\} \qquad (6.4.0.83)$$

with the two singular surfaces

$$\mathbb{Z}_2 : \quad \mathbb{P}_2 : \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}, \qquad \mathbb{Z}_3 : \quad \mathbb{P}_3[3] : \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{6.4.0.84}$$

When resolved, these lead to the 3–folds

$$\mathbb{P}_2 \times \mathbb{P}_1, \qquad \mathbb{P}_3[3] \times \mathbb{P}_1 \tag{6.4.0.85}$$

respectively. Using Künneth's formula these 3–folds lead the Hodge diamond of the resolved space

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 3 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 9 & & 0 & & 0 & \\
0 & & 1 & & 35 & & 35 & & 1 & & 0 \\
 & 0 & & 0 & & 9 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 3 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

which determine the Euler number to be

$$\chi = 2(b_0 - b_1 + b_2 - b_3 + b_4) - b_5 = -46. \tag{6.4.0.86}$$

This example is misleading in one respect however: all of the critical generations are parametrized by the chiral ring itself, i.e. the monomials of charge 1. This translates into the fact that all complex deformations of the critical Calabi–Yau space

$$\begin{array}{c} \mathbb{P}_2 \\ \mathbb{P}_3 \end{array} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \tag{6.4.0.87}$$

which derives from this Fano fold via our construction are indeed parametrized by the monomials of the polynomials.

In many cases however these monomials fail to parametrize all complex deformations, a simple example furnished by the complete intersection Calabi–Yau manifold [93]

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_4 \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}. \tag{6.4.0.88}$$

This critical manifold can be derived from the higher dimensional space

$$\mathbb{P}_{(2,2,2,2,2,3,3)}[8] \ni \left\{ \sum_{i=1}^{2} \left(x_i^4 + x_i y_i^2 \right) + x_3^4 + x_4^4 + x_5^4 = 0 \right\} \tag{6.4.0.89}$$

the analysis of which shows that indeed the 'missing' complex deformations, not parametrized by the complex deformations, originate from the singularities of the higher dimensional space.

It must be emphasized that the computation of the critical spectrum from the higher dimensional space does not depend on the ability to find an embedding of a 3–dimensional CY space. Consider, e.g.

$$\mathbb{P}_{(1,1,2,2,2,2,2)}[6] \ni \{z_1^6 + z_2^6 + \sum_i z_i^3 = 0\} \tag{6.4.0.90}$$

for which RH leads to the singular Euler number

$$\chi_s = 3\cdot 6 - 2\chi(\mathbb{P}_{1,1,2,2,2,2)}[6]) = 18 - 2\{3\cdot 5 - 2\chi(\mathbb{P}_{(1,1,2,2,2)}[6])\} = \cdots = -132 \tag{6.4.0.91}$$

which leads to the resolved Euler number

$$\chi = -132 + \chi(\mathbb{P}_4[3]) = -138 \tag{6.4.0.92}$$

after blowing up the singular cubic 3–fold.

From the singular cubic 3–fold, which has Euler number -6 one obtains immediately that the number of critical (1,1)–forms is 1 and that there are 5 additional critical (2,1)–modes coming from the singularity, in addition to the 68 number of monomial deformations. Thus for the total critical spectrum one obtains

$$(h^{(1,1)}_{crit}, h^{(2,1)}_{crit})(\mathbb{P}_{(1,1,2,2,2,2,2)}[6]) = (1, 73). \tag{6.4.0.93}$$

It is also of interest to compute the cohomology of the noncritical manifold. First one might check the Euler number via a Chern computation: with $c_5 = -688h^5$ one obtains for the singular space $\chi_s = -129$ which, after resolving the cubic 3–fold $\mathbb{P}_4[3]$ (which has Euler number -6) leads to $\chi = -138$ as computed with RH.

In order to get the cohomology of the ambient space it is just necessary to compute, via Künneth the Hodge diamond of the bundle $\mathbb{P}_3[4] \times \mathbb{P}_1$, leading to one additional (1,1)–form. Using the result for the Euler number one finds

$$-138 = \chi = 2(b_0 + b_2 + b_4) - b_5 = 10 - 2h^{(4,1)} - 146 \tag{6.4.0.94}$$

and hence $h^{(4,1)} = 1$, as expected. Thus the Hodge diamond

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
0 & & 1 & & 73 & & 73 & & 1 & & 0 \\
 & 0 & & 0 & & 2 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

is obtained.

The general Hodge–diamond for odd–dimensional manifolds $dim_{\mathbb{C}} M = 2p+1$

$$\begin{array}{c}
1 \\
0 \qquad\qquad 0 \\
0 \qquad\quad h^{(1,1)} \qquad\quad 0 \\
0 \qquad 0 \qquad\qquad 0 \qquad 0 \\
0 \qquad 0 \qquad h^{(2,2)} \qquad 0 \qquad 0 \\
0 \qquad 0 \qquad 0 \qquad\qquad 0 \qquad 0 \qquad 0 \\
\vdots \\
\vdots \\
0 \quad \cdots \quad 0 \qquad h^{(p,p)} \qquad 0 \quad \cdots \quad 0 \\
0 \quad \cdots \quad 0 \quad n \quad h^{(p+1,p)} \qquad h^{(p,p+1)} \quad n \quad 0 \quad \cdots \quad 0
\end{array}$$

whereas for even–dimensional manifolds $\dim_{\mathbb{C}} = 2p$ it is of the form

$$\begin{array}{c}
1 \\
0 \qquad\qquad 0 \\
0 \qquad h^{(1,1)} \qquad 0 \\
0 \qquad 0 \qquad\qquad 0 \qquad 0 \\
0 \qquad 0 \qquad h^{(2,2)} \qquad 0 \qquad 0 \\
0 \qquad 0 \qquad 0 \qquad\qquad 0 \qquad 0 \qquad 0 \\
\vdots \\
\vdots \\
0 \quad \cdots \quad 0 \quad n \quad h^{(p,p)} \quad n \quad 0 \quad \cdots \quad 0
\end{array}$$

Even though, from an abstract point of view, it is not knownOne universal featur e that the geometrical construction tells us is that in the transition from the higher dimensional spectrum to the critical spectrum, described by the generations and antigenerations, the cohomology of the ambient projective space does not contribute.

6.5 When $h^{D_{crit}+Q-1,Q-1)} > 1$: the degeneration phenomenon.

It is tempting to try a cohomological definition of the special class of Fano folds (6.1.0.4) as a class of spaces for which there exists precisely one analog of the holomorphic three–form which exists in critical string vacua. This however would be fallacious.

Consider the example

$$\mathbb{P}_{(2,2,2,2,2,5,5)}[6] \ni \left\{\sum_i x_i^3 + \sum y_j^2 = 0\right\} \tag{6.5.0.95}$$

whose singular sets consist of the 3–fold

$$\mathbb{Z}_2: \quad \mathbb{P}_4[5], \quad \chi = -200 \tag{6.5.0.96}$$

and the two points

$$\mathbb{Z}_5: \quad \mathbb{P}_1[2] = 2 \text{ pts.} \tag{6.5.0.97}$$

Thus the fiber collapses to two points in this example and the critical manifold becomes

$$\begin{array}{c} \mathbb{P}_4 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \tag{6.5.0.98}$$

i.e. it consists of two copies of the quintic.

This *degeneration phenomenon* also shows up through an analysis of the Hodge structure of the manifold. The noncritical spectrum can be obtained as follows: first one computes via RH the Euler number, which turns out to be

$$\chi_s = 2 \cdot 6 - \chi(\mathbb{P}_{2,2,2,2,2,3)}[10]) = -198 \tag{6.5.0.99}$$

which, when resolved, leads to

$$\chi = -\chi_s + \sum_i (n_i - 1)\chi(S_i) = -198 + (-200) + 4 \cdot 2 = -390. \tag{6.5.0.100}$$

This result can be checked with the resolution formula for the Euler number: with $c_5 = -7968h^5$ it follows that

$$\chi = -\frac{498}{5} - \frac{-200}{2} + 2(-200) - \frac{2}{5} + 5 \cdot 2 = -390. \tag{6.5.0.101}$$

In more detail, by resolving these singularities one is locally lead to the 4–fold

$$\mathbb{P}_4[5] \times \mathbb{P}_1 \tag{6.5.0.102}$$

which, via Künneth, leads to one additional (1,1)–form whereas the resolution of the two $\mathbb{Z}_5$ points introduces 2 additional (1,1)–forms. Using the above result for the Euler number then leads to

$$\chi = -390 = 2(1 + 4 + 4) - 2h^{(4,1)} - 404 \tag{6.5.0.103}$$

and hence to $h^{(4,1)} = 2$.

Thus the Hodge diamond of the 5–folds is completely determined and leads to

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 4 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 4 & & 0 & & 0 & \\
0 & & 2 & & 202 & & 202 & & 2 & & 0 \\
 & 0 & & 0 & & 4 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 4 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}
$$

(including the 101 monomials of charge unity).

This example is important because it shows two things: the first is that in fact $h^{(4,1)} = 2$! This indicates that, contrary to what one might expect, the higher dimensional analog of the critical holomorphic $(D_{crit}, 0)$ is not always 1–dimensional, the simple reason being that multiple copies of the critical manifold can be embedded in the higher dimensional space. The second point is that contrary to the case of smooth noncritical manifolds singular spaces have a much more complicated cohomology and hence such manifolds lead, in general, to many more states than the critical spectrum.

6.6 $Q \in \mathbb{N}$: Splitting and Contraction

It is shown in thhis section is to point out that the relation between higher dimensional manifolds and critical vacua is not 1–1. Instead many higher dimensional manifolds are related to one and the same critical vacuum. In general even an infinite number.

One way to see this is via the process of splitting and contraction introduced in [31]. The discussiong of ref. [31] is restricted to Calabi–Yau manifolds embedded in products of ordinary projective spaces but it readily generalizes to the more general framework of weighted projective spaces.

It is useful to start with an example. Consider the 7–fold

$$\mathbb{P}_{(1,1,10,10,2,2,2,2,6)}[12] \ni \left\{ \sum_{i=1}^{2} \left(x_i^2 y_i + y_i z_i + z_i^6 \right) + z_3^6 + z_4^6 + z_5^2 = 0 \right\} \qquad (6.6.0.104)$$

where $(x_1, x_2, y_1, y_2, z_1, z_2, ..., z_5)$ are the coordinates of the ambient weighted space. This space leads, via an iterative construction, as described in the previous section, to the critical manifold

$$\begin{array}{l} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 6 \end{bmatrix} \tag{6.6.0.105}$$

with the by now familiar split of the polynomial.

Using the same analysis as described in [31] it is however easy to see that this critical manifold is in fact isomorphic to the Calabi–Yau manifold

$$\begin{array}{l} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix} \tag{6.6.0.106}$$

described by the polynomials

$$\begin{aligned} p_1 &= x_1^2 y_1 + x_2^2 y_2 \\ p_2 &= y_1^6 + y_2^6 + y_3^6 + y_4^6 + y_5^2 \end{aligned} \tag{6.6.0.107}$$

which can be derived via our construction from the higher dimensional manifold

$$\mathbb{P}_{(5,5,2,2,2,2,6)}[12] \ni \left\{ \sum_{i=1}^{2} \left(x_i^2 y_i + y_i^6 \right) . + y_3^6 + y_4^6 + y_5^2 = 0 \right\} \tag{6.6.0.108}$$

where $(x_1, x_2, y_1, y_2, ..., y_5)$ are the homogeneous coordinates of the 6–dimensional weighted space. Manifold (6.6.0.105) namely is the so–called [31] split of the manifold (6.6.0.106) and in special circumstances, which arise in this case, the splitting or contraction process does not change the theory but rather provides different representations of the same manifold.

It turns out in fact that this manifold can be rewritten via ineffective splits into an infinite sequence of different representations as

$$\begin{array}{l} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix} \longrightarrow \begin{array}{l} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 6 \end{bmatrix} \longrightarrow \begin{array}{l} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,3)} \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix} \longrightarrow \cdots \tag{6.6.0.109}$$

which can be derived from a corresponding sequence of higher dimensional spaces

$$\mathbb{P}_{(5,5,2,2,2,2,6)}[12] \longrightarrow \mathbb{P}_{(1,1,10,10,2,2,2,2,6)}[12] \longrightarrow \mathbb{P}_{(5,5,2,2,10,10,2,2,2,2,6)}[12] \longrightarrow \cdots \tag{6.6.0.110}$$

i.e. the infinite sequence belongs to the spaces

$$\mathbb{P}_{(5,5,2,2,10,10,2,2,10,10,...,2,2,10,10,2,2,6)}[12] \tag{6.6.0.111}$$

where the part $(2, 2, 10, 10)$ occurs k times, and

$$\mathbb{P}_{(1,1,10,10,2,2,10,10,2,2,...10,10,2,2,2,2,6)}[12]. \tag{6.6.0.112}$$

where $(10, 10, 2, 2)$ occurs k times.

The construction above easily generalizes to a number of examples which all belong to a class of spaces [159] discussed above. Consider manifolds embedded in

$$\begin{array}{c} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{array} \begin{bmatrix} 2 & 0 \\ k_1 & k \end{bmatrix}, \tag{6.6.0.113}$$

where $k = k_1 + k_3 + k_4 + k_5$. These spaces can be split into the infinite sequences

$$\begin{array}{c} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{array} \begin{bmatrix} 2 & 0 \\ k_1 & k \end{bmatrix} \longrightarrow \begin{array}{c} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & k_1 & k \end{bmatrix} \longrightarrow \begin{array}{c} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k_1,k_1,k_3,k_4,k_5)} \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k_1 & k \end{bmatrix} \longrightarrow \cdots \tag{6.6.0.114}$$

with the corresponding sequence of higher dimensional spaces given by

$$\mathbb{P}_{(k-k_1,k-k_1,2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \longrightarrow \mathbb{P}_{(k_1,k_1,2(k-k_1),2(k-k_1),2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \longrightarrow \cdots \tag{6.6.0.115}$$

i.e. they belong to the set of spaces of type

$$\mathbb{P}_{((k-k_1),(k-k_1),2k_1,2k_1,2(k-k_1),2(k-k_1),....,2k_1,2k_1,2k_3,2k_4,2k_5)}[2k] \tag{6.6.0.116}$$

where the part $(2k_1, 2k_1, 2(k-k_1), 2(k-k_1))$ occurs p times, and

$$\mathbb{P}_{(k_1,k_1,2(k-k_1),2(k-k_1),2k_1,2k_1,...,2k_3,2k_4,2k_5)}[2k]. \tag{6.6.0.117}$$

where $(2(k-k_1), 2(k-k_1), 2k_1, 2k_1)$ occurs p times.

6.7 Topological Identities between Critical Manifolds

It is worth remarking that via this construction it is possible to also discuss manifolds that do not appear to admit an analysis along these lines. The reason are topological identities [31] which allow to represent complete intersection Calabi–Yau manifolds in different ways. Consider again the codimension–four Calabi–Yau manifold

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{6.7.0.118}$$

from chapter 3 (3.3.0.31), with the defining polynomials

$$\begin{array}{rclrcl} p_1 & = & \sum_{i=1}^{2} u_i^2 v_i, & p_2 & = & \sum_{i=1}^{3} v_i^2 w_i \\ p_3 & = & \sum_{i=1}^{3} w_i^2 x_i, & p_4 & = & \sum_{i=1}^{3} x_i^2 \end{array} \tag{6.7.0.119}$$

which can be derived from

$$\mathbb{P}_{(5,5,6,6,6,4,4,4,8,8,8)}[16] \tag{6.7.0.120}$$

as discussed in Section 4.

This manifold can be represented in different ways by using a well–known representation of the sphere as a quadric in the projective plane

$$S^2 = \mathbb{P}_1 = \mathbb{P}_2[2]. \tag{6.7.0.121}$$

Using this, the manifold can be rewritten as

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}. \tag{6.7.0.122}$$

Furthermore one can use the surface identity [31]

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \end{array} \tag{6.7.0.123}$$

via the application rule

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ X \end{array} \begin{bmatrix} 2 & 0 \\ 1 & a \\ 0 & M \end{bmatrix} = \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ X \end{array} \begin{bmatrix} a \\ a \\ M \end{bmatrix} \tag{6.7.0.124}$$

to obtain the representation

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \tag{6.7.0.125}$$

whose defining polynomials are some deformations of

$$\begin{aligned} p_1 &= \sum_i u_i^2 v_i^2 y_i \\ p_2 &= \sum_i x_i^2 y_i^2. \end{aligned} \tag{6.7.0.126}$$

The natural ansatz for the higher dimensional polynomial

$$p = p_1 + p_2 \tag{6.7.0.127}$$

clearly does not have an isolated singularity and hence does not seem to have an interpretation as defining a quasihomogeneous hypersurface in some weighted projective space. In fact it is not even possible to assign weights to the order parameters such that the central charge of this model is nine and hence describes a critical vacuum. But because this manifold is diffeomorphic to the variety (6.2.0.36) it does admit, via this equivalence, an embedding into a higher dimensional space.

6.8 Toric Considerations

The simplest of all Calabi–Yau spaces surely is the set

$$\mathbb{P}_1[2] \tag{6.8.0.128}$$

which happens to consist of two points exactly. The higher dimensional noncritical manifold of this space is $\mathbb{P}_3[2]$ whose toric analysis is what follows.

Writing

$$\begin{aligned} p &= x_1^2 + \cdots + x_4^2 + x_1x_2 + \cdots x_3x_4 \\ &= z_3z_4\left(\frac{z_1^2}{z_3z_4} + \frac{z_2^2}{z_3} + \frac{z_3}{z_4} + \frac{z_4}{z_3} + \frac{z_1z_2}{z_3z_4} + \frac{z_1}{z_4} + \frac{z_1}{z_3} + \frac{z_2}{z_3} + 1\right) \end{aligned} \tag{6.8.0.129}$$

leads (in $U_4 = \{x_4 = 1\}$) to the integral points

$$(2,0,-1),(0,2,-1),(0,0,1),(0,0,-1),(1,1,-1),(1,0,0),(1,0,-1),(0,1,0),(0,1,-1),(0,0,0) \tag{6.8.0.130}$$

which span by a polyhedron whose vertices are

$$\Delta = \{(2,0,-1),(0,2,-1),(0,0,1),(0,0,-1)\}. \tag{6.8.0.131}$$

This polyhedron has no integral point in its interior and hence one deduces from it that $h^{(2,0)} = 0$ for the surface which is good (and which is something we knew because the surface has positive first Chern class.) To obtain

$$h^{(1,1)}(\bar{Z}) = l^*(2\Delta) - 4l^*(\Delta) - 3 - \sum_{codim\ \Gamma=1} (l^*(\Gamma) - 1) \tag{6.8.0.132}$$

the first task is to count the number of integral points in the interior of the polyhedron 2Δ which is panned by

$$2\Delta = \{(4,0,-2),(0,4,-2),(0,0,2),(0,0,-2)\} \tag{6.8.0.133}$$

After some computing one finds that there is only on interior integral point $(1,1,-1)$ and hence

$$l^*(2\Delta) = 1 \tag{6.8.0.134}$$

There are 4 two–dimensional faces with no interior integral points and therefore

$$h^{(1,1)}(\bar{Z}) = 1 - 3 - 4(-1) = 2 \tag{6.8.0.135}$$

which we also knew because $\mathbb{P}_3[2] = \mathbb{P}_1 \times \mathbb{P}_1$.

The analysis of the noncritical manifolds suggests that the noncritical manifolds defined above lend themselves to a toric description in terms of polyhedra Δ which are characterized by the condition that the Q–fold has precisely one interior integral point

$$l^*(Q\Delta) = 1. \tag{6.8.0.136}$$

6.9 The inear σ–Model

It clearly would be of interest to find a σ–model theoretic framework for our construction. In [179] there is an analysis of models that do not correspond to CY spaces. This analysis pertains to models which may be closely related to **noncritical** manifolds. Since the analysis there is only concerned ed with Kähler modes originating from the ambient space it is simples to restrict attention to smooth manifolds. These contain, of course, manifolds describing mirrors of rigid Calabi–Yau manifolds and therefore are the most puzzling.

The starting point of the analysis in [179] is a U(1) gauge theory in N=2 superspace, extending the standard Landau–Ginzburg action for the chiral N=2 superfields to

$$L = L_{kin,V} + L_{kin,\Phi_i} + L_{W,\Phi_i} + L_{D,\theta}. \tag{6.9.0.137}$$

Consider, in the notation of [179], the gauge invariant field strength

$$\Sigma = \frac{1}{2\sqrt{2}}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\} \tag{6.9.0.138}$$

the kinetic term of which is given by

$$\mathcal{A}_g = -\frac{1}{4e^2}\int d^2z d^4\theta\ \bar{\Sigma}\Sigma. \tag{6.9.0.139}$$

There are two possible interactions, the θ angle and the Fayet–Illiopoulos D–term. These can be writen as

$$\mathcal{A}_{\mathrm{D},\theta} = \frac{it}{2\sqrt{2}}\int d^2z d\theta^+ d\bar{\theta}^-\ \Sigma + h.c. \tag{6.9.0.140}$$

where

$$t = ir + \frac{\theta}{2\pi} \tag{6.9.0.141}$$

where r is the coefficient of the D–term.

To this are added N chiral superfields with U(1)–charge Q_i. The kinetic energy of these fields is chosen to be

$$\mathcal{A}_m = \int d^2z d^4\theta\ \sum_i \Phi_i \Phi_i \tag{6.9.0.142}$$

and the superpotential is assumed to be of gauge invariant form

$$\mathcal{A}_W = -\int d^2 z d^2\theta \; W(\Phi_i) - h.c. \tag{6.9.0.143}$$

which is supersymmetric because the Φ_i are chiral and W is holomorphic.

The constant part of the lowest components of the superfields Φ_i will be thought of as parametrizing $\mathbb{C}_n$. Assuming, as is done in [179], the Kähler metric in the kinetic term of the Φ_i should be flat.

The bosonic equations of motion for the auxiliary fields D_a and F_i become

$$D = -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right) \tag{6.9.0.144}$$

$$F_i = \frac{\partial W}{\partial \phi_i}. \tag{6.9.0.145}$$

The bosonic potential that one obtains in terms of the matter fields ϕ_i and the auxiliary fields D_a and F_i is

$$U(\phi_i,\sigma) = \frac{1}{2e^2}D^2 + \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + 2|\sigma|^2 \sum_i Q_i^2|\phi_i|^2. \tag{6.9.0.146}$$

Including the one one–loop correction of the theory leads to a σ–dependent effective r

$$r_{\text{eff}} \sim r + \frac{\sum_i Q_i}{2\pi} ln\left(\frac{|\sigma|}{\mu}\right). \tag{6.9.0.147}$$

Assume now that the superpotential takes the form

$$W(\Phi_i) = \Phi_0 \widetilde{W}(\Phi_1, ..., \Phi_N) \tag{6.9.0.148}$$

where $\widetilde{W}$i is a quasihomogeneous polynomial which is assumed to be transverse, i.e. the equations

$$\frac{\partial \widetilde{W}}{\partial \Phi_i} = 0 \tag{6.9.0.149}$$

can only be solved at the origin.

With this potential the bosonic potential becomes

$$U(\phi_i,\sigma) = \frac{1}{2e^2}D^2 + \left|\widetilde{W}\right|^2 + |\phi_0|^2 \sum_i \left|\frac{\partial \widetilde{W}}{\partial \phi_i}\right|^2 + 2|\sigma|^2\left(\sum_i k_i^2|\phi_i|^2 + k_0^2|\phi_0|^2\right) \tag{6.9.0.150}$$

with

$$D = -e^2\left(\sum_i k_i|\phi_i|^2 - k_0\Phi_0\bar{\Phi}_0 - r_{\text{eff}}\right). \tag{6.9.0.151}$$

Note that all terms in (6.9.0.150) are ≥ 0. Thus in order to minimize the potential U one has to minimize D^2 which means different things, depending on what the variable r does. Assume now, that all chiral superfields ϕ_i have charge unity under the U(1) gauge group.

The $r_{\text{eff}} = r >> 0$ Phase: In this case not all ϕ_i can be zero. But since the polynomial p is transverse everywhere except at the origin this means, because $\partial\widetilde{W}$ is nonzero for some i, that ϕ_0 must be zero. Thus $D = 0$ leads to

$$\sum_{i=1}^{N}\bar{\phi}_i\phi_i = r. \tag{6.9.0.152}$$

This simply defines a sphere $S^{2N-1} \subset \mathbb{C}_N$. Recalling that one has to mod out the U(1) gauge group and also that the sphere S^{2N-1} can be Hopf–fibered $S^1 \longrightarrow S^{2N-1} \longrightarrow \mathbb{P}_{N-1}$ leads to the condition that the constant bosonic components of Φ_i parametrize a projective space $\mathbb{P}_{N-1} = S^{2N-1}/U(1)$. Furthermore the vanishing of $\widetilde{W}$ leads to a geometry for the space of groundstates described by a hypersurface of degree $\widetilde{W}$ embedded in $\mathbb{P}_{N-1}$.

The $r_{\text{eff}} = r << 0$ Phase: In this case the vanishing of D leads to $\phi_0 \neq 0$ and hence the term $|\phi_0|^2\sum_i|\partial_i\widetilde{W}|^2$ enforces that $\phi_i = 0$ since this is the only place where the partials are allowed to vanish, because of transversality. This fixes the modulus of ϕ_0

$$|\phi_0| = \sqrt{-r}N. \tag{6.9.0.153}$$

Because of the gauge invariance the classical vacuum is in fact unique, modulo these gauge transformations. Expanding around this vacuum leads to massless ϕ_i (for $N \geq 3$). To find the potential for these massless fields one has to integrate out the massive field ϕ_0. Integrating out ϕ_0 simply means in this case setting ϕ_0 to its expectation value. Thus the effective superpotential of the low energy theory is

$$\widetilde{W} = \sqrt{-r}W(\phi_i). \tag{6.9.0.154}$$

The factor $\sqrt{-r}$ is inessential since it can be absorbed by rescaling the ϕ_i. Since the origin is a multicritical point, this describes a Landau–Ginzburg theory. More precisely it describes a Landau–Ginzburg orbifold.

Applying this analysis to the two smooth higher dimensional manifolds $\mathbb{P}_8[3]$ and $\mathbb{P}_7[4]$ leads to the conclusion that there are 6, respectively 4, additional modes in these theories that would not be present if we were to consider Calabi–Yau spaces described by polynomials of degree 9 and 8. These numbers seems to fit perfectly when considering the Hodge diamond decomposition of noncritical into critical cohomology: for the cubic in $\mathbb{P}_8$ this leads to the split

$$
\begin{array}{ccccccccccccccc}
&&&&&&&1\\
&&&&&&0&&0\\
&&&&&0&&1&&0\\
&&&&0&&0&&0&&0\\
&&&0&&0&&1&&0&&0\\
&&0&&0&&0&&0&&0&&0\\
&0&&0&&0&&1&&0&&0&&0\\
0&&0&&1&&84&&84&&1&&0&&0\\
&0&&0&&0&&1&&0&&0&&0\\
&&0&&0&&0&&0&&0&&0\\
&&&0&&0&&1&&0&&0\\
&&&&0&&0&&0&&0\\
&&&&&0&&1&&0\\
&&&&&&0&&0\\
&&&&&&&1
\end{array}
$$

whereas the quartic in $\mathbb{P}_7$ leads to

$$
\begin{array}{ccccccccccc}
&&&&&1\\
&&&&0&&0\\
&&&0&&1&&0\\
&&0&&0&&0&&0\\
&0&&0&&1&&0&&0\\
0&&1&&90&&90&&1&&0\\
&0&&0&&1&&0&&0\\
&&0&&0&&0&&0\\
&&&0&&1&&0\\
&&&&0&&0\\
&&&&&1
\end{array}
$$

These numbers of course are only special cases of a general property: for a smooth noncritical manifolds of dimension $D_{crit} + 2Q$ the diagonal cohomology contains (for $D_{crit} = 3$)

$$H^{(1,1)}, \ldots, H^{(1+Q,1+Q)}. \tag{6.9.0.155}$$

Embedding the critical Kähler form thus leaves Q forms unaccounted for.

Thus the counting of states described in [179] accounts precisely for the additional cohomology groups found in the special class of Fano folds (6.1.0.4) which suggests that it may be the proper σ–model theoretic framework for these manifolds.

6.10 Noncritical Complete Intersection Manifolds

The discussion in Section 3 of special Fano–folds which lead to critical manifolds of codimension three and larger, suggests that the class of (6.1.0.4) admits a generalization to noncritical complete intersections.

To be concrete consider the 7–fold

$$\mathbb{P}_{(5,6,4,5,6,4,4,6,8)}[16] \ni \left\{\sum_{i=1}^{2}\left(x_i^2 y_i + y_i^2 z_i + z_i^4\right) + y_3^2 z_3 + z_3^4 = 0\right\} \tag{6.10.0.156}$$

the $\mathbb{Z}_2$–fibering of which leads to the weighted complete intersection

$$\begin{matrix}\mathbb{P}_1 \\ \mathbb{P}_{(3,3,2,2,2,3,4)}\end{matrix}\begin{bmatrix}2 & 0\\ 3 & 8\end{bmatrix}. \tag{6.10.0.157}$$

This variety is not a Calabi–Yau space since its first Chern class is nonvanishing

$$c_1 = 8h_2, \tag{6.10.0.158}$$

where h_2 is the pullback of the Kähler class of the weighted ambient factor.

The $\mathbb{Z}_3$ fibering of this space leads to the space

$$\begin{matrix}\mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_{(1,1,1,2)}\end{matrix}\begin{bmatrix}2 & 0 & 0\\ 1 & 2 & 0\\ 0 & 1 & 4\end{bmatrix} \ni \left\{\begin{array}{ll} p_1 = & \sum_{i=1}^{2} x_i^2 y_i = 0 \\ p_2 = & \sum_{i=1}^{3} y_i z_i = 0 \\ p_3 = & \sum_{j=1}^{3} z_i^4 + z_4^2 = 0\end{array}\right\} \tag{6.10.0.159}$$

which does define a critical configuration.

It is therefore natural to consider the class of manifolds defined as

$$\begin{matrix}\mathbb{P}_{(k_1^1,\ldots,k_{n_1+1}^1)} \\ \mathbb{P}_{(k_1^2,\ldots,k_{n_2+1}^2)} \\ \vdots \\ \mathbb{P}_{(k_1^F,\ldots,k_{n_F+1}^F)} \\ \mathbb{P}_{(k_1^{F+1},\ldots,k_{n_{F+1}+1}^{F+1})} \\ \vdots \\ \mathbb{P}_{(k_1^L,\ldots,k_{n_L+1}^L)}\end{matrix}\begin{bmatrix} d_1^1 & d_2^1 & \ldots & d_f^1 & 0 & \ldots & 0 \\ d_1^2 & d_2^2 & \ldots & d_f^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \ldots & 0 \\ d_1^F & d_2^F & \ldots & d_f^F & 0 & \ldots & 0 \\ 0 & \ldots & 0 & d_f^{F+1} & d_{f+1}^{F+1} & \ldots & d_{f+1}^N \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & d_f^L & \ldots & \ldots & d_{f+K}^N \end{bmatrix} \tag{6.10.0.160}$$

such that

$$c_1(M) = (Q - 1 - f)\sum_{i=f+1}^{N} c_1(\mathcal{N}_i) \tag{6.10.0.161}$$

A further simple example in this class is the noncritical manifold

$$\begin{matrix}\mathbb{P}_1 \\ \mathbb{P}_{(1,1,1,2,2,2)}\end{matrix}\begin{bmatrix}2 & 0\\ 1 & 4\end{bmatrix} \tag{6.10.0.162}$$

with first Chern class

$$c_1 = 4h_2. \tag{6.10.0.163}$$

Applying the paradigm leads to the critical complete intersection

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \tag{6.10.0.164}$$

with first Chern class

$$c_1 = 4h_2. \tag{6.10.0.165}$$

Applying the paradigm leads to the critical complete intersection

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{array} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \tag{6.10.0.166}$$

which describes a K3 surface. It should be noted that the noncritical complete intersection comes from the noncritical hypersurface

$$\mathbb{P}_{(3,3,2,2,2,4,4,4)}[8] \tag{6.10.0.167}$$

via the standard construction.

Chapter 7

(0,2)–Triality of the Heterotic String

7.1 Introduction

At this point there exists overwhelming evidence that mirror symmetry exists, at least in the moduli space of (2,2) supersymmetric ground states. This class in fact has taken center stage ever since the revival of superstring theory some ten years ago. This has been the case despite the fact that the main motivation for considering supersymmetric ground states, the hierarchy problem, actually necessitates only the consideration of (0,2) conformal field theories [11]. The reason for this focus on an apparently rather special class of theories is to be found not so much in the fact that (0,2) models were somewhat hard to come by. Indeed, (0,2) models have been constructed a number of years ago by several authors [118, 133, 150, 78, 157] [1]. More important is the fact that (2,2) theories are of phenomenological interest, leading to three–generation models with small gauge groups at the Planck scale [158, 160] and that they raise a number of fundamental and intriguing questions. A short and incomplete list of such interesting aspects might comprise the possibility of conifold phase transitions [31, 171, 100] in the early universe, the exact solution of Calabi–Yau σ–models [87, 144, 176, 179, 180] as well as the fact that the moduli space features a number of important properties, such as special geometry [51, 49, 66, 170], world sheet mirror symmetry [41, 97, 141], spacetime mirror symmetry [112, 74] and scaling [163]. The full understanding of these problems has been slow to emerge and most of them have not yet been resolved in a completely satisfactory fashion even in the simpler context of (2,2) theories.

The fact remains, however, that (0,2) theories do provide a phenomenologically appealing framework [178] and a tantalizing question has been for some time what the generic features of

[1] More recently (0,2) exactly solvable theories based on coset theories have been considered in [20].

the space of (0,2) vacua are and which, if any, of the features mentioned above survive in this more general context. Part of the problem has been that even though exactly solvable (0,2) models have been known for many years, their relation to (0,2) σ–models has remained obscure. Unlike the situation in the framework of (2,2) compactifications, where the conformal fixed points of particularly simple types of Calabi–Yau σ–models have been shown [87, 96, 160, 137, 138, 139, 140, 82, 83] to be described by Gepner models [86] or Kazama–Suzuki models [120, 135, 142, 155], no such understanding has emerged in the context of (0,2) theories. It is this question of the existence and nature of the exact theory describing the conformal fixed points of (0,2) σ–models which we address in this chapter .

In [179] Witten formulated a framework which has been employed in [56] to formulate a class of (0,2) supersymmetric Landau–Ginzburg models, generalizing the construction of the class of (2,2) Landau–Ginzburg theories. This Landau–Ginzburg formulation allows us to address the problem of destabilization of (0,2) σ–models [53, 70, 54, 55, 56, 169] and facilitates the computation of some important characteristics of these theories, but falls short of addressing the important problem of identifying the exactly solvable structure of the superconformal fixed points. In [23] a class of exactly solvable (0,2) models with $(c, \bar{c}) = (6 + r, 9)$, $r = 3, 4, 5$ and gauge group of rank $(9 - r)$ was constructed by generating new modular invariants from the class of Gepner models via a slight modification of the simple current method of [156, 157]. By considering simple currents which break part of the supersymmetry one finds that in general there exist several (0,2) daughter theories which can be built from any of the Gepner models. The resulting class of theories thus is much richer than the original class of (2,2) tensor models. Furthermore, even for theories with an E_6 gauge group one obtains spectra which do not appear among the class of Gepner models [86], the complete list of which has been constructed in [137, 138, 139, 140, 82, 83] [2]. This supports the expectation that (2,2) theories describe but a small part of the total space of all vacua with N=1 spacetime supersymmetry.

In the present chapter we generalize the (2,2) triality of exactly solvable models, Landau–Ginzburg theories and Calabi–Yau manifolds, to the context of (0,2) string vacua by establishing a relation between the linear σ–models considered in [56] and the exactly solvable models of [23]. The technique we use is reminiscent of the construction introduced in [141]. It was shown there that for a particular class of discrete symmetries acting on a (2,2) exactly solvable model or its Landau–Ginzburg counterpart, it is possible not only to derive the anomalous dimensions of the

[2] They do appear in the class of all Landau–Ginzburg theories [122, 130] and therefore might describe (0,2) deformations of known (2,2) theories. The complete list of Landau–Ginzburg theories can be accessed on the web at the Calabi–Yau pages at
http://www.math.okstate.edu/˜katz/CY
and its European mirror
http://thew02.physik.uni-bonn.de/˜netah/cy.html.
(These pages are in an experimental stage and prone to changes.)

chiral primary fields of the orbifold, but also to find a superpotential which describes the orbifold theory. The implementation of simple currents is similar to the action of discrete symmetries in that they also change the dimensions of the original fields. We show that by considering the detailed structure of the massless spectrum it is possible to extract the anomalous dimensions of the scaling fields of the (0,2) theory and to derive the chiral ring describing the (0,2) theory. We furthermore show that our identification is correct by computing the Yukawa couplings of the exactly solvable models as well as the product structure of the chiral ring. Comparing the two emerging patterns leads to perfect agreement up to a change of basis.

This chapter is organized as follows. In Section 2 we review the basic ingredients of the construction of [23] before we proceed in Sections 3 and 4 to establish in some detail the relation between a particularly simple exactly solvable (0,2) model and a linear (0,2) σ–model. To this end we first determine in Section 3 the massless spectrum and compute the Yukawa couplings in the exact theory. In Section 4 we derive the chiral ring from the structure of the generations of the exact model and analyze its product structure, the comparison of which with the exact Yukawa couplings leads to complete agreement. We close this Section by showing that the resulting chiral ring can be derived from a particular (0,2) linear σ–model defining a stable bundle over a smooth Calabi–Yau manifold. This linear σ–model features both a Landau–Ginzburg phase and a (0,2) Calabi–Yau phase and therefore the exactly solvable model described in Section 3 indeed provides the underlying exact conformal field theory of a (0,2) Calabi–Yau manifold at a particular point in the moduli space. It then follows from the work of [56, 169] that the theories contained in a neighborhood of this point in moduli space are also conformally invariant. In the remaining Sections we extend our considerations to further models.

7.2 Exactly solvable theories

7.2.1 New models from old via simple currents

The class of exactly solvable models which we will focus on has been described in some detail in [23].

The basic tool for the construction is the simple current technique [156, 157] for building new modular invariants from old ones. Briefly, one considers a rational conformal field theory with a given modular invariant partition function which is supposed to contain a simple current J of index N and monodromy parameter R, i.e. a unipotent field ($\mathrm{J}^N = 1$) such that for primary fields

Φ_i of the theory

$$\mathrm{J} \times \Phi_i = \Phi_j, \qquad \Delta(\mathrm{J}) = \frac{R(N-1)}{2N} \mod 1. \tag{7.2.1.1}$$

R then is defined modulo N for N odd and modulo $2N$ for N even. Furthermore one introduces a monodromy charge Q_J for primary fields associated with a given simple current J

$$Q_\mathrm{J}(\Phi_i) = \Delta(\Phi_i) + \Delta(\mathrm{J}) - \Delta(\mathrm{J} \cdot \Phi_i) \mod 1, \tag{7.2.1.2}$$

which takes values $\frac{t}{N}$, $t \in \mathbb{Z}$. Of importance to the construction will be a slightly modified monodromy charge defined on the element of each orbit by

$$\hat{Q}(\mathrm{J}^p\Phi_i) = \frac{t + pR}{2N} \mod 1. \tag{7.2.1.3}$$

The simple current and its iterative application defines orbits of all the primary fields Φ_i, J Φ_i, $\cdots$, $\mathrm{J}^d\Phi_i$, where d is a divisor of N. If R is even the matrix

$$M_{ij}(\mathrm{J}) := \delta^1\left(\hat{Q}(\Phi_i) + \hat{Q}(\Phi_j)\right) \sum_{p=1}^{\mathrm{N}} \delta(\Phi_i, \mathrm{J}^p\Phi_j) \tag{7.2.1.4}$$

with $\delta^1(x) = 1$ for $x \in \mathbb{Z}$ and zero otherwise, defines a new modular invariant partition function

$$Z(\tau, \bar{\tau}) = \sum_{i,j} \chi_i(\tau) M_{ij} \chi_j(\bar{\tau}). \tag{7.2.1.5}$$

This procedure allows for an iteration procedure with a whole bunch of simple currents by considering

$$Z(\tau, \bar{\tau}) \sim \vec{\chi}(\tau) M(\mathrm{J}_n) \cdots M(\mathrm{J}_2) M(\mathrm{J}_1) \vec{\chi}(\bar{\tau}), \tag{7.2.1.6}$$

where $\sim$ indicates equality up to an overall factor originating from a universal multiplicity factor.

7.2.2 (0,2) Simple current modular invariants leading to E_{9-r}

We now wish to apply the above general considerations to the particular case, where the final theory we end up with has the following properties: a) The gauge group is any of the groups $\mathrm{E}_{9-r} \ni \{\mathrm{E}_6, \mathrm{E}_5 = \mathrm{SO}(10), \mathrm{E}_4 = \mathrm{SU}(5), \mathrm{E}_3 = \mathrm{SU}(3) \times \mathrm{SU}(2)\}$. b) The central charges in the two internal sectors are $(c, \bar{c}) = (6 + r, 9)$. c) Besides the right moving $\mathrm{U}(1)_\mathrm{R}$ current which is part of the right moving N=2 superconformal theory, there exists also a left moving $\mathrm{U}(1)_\mathrm{L}$ current J_L satisfying the operator product expansion

$$\mathrm{J}_\mathrm{L}(z)\mathrm{J}_\mathrm{L}(w) = \frac{r}{(z-w)^2} + \mathrm{reg}. \tag{7.2.2.1}$$

d) Only the subset $\mathrm{SO}(16-2r) \times \mathrm{U}(1)_\mathrm{L} \subset \mathrm{E}_{9-r}$ of the gauge group is linearly realized, the full E_{9-r} being generated by taking orbits with respect to the spectral flow operator of conformal dimension

$(\Delta, Q) = (\frac{r}{8}, \frac{r}{2})$. It is convenient to describe the construction in the following left–right symmetric way, where the asymmetry between the left and right sector is achieved at the end of the day by throwing away part of the right moving current algebra in such a way as not to endanger modular invariance. Consider then an internal conformal field theory with the ingredients in Table 1.

	Left Sector c	Right Sector $\bar{c}$
N=2 SCFT	9	9
$(\mathrm{U}(1)_2)^{r-3}$	$r-3$	$r-3$
$\mathrm{SO}(16-2r)\times\mathrm{E}_8$	$16-r$	$16-r$

Table 1: *Ingredients for the construction of the internal theory.*

The crucial, new ingredient, as compared to Gepner's tensor model construction, is the free boson ϕ compactified on a circle of radius R=2, denoted by $\mathrm{U}(1)_2$. The diagonal partition function of this theory is simply

$$Z_{\mathrm{U}(1)_2}(\tau,\bar{\tau}) = \sum_{m=-1}^{2} \Theta_{m,2}(\tau)\Theta_{m,2}(\bar{\tau}). \tag{7.2.2.2}$$

The conformal dimension and charge of the primary fields $\Phi_{m,2}^{\mathrm{U}(1)_2}$ are

$$(\Delta, Q)\,(\Phi_{m,2}^{\mathrm{U}(1)_2}) = \left(\frac{m^2}{8}, \frac{m}{2}\right). \tag{7.2.2.3}$$

The current $j_{\mathrm{U}(1)_2} = i\partial_z\phi$ satisfies the following OPE

$$j_{\mathrm{U}(1)_2}(z) j_{\mathrm{U}(1)_2}(w) = \frac{1}{(z-w)^2} + \text{reg.} \tag{7.2.2.4}$$

Consider first the left moving sector. Even though the $\mathrm{U}(1)_2$ theory is not N=2 supersymmetric, it does feature a spectral flow between the m even sectors and the m odd sectors. Since this boson describes a Dirac fermion the spectral flow operator $e^{i\phi(z)/2}$, with conformal dimension and charge $(\frac{1}{8}, \frac{1}{2})$, relates the NS sector and the R sector of the Dirac fermion. The introduction of the $\mathrm{U}(1)_2$ current algebra is motivated by the fact that by combining its current $j_{\mathrm{U}(1)_2}$ with the U(1) current $j_{c=9} = i\sqrt{3}\partial\Phi$ of the N=2, c=9 theory one obtains the OPE (7.2.2.1) for $\mathrm{U}(1)_{\mathrm{L}}$. Furthermore, putting together the spectral flow operators of the N=2 conformal field theory with the $\mathrm{U}(1)_2$ one obtains the spectral flow operator of the $c = (6+r)$ left moving sector

$$\Sigma_{c=6+r}(z) = e^{i\frac{\sqrt{3}}{2}\Phi(z)} \cdot \prod_{j=1}^{r-3} e^{\frac{i}{2}\phi_j(z)}. \tag{7.2.2.5}$$

It can be shown that this operator does indeed extend the group $\mathrm{SO}(16-2r)\times$ $\mathrm{U}(1)_{\mathrm{L}}$ to E_{9-r}.

Before turning to the right moving sector, it is useful to review a few facts about the representations of SO($2n$) Kac–Moody algebras at level $k=1$. Recall that the representations of an SO($2n$) algebra decompose into scalars (0), spinors (s), antispinors (c) and vectors (v). The characters and the quantum numbers of the corresponding primary fields are collected in Table 2.

Character	Conformal Dimension Δ	Q mod 2	Degeneracy
$\chi_0=\frac{1}{2}\left(\left(\frac{\vartheta_3}{\eta}\right)^n+\left(\frac{\vartheta_4}{\eta}\right)^n\right)$	0	0	1
$\chi_v=\frac{1}{2}\left(\left(\frac{\vartheta_3}{\eta}\right)^n-\left(\frac{\vartheta_4}{\eta}\right)^n\right)$	$\frac{1}{2}$	1	$2n$
$\chi_s=\frac{1}{2}\left(\left(\frac{\vartheta_2}{\eta}\right)^n+\left(\frac{\vartheta_1}{\eta}\right)^n\right)$	$\frac{n}{8}$	$\frac{n}{2}$	2^{n-1}
$\chi_c=\frac{1}{2}\left(\left(\frac{\vartheta_2}{\eta}\right)^n-\left(\frac{\vartheta_1}{\eta}\right)^n\right)$	$\frac{n}{8}$	$\frac{n}{2}-1$	2^{n-1}

Table 2: *Characters of SO(2n). The charge Q is taken with respect to the sum of all Cartan elements of the Lie algebra and ϑ_i denote the Jacobi theta functions.*

Because our goal is to eventually turn the bosonic theory into a heterotic theory by applying the bosonic string map as considered by Gepner [86]

$$\begin{array}{ll}\chi_0^{\mathrm{SO(10)\times E_8}}\longrightarrow\chi_v^{\mathrm{SO(2)}} & \chi_v^{\mathrm{SO(10)\times E_8}}\longrightarrow\chi_0^{\mathrm{SO(2)}}\\ \chi_s^{\mathrm{SO(10)\times E_8}}\longrightarrow-\chi_c^{\mathrm{SO(2)}} & \chi_c^{\mathrm{SO(10)\times E_8}}\longrightarrow-\chi_s^{\mathrm{SO(2)}},\end{array}\tag{7.2.2.6}$$

we wish to extend the SO($16{-}2r$)$\times$U$(1)_2^{r-3}$ to SO(10). In the right sector this is achieved by considering $(r-3)$ simple currents of the form

$$\mathrm{J_{ext}}=\Phi_{2,2}^{\mathrm{U(1)_2}}\otimes\Phi_v^{\mathrm{SO}(16-2r)},\tag{7.2.2.7}$$

with $\Delta(\mathrm{J_{ext}})=1$, which generate orbits of SO(10). For $r=4$, say, these orbits take the form

$$\begin{array}{ll}\chi_0^{\mathrm{SO(10)}}=\chi_0^{\mathrm{SO(8)}}\Theta_{0,2}+\chi_v^{\mathrm{SO(8)}}\Theta_{2,2} & \chi_v^{\mathrm{SO(10)}}=\chi_0^{\mathrm{SO(8)}}\Theta_{2,2}+\chi_v^{\mathrm{SO(8)}}\Theta_{0,2}\\ \chi_s^{\mathrm{SO(10)}}=\chi_s^{\mathrm{SO(8)}}\Theta_{1,2}+\chi_c^{\mathrm{SO(8)}}\Theta_{-1,2} & \chi_c^{\mathrm{SO(10)}}=\chi_c^{\mathrm{SO(8)}}\Theta_{0,2}+\chi_s^{\mathrm{SO(8)}}\Theta_{2,2}.\end{array}\tag{7.2.2.8}$$

So far our considerations have been completely general. From this point on we focus our attention on products of N=2 minimal tensor models a la Gepner.

7.2.3 (0,2) Theories from Gepner models

Recall that for the minimal models the conformal anomaly is

$$c=\frac{3k}{k+2}\tag{7.2.3.1}$$

and the dimensions and charges of the chiral primary fields $\Phi^{l,q,s}$ are given by

$$\Delta^{l,q,s} = \frac{l(l+2)-q^2}{4(k+2)} + \frac{s^2}{8}, \qquad Q^{l,q,s} = -\frac{q}{k+2} + \frac{s}{2}, \tag{7.2.3.2}$$

where the level $k \in \mathbb{N}$, $0 \leq l \leq k$, $l - q + s \in 2\mathbb{Z}$ and $|q - s| \leq l$. Here the range of the various quantum numbers is $l = 0, ..., k$, $q \sim q + 2(k+2)$, $s \sim s + 4$ and we will employ the following notation for the complete fields

$$\Phi^{l,q,s}_{\bar{l},\bar{q},\bar{s}} = \begin{bmatrix} l & q & s \\ \bar{l} & \bar{q} & \bar{s} \end{bmatrix}, \tag{7.2.3.3}$$

which, if $l = \bar{l}$, simplifies into

$$\Phi^{l,q,s}_{l,\bar{q},\bar{s}} = \begin{bmatrix} l \begin{matrix} q & s \\ \bar{q} & \bar{s} \end{matrix} \end{bmatrix}. \tag{7.2.3.4}$$

Important, finally, is the identification

$$\begin{bmatrix} l & q & s \\ \bar{l} & \bar{q} & \bar{s} \end{bmatrix} \sim \begin{bmatrix} k-l & q+(k+2) & s+2 \\ k-\bar{l} & \bar{q}+(k+2) & \bar{s}+2 \end{bmatrix}. \tag{7.2.3.5}$$

In [23] the tensor models $\bigotimes_{i=1}^{n} k_i$ have been used to construct (0,2) theories with the aforementioned properties a) – d) by breaking the left moving supersymmetry as well as the E_6 gauge group present in Gepner's construction. The following steps do the job:

- The simple currents extending $SO(16-2r)\times U(1)_2^{r-3}$ to $SO(10)$ in the tensor model can be written as

$$J^j_{\text{ext}} = [0\ 0\ 0]^{\otimes n} \otimes [0]^{j-1}[2][0]^{r-3-j} \otimes [v], \quad j = 1, ..., r-3, \tag{7.2.3.6}$$

  where $[v]$ denotes the vector representation of $SO(16-2r)$.

- Next one has to make sure, as usual, that only fields of same character couple, i.e. Neveu–Schwarz ($s \in \{0, 2\}$) to Neveu–Schwarz and Ramond ($s \in \{1, 3\}$) to Ramond. This is achieved via the projection operators $J_i = G_i \otimes \Phi_v^{SO(16-2r)}$, where the $G_i = \Phi^{k,k+2,4}$ are the supercurrents in the i'th factor of the tensor model. The appropriate fields in a tensor model with n factors take the (chiral) form

$$J_i = [0\ 0\ 0]^{\otimes(i-1)} \otimes [k\ (k+2)\ 4] \otimes [0\ 0\ 0]^{\otimes(n-i)} \otimes [0]^{r-3} \otimes [v], \quad i = 1, ..., n. \tag{7.2.3.7}$$

- The next step is to implement the right moving GSO projection onto states with even overall charge. This is achieved via the simple current

$$\bar{J}_{\text{GSO}} = \bar{\Sigma}_{c=9} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes(r-3)} \otimes \begin{bmatrix} 0 \\ s \end{bmatrix}, \tag{7.2.3.8}$$

where

$$\bar{\Sigma}_{c=9} = \begin{bmatrix} 0 & \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \end{bmatrix}^{\otimes n} \tag{7.2.3.9}$$

is the right moving spectral flow operator of dimensions $(\Delta, \bar{\Delta})(\bar{\Sigma}_{c=9}) = (0, \frac{3}{8})$ and charges $(Q, \bar{Q})(\bar{\Sigma}_{c=9}) = (0, \frac{3}{2})$. Implementing these projections leads to the partition function

$$Z \sim \vec{\chi}(\tau) M(\bar{\mathrm{J}}_{\mathrm{GSO}}) \left(\prod_{i=1}^{n} M(\mathrm{J}_i) \right) \left(\prod_{j=1}^{r-3} M(\mathrm{J}^j_{\mathrm{ext}}) \right) \vec{\chi}(\bar{\tau}). \tag{7.2.3.10}$$

At this point all the conditions for a (2,2) supersymmetric theory have been implemented. After turning this left–right symmetric theory into a class of heterotic string vacua by applying the bosonic string map in the way described above, we should expect the procedure described thus far to provide an alternative construction of Gepner's models. The fact that the partition function (7.2.3.10) indeed reproduces the expected spectra provides a nice check of the implementation.

- Our goal, however, was to obtain an asymmetric (0,2) CFT and in order to do so one simply has to introduce further simple currents in the left moving sector, denoted by Υ_i in the following, which do not commute with the simple currents that appear in (7.2.3.10). Of particular interest are simple currents Υ_i which break both the left moving N=2 supersymmetry and the E_6 gauge group which results from the $\bar{\mathrm{J}}_{\mathrm{GSO}}$ projection. It turns out that for each tensor model one can find a multitude of such simple currents, making this class an interesting framework. Given the existence of fields Υ_i with the appropriate properties the only remaining ingredient is the left moving GSO projection which is implemented by the simple current

$$\mathrm{J}_{\mathrm{GSO}} = \Sigma_{c=6+r} \otimes \begin{bmatrix} s \\ 0 \end{bmatrix}, \tag{7.2.3.11}$$

where

$$\Sigma_{c=6+r} = \begin{bmatrix} 0 & \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \end{bmatrix}^{\otimes n} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes (r-3)}. \tag{7.2.3.12}$$

The crucial effect of the simple currents Υ_i is that they prevent the extension of the gauge group to E_6 via the currents J_i and $\mathrm{J}_{\mathrm{ext}}$. Instead the chiral spectral flow is $\Sigma^2_{c=6+r}$ and thus is an operator of dimension $(\Delta, \bar{\Delta}) = (\frac{r}{2}, 0)$ and charges $(Q, \bar{Q}) = (r, 0)$. It is this operator which extends the $\mathrm{SO}(16-2r)\times\mathrm{U}(1)_{\mathrm{L}}$ to E_{9-r}.

Putting everything together then leads to the final form of the partition function [23]

$$Z \sim \vec{\chi}(\tau) M(\mathrm{J}_{\mathrm{GSO}}) \left(\prod_{i} M(\Upsilon_i) \right) M(\bar{\mathrm{J}}_{\mathrm{GSO}}) \left(\prod_{i=1}^{r} M(\mathrm{J}_i) \right) \left(\prod_{j=1}^{r-3} M(\mathrm{J}^j_{\mathrm{ext}}) \right) \vec{\chi}(\bar{\tau}). \tag{7.2.3.13}$$

This partition function exhibits all the desired features in order to be of use in the exploration of exactly solvable models which possibly describe particular points in the moduli space of Landau–Ginzburg theories constructed in [56]. In short one might summarize the structure of these vacua as described in Table 3.

	Left Sector	Right Sector
Ghosts b, c	-26	-26
Super Ghosts β, γ	–	11
Spacetime Coordinates X^μ	4	4
Superpartners ψ^μ of X^μ	–	2
Internal CFT	$6{+}r$	9
Gauge Group SO($16{-}2r$)$\times$E$_8$	$16{-}r$	–

Table 3: *Anomaly structure of the complete (0,2) theory.*

7.3 An (80,0) SO(10) (0,2)–model

The exact theories we focus on in the following are all derived from the parent 'quintic' tensor model defined by considering the product of five N=2 minimal factors at level k=3. We will denote this theory by $3^{\otimes 5}$. The different spectra and gauge groups will be obtained by applying various simple currents Υ_i.

7.3.1 The massless spectrum

Our first exact model is based on the simple current

$$\Upsilon_1 = [3\ 0\ \ -1] \otimes [0\ 0\ 0]^{\otimes 4} \otimes [1] \otimes [0], \tag{7.3.1.1}$$

which only affects the first of the five minimal N=2 factors and turns out to break the E$_6$ of the parent Gepner model down to SO(10). The spectrum therefore is arranged into representations of SO(10) and we have summarized the relevant multiplicities of the massless sector in Table 4.

SO(10) Representation:	**0**	**10**	**16**	$\overline{\mathbf{16}}$
Spin 0:	350	74	80	0
Spin 1:	7	0	0	0

Table 4: *Massless spectrum of the (80,0) SO(10) daughter of the $3^{\otimes 5}$ model.*

Of particular relevance for the following are the generations, which can be represented in a number of ways. Considering the internal part of a generation only, the **16** of the SO(10) in the (–1) ghost picture as a spacetime scalar leads to a vertex operator of the form (at zero momentum)

$$V_{-1}^a = e^{-\rho(\bar{z})}\mathcal{O}_{16}(z,\bar{z})\lambda^a, \tag{7.3.1.2}$$

where the internal operator $\mathcal{O}_{16}(z,\bar{z})$ has U(1) charge $(Q,\bar{Q}) = (-1,-1)$, and $\rho(\bar{z})$ is the bosonized supersymmetry ghost, whereas the fermions λ^a generate the SO(10) current algebra. The operator has $\Delta = \bar{\Delta} = \frac{1}{2}$ and therefore defines a chiral primary field. In this representation the generations take the form exhibited in Table 5, in which we have used the abbreviations

$$g_0 = \begin{bmatrix} 2 & \begin{matrix} 2 & 0 \\ 2 & 0 \end{matrix} \end{bmatrix}, \quad g_1 = \begin{bmatrix} 2 & \begin{matrix} -3 & -1 \\ 2 & 0 \end{matrix} \end{bmatrix} \tag{7.3.1.3}$$

for the two states that appear in the first minimal factor and

$$u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{7.3.1.4}$$

for the U(1)–part of the generations.

Type							Field	Number
I	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 2 & \begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 3 & \begin{smallmatrix} 3 & 0 \\ 3 & 0 \end{smallmatrix} \end{bmatrix}$	u_0	$\begin{bmatrix} v \\ v \end{bmatrix}$	12
II	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 3 & \begin{smallmatrix} 3 & 0 \\ 3 & 0 \end{smallmatrix} \end{bmatrix}$	u_0	$\begin{bmatrix} v \\ v \end{bmatrix}$	12
III	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 2 & \begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 2 & \begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix} \end{bmatrix}$	u_0	$\begin{bmatrix} v \\ v \end{bmatrix}$	12
IV	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 2 & \begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix} \end{bmatrix}$	u_0	$\begin{bmatrix} v \\ v \end{bmatrix}$	4
V_i	g_i	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 3 & \begin{smallmatrix} 3 & 0 \\ 3 & 0 \end{smallmatrix} \end{bmatrix}$	u_i	$\begin{bmatrix} v \\ v \end{bmatrix}$	8
VI_i	g_i	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 2 & \begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix} \end{bmatrix}$	u_i	$\begin{bmatrix} v \\ v \end{bmatrix}$	24
VII_i	g_i	$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	$\begin{bmatrix} 1 & \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$	u_i	$\begin{bmatrix} v \\ v \end{bmatrix}$	8

Table 5: *Generations of the (80,0) SO(10) daughter of* $3^{\otimes 5}$.

The **10** of SO(10) can be counted by enumerating the singlet part of the decomposition $\mathbf{10} = \mathbf{1}_{-2} \oplus \mathbf{8}_0^s \oplus \mathbf{1}_2$ of the vectors of SO(10) with respect to the maximal subgroup SO(8)×U(1) which, using abbreviations

$$v_0 = \begin{bmatrix} 2 & \begin{matrix} -2 & 0 \\ -4 & -2 \end{matrix} \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 & \begin{matrix} 3 & 1 \\ -4 & -2 \end{matrix} \end{bmatrix} \tag{7.3.1.5}$$

for the relevant primary fields in the first factor and u_i^+ for the charge conjugate of u_i, leads to the list of vectors contained in Table 6.

Type							Field	Number
A	$\left[0\begin{matrix}0 & 0\\ -2 & -2\end{matrix}\right]$	$\left[1\begin{matrix}-1 & 0\\ -3 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	u_0	$\begin{bmatrix}0\\ v\end{bmatrix}$	4
B	$\left[0\begin{matrix}0 & 0\\ -2 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	u_0	$\begin{bmatrix}0\\ v\end{bmatrix}$	6
C_i	v_i	$\left[0\begin{matrix}0 & 0\\ -2 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	u_i^+	$\begin{bmatrix}0\\ v\end{bmatrix}$	24
D_i	v_i	$\left[1\begin{matrix}-1 & 0\\ -3 & -2\end{matrix}\right]$	$\left[1\begin{matrix}-1 & 0\\ -3 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	u_i^+	$\begin{bmatrix}0\\ v\end{bmatrix}$	12
E_i	v_i	$\left[1\begin{matrix}-1 & 0\\ -3 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[3\begin{matrix}-3 & 0\\ -5 & -2\end{matrix}\right]$	u_i^+	$\begin{bmatrix}0\\ v\end{bmatrix}$	24
F_i	v_i	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}-2 & 0\\ -4 & -2\end{matrix}\right]$	u_i^+	$\begin{bmatrix}0\\ v\end{bmatrix}$	2
G_1	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\begin{bmatrix}2\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\ v\end{bmatrix}$	1
G_2	$\left[2\begin{matrix}-1 & 1\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	$\left[2\begin{matrix}4 & 2\\ -4 & -2\end{matrix}\right]$	u_1^+	$\begin{bmatrix}0\\ v\end{bmatrix}$	1

Table 6: *Vectors of the SO(10) (80,0) daughter of* $3^{\otimes 5}$.

With the explicit form of the massless spectrum at our disposal we can now explore this model in greater depth by computing the Yukawa couplings. This is what we will turn to in the next Subsection.

7.3.2 The exact Yukawa couplings

Suppose we wish to compute the Yukawa couplings

$$< \mathbf{10}\cdot\mathbf{16}\cdot\mathbf{16} > . \tag{7.3.2.1}$$

Then we have to specify the vertex operators and the picture in which they live. To get the ghostnumber -2 of string theory tree level correlation functions [81] one might consider vertex operators

$$< V^{\mathbf{10}}_{-1}\, V^{\mathbf{16}}_{-1/2}\, V^{\mathbf{16}}_{-1/2} > . \tag{7.3.2.2}$$

Using the decompositions

$$\mathbf{10} = \mathbf{1}_{-2}\oplus \mathbf{8}^s_0 \oplus \mathbf{1}_2, \qquad \mathbf{16} = \mathbf{8}^v_{-1}\oplus\mathbf{8}^c_1 \tag{7.3.2.3}$$

of the SO(10) representations with respect to the maximal subgroup SO(8) $\times$ U(1) $\subset$ SO(10), the couplings $< \mathbf{10}\cdot\mathbf{16}\cdot\mathbf{16} >$ decompose into

$$\left\langle \begin{pmatrix}\mathbf{1}_{-2}\\ \mathbf{8}^s_0\\ \mathbf{1}_2\end{pmatrix}\begin{pmatrix}\mathbf{8}^v_{-1}\\ \mathbf{8}^c_1\end{pmatrix}\begin{pmatrix}\mathbf{8}^v_{-1}\\ \mathbf{8}^c_1\end{pmatrix}\right\rangle =< \mathbf{1}_2\cdot\mathbf{8}^v_{-1}\cdot\mathbf{8}^v_{-1} >=< \mathbf{8}^s_0\cdot\mathbf{8}^c_1\cdot\mathbf{8}^v_{-1} > . \tag{7.3.2.4}$$

The sum over all right moving charges also has to vanish, a condition which is not satisfied for the fields listed in Tables 5 and 6. Using the supersymmetry charge operator $\bar{\Sigma}_{c=9}$ (7.2.3.9) of the Gepner parent model one obtains the desired form

$$V^a_{1/2} = \bar{\Sigma}\ V^a_{-1} \tag{7.3.2.5}$$

with which the Yukawa couplings can be written as

$$< V^{\mathbf{10}}_{-1}\ V^{\mathbf{16}}_{-1/2}\ V^{\mathbf{16}}_{-1/2} > = < \bar{\Sigma}^2 (\mathbf{1}_2)_{-1}\ (\mathbf{8}^v_{-1})_{-1}\ (\mathbf{8}^v_{-1})_{-1} > . \tag{7.3.2.6}$$

The first obstacle any correlation function has to overcome, of course, is charge conservation, according to which the U(1) charges must cancel in every factor. Now, according to (7.3.2.6), every coupling of the type (7.3.2.1) contains per construction the operator $\bar{\Sigma}^2$. We consider first the vectors. Starting with the vector A of Table 6 we need to check which pair of generations we can multiply to get something nonvanishing. Computing the charges of $\left(\bar{\Sigma}^2\ A\right)$ results in

$$\begin{pmatrix} Q \\ \bar{Q} \end{pmatrix} (\bar{\Sigma}^2\ \mathrm{A}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix} \otimes \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix} \otimes \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix} \otimes \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix}, \tag{7.3.2.7}$$

hence pairs of generations are needed whose product leads to the negative of this charge array. Looking back at the generations we see that there are members of the pair families (I, II) on the one hand, and (II,III) and (III, IV) on the other, which lead to the appropriate charges. Thus we have the following potentially nonvanishing Yukawa couplings

$$< \mathrm{A} \cdot \mathrm{I} \cdot \mathrm{II} >, \quad < \mathrm{A} \cdot \mathrm{II} \cdot \mathrm{III} >, \quad < \mathrm{A} \cdot \mathrm{III} \cdot \mathrm{IV} > \tag{7.3.2.8}$$

for the simple reason that I – IV are the only generations with charge zero in the first factor (the only N=2 minimal factor affected by the simple current). Proceeding in the same manner one finds the remaining potentially nonvanishing couplings.

In order to compute the actual values of these couplings one uses the fact that a primary field in the N=2 minimal superconformal theory at level k, is just a product of a parafermionic field and a scalar field $\varphi(z)$

$$\Phi^{l,q,s}_{\bar{l},\bar{q},\bar{s}}(z,\bar{z}) = \phi^{l,q-s}_{\bar{l},\bar{q}-\bar{s}}(z,\bar{z}) e^{i(\alpha_{qs}\varphi(z) + \alpha_{\bar{q}\bar{s}}\varphi'(\bar{z}))}, \tag{7.3.2.9}$$

where

$$\alpha_{qs} = \frac{-q + \frac{s}{2}(k+2)}{\sqrt{k(k+2)}}. \tag{7.3.2.10}$$

The parafermionic field $\phi^{l,m}$ in turn can be expressed in terms of SU(2) primary fields and a further scalar field $\tilde{\varphi}$ as

$$\phi^{l,m} = G^{\frac{l}{2},\frac{m}{2}}\ e^{-\frac{m}{2\sqrt{2}}\tilde{\varphi}}. \tag{7.3.2.11}$$

Thus the only nontrivial correlation functions one has to know in order to compute the Yukawa couplings of the superconformal model are the three–point functions of the SU(2) theory. These have been obtained by Zamolodchikov and Fateev [187], who found

$$< G^{j_1,m_1}_{j_1,\bar m_1} G^{j_1,m_1}_{j_1,\bar m_1} G^{j_1,m_1}_{j_1,\bar m_1} >_{\mathrm{SU}_k} = \frac{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \bar m_1 & \bar m_2 & \bar m_3 \end{pmatrix} \rho_k(j_1,j_2,j_3)}{f(z_{12}\bar z_{12}, z_{13}\bar z_{13}, z_{23}\bar z_{23})}, \tag{7.3.2.12}$$

where $\left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ are the Wigner $3j$–symbols

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sqrt{\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} \prod_{i=1}^{3} (j_i+m_i)\ (j_i-m_i)!} \times$$
$$\times \sum_{z\in\mathbf{Z}} \frac{(-1)^{z+j_1-j_2-m_3}}{z!(j_1+j_2-j_3-z)!(j_1-m_1-z)!(j_2+m_2-z)!(j_3-j_2+m_1+z)!(j_3-j_1-m_2+z)!} \tag{7.3.2.13}$$

and

$$\rho_k(j_1,j_2,j_3) = F_k(j_1,j_2,j_3) \sqrt{\frac{\Gamma\left(\frac{k+3}{k+2}\right)}{\Gamma\left(\frac{k+1}{k+2}\right)} \prod_{r=1}^{3} (2j_r+1) \frac{\Gamma\left(1-\frac{2j_r+1}{k+2}\right)}{\Gamma\left(1+\frac{2j_r+1}{k+2}\right)}}, \tag{7.3.2.14}$$

where

$$F_k(j_1,j_2,j_3) = \frac{\pi_k(j_1+j_2+j_1+1)\ \pi_k(j_1+j_2-j_3)\ \pi_k(j_2+j_3-j_1)\pi_k(j_3+j_1-j_2)}{\pi_k(2j_1)\ \pi_k(2j_2)\ \pi_k(2j_3)} \tag{7.3.2.15}$$

with, finally,

$$\pi_k(j) = \prod_{r=1}^{j} \frac{\Gamma\left(1+\frac{r}{k+2}\right)}{\Gamma\left(1-\frac{r}{k+2}\right)}. \tag{7.3.2.16}$$

Plugging in these ingredients we can compute the necessary Yukawa couplings of the individual minimal theories. The relevant couplings for the level $k=3$ theory are

$$\begin{aligned}
&\left\langle \begin{bmatrix} 1 & {-1} & 0 \\ & {-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \end{bmatrix} \right\rangle = 1 \qquad
&&\left\langle \begin{bmatrix} 2 & {-2} & 0 \\ & {-2} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \end{bmatrix} \right\rangle = 1 \\
&\left\langle \begin{bmatrix} 3 & {-3} & 0 \\ & {-3} & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \end{bmatrix} \right\rangle = 1 \qquad
&&\left\langle \begin{bmatrix} 3 & {-3} & 0 \\ & {-3} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \end{bmatrix} \right\rangle = 1 \\
&\left\langle \begin{bmatrix} 2 & {-1} & 0 \\ & {-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \end{bmatrix} \right\rangle = \kappa \qquad
&&\left\langle \begin{bmatrix} 2 & 3 & 1 \\ & {-2} & 0 \end{bmatrix} \begin{bmatrix} 2 & {-3} & {-1} \\ & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \end{bmatrix} \right\rangle = 1,
\end{aligned} \tag{7.3.2.17}$$

where

$$\kappa = \left(\frac{\Gamma\left(\frac{3}{5}\right)^3 \Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\frac{2}{5}\right)^3 \Gamma\left(\frac{4}{5}\right)} \right)^{1/2}. \tag{7.3.2.18}$$

With the couplings for the individual N=2 factors at hand we finally arrive at the Yukawa couplings of the full theory, the results of which are contained in Table 7.

10	Generations
A·	I·II$=1$, II·III$=1$, III·IV$=1$
B·	$\mathrm{I}^2=1$, I·III$=1$, $\mathrm{II}^2=\kappa^2$, II·III$=\kappa$, $\mathrm{III}^2=1$, III·IV$=\kappa$, $\mathrm{IV}^2=\kappa^2$
C_i·	I·$\mathrm{V}_i=1$, I·$\mathrm{VI}_i=1$, II·$\mathrm{VI}_i=\kappa$, III·$\mathrm{VI}_i=1$, III·$\mathrm{VII}_i=\kappa$
D_i·	I·$\mathrm{VII}_i=1$, II·$\mathrm{V}_i=1$, II·$\mathrm{VI}_i=1$, III·$\mathrm{VII}_i=1$, IV·$\mathrm{VI}_i=1$
E_i·	I·$\mathrm{VI}_i=1$, II·$\mathrm{VI}_i=\kappa$, II·$\mathrm{VII}_i=\kappa^2$, III·$\mathrm{V}_i=1$, III·$\mathrm{VI}_i=1$, III·$\mathrm{VII}_i=\kappa$, IV·$\mathrm{VI}_i=\kappa$, IV·$\mathrm{VII}_i=\kappa^2$
F_i·	III·$\mathrm{VI}_i=\kappa$, IV·$\mathrm{VII}_i=\kappa^3$

Table 7: *Yukawa couplings of the generations of the SO(10) (80,0) daughter of* $3^{\otimes 5}$.

7.4 Chiral ring and σ–model structure of the (80,0) theory

In this Section we determine the linear σ–model whose fixed point is described by the exactly solvable model analyzed in depth in the previous Section. This problem naturally splits into two parts. The first is to determine the chiral ring of the (0,2) of the theory, defining a monomial algebra, the product structure of which captures the behaviour of the Yukawa couplings. In the context of (0,2) theories based either on Calabi–Yau manifolds or Landau–Ginzburg theories, this would appear to furnish the appropriate framework in which one eventually should understand the complete structure of the underlying string model, the reason being that Calabi–Yau manifolds are algebraic projective varieties and thus are embedded in a simple space, $\mathbb{P}_n$, although by a rather complicated map.

Being able to derive the ring structure from one or more superpotentials, in order to obtain a complete intersection, simplifies the analysis considerably, but does restrict the focus to a rather narrow framework, the confines of which has to be overcome eventually. In the context of (2,2) compactifications a well known technique for dealing with rings more general than those originating from complete intersections is furnished by toric geometry and an interesting problem for future work would be to formulate (0,2) theories based on toric varieties. Even toric geometry, however, is a rather restrictive framework and it would be of great interest to explore Calabi–Yau manifolds in purely algebraic, ring theoretic, terms.

Here our interest is a different one, however, and we focus on the particular exactly solvable theory we have been analyzing in some detail in the previous Section precisely because its massless sector is simplified by the fact that it does not contain antigenerations. Because of this we might hope that its σ–model is described by a simple geometric structure. We will see below that this is indeed the case and that we are lead to a rank four vector bundle on a smooth codimension two complete intersection Calabi–Yau manifold. First we derive the chiral ring.

7.4.1 The (0,2) chiral ring of the (80,0) model

In order to extract the chiral ring we consider the detailed structure of the generations. Because the simple current acts only on the first factor of the tensor product of N=2 minimal theories, we expect the chiral primary fields of the unaffected minimal theories to remain unchanged. From the structure of the generations of type I – IV in Table 5 we indeed see that the basic field which appears in the last four minimal factors is the chiral primary field $\begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix}$. Thus we see that in order to relate these exact states to the complex coordinates of a ring we should make the identification

$$x_i \sim \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0\\ 0\end{bmatrix}, \qquad (7.4.1.1)$$

where the nonzero state is in the $(i+1)st$ factor of the 5 individual factors. From the remaining generation families V – VII of Table 5 we can then read off that the variables

$$\begin{aligned} y_1 &\sim \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0\\ 0\end{bmatrix} \\ y_2 &\sim \begin{bmatrix}2 & \begin{matrix}-3 & -1\\ 2 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}-1\\ 0\end{bmatrix} \end{aligned} \qquad (7.4.1.2)$$

must have degree $\deg(y_i) = 2$, in agreement with the anomalous weights of the states. It is important to note that for the 'twisted' state y_2 the contribution of the $U(1)_2$ is crucial in order for these fields to be chiral primary fields. Thus we have two types of coordinates x_i, $i = 1,2,3,4$, coming from the inert minimal factors and y_l, $l = 1,2$, generated by the simple current Υ. Using the relations (7.4.1.1) and (7.4.1.2) maps the generations into the monomial representations listed in Table 8.

Family	Monomial Representative	Degeneracy
I	$x_i^2 x_j^3$	12
II	$x_i x_j x_k^3$	12
III	$x_i x_j^2 x_k^2$	12
IV	$x_i x_j x_k x_l^2$	4
V_l	$x_i^3 y_l$	8
VI_l	$x_i x_j^2 y_l$	24
VII_l	$x_i x_j x_k y_l$	8

Table 8: *Monomial representation of the generations of the (80,0) model.*

The chiral ring $\mathcal{R} = \frac{\mathbb{C}[x_i, y_j]}{\mathcal{I}}$ determined by these states is generated from free ring

$$\mathbb{C}[x_1, x_2, x_3, x_4, y_1, y_2] \qquad (7.4.1.3)$$

by considering equivalence relations defined via the ideal

$$\mathcal{I}[x_1^4, x_2^4, x_3^4, x_4^4, y_1^2, y_2^2, y_1 y_2]. \qquad (7.4.1.4)$$

The question then arises whether this ideal can be derived from a stable bundle over a Calabi–Yau threefold. We will show in the next Subsection that this is indeed possible. First, however, we complete our discussion of the spectrum by considering the vectors in $\mathrm{H}^1(M, \Lambda^2 V)$, which can be identified with monomials of degree ten. There are a total of 72 of such elements, the explicit form of which is contained in Table 9.

Family	Monomial Representative	Degeneracy
A	$x_i x_j^3 x_k^3 x_l^3$	4
B	$x_i^2 x_j^2 x_k^3 x_l^3$	6
C_m	$y_m x_i^2 x_j^3 x_k^3$	24
D_m	$y_m x_i x_j x_k^3 x_l^3$	12
E_m	$y_m x_i x_j^2 x_k^2 x_l^3$	24
F_m	$y_m x_i^2 x_j^2 x_k^2 x_l^2$	2

Table 9: *Monomial representation of the vectors of the (80,0) model.*

The precise relation of these monomial forms to the fields of the exactly solvable theory is indicated by the types A–F, referring back to Table 6. The two vectors G_1, G_2 of Table 6 do not have a monomial representation. As we have seen in the preceding Section they also do not lead to nonvanishing Yukawa couplings. The product structure of the ring $\mathcal{R}$ is then given by the nonvanishing relations described in Table 10.

Vector	Generations
A:	$\mathrm{I}\cdot\mathrm{II}$, $\mathrm{II}\cdot\mathrm{III}$, $\mathrm{III}\cdot\mathrm{IV}$
B:	I^2, $\mathrm{I}\cdot\mathrm{III}$, II^2, $\mathrm{II}\cdot\mathrm{III}$, III^2, $\mathrm{III}\cdot\mathrm{IV}$, IV^2
C_l:	$\mathrm{I}\cdot\mathrm{V}_l$, $\mathrm{I}\cdot\mathrm{VI}_l$, $\mathrm{II}\cdot\mathrm{VI}_l$, $\mathrm{III}\cdot\mathrm{VI}_l$, $\mathrm{III}\cdot\mathrm{VII}_l$
D_l:	$\mathrm{I}\cdot\mathrm{VII}_l$, $\mathrm{II}\cdot\mathrm{V}_l$, $\mathrm{II}\cdot\mathrm{VI}_l$, $\mathrm{III}\cdot\mathrm{VII}_l$, $\mathrm{IV}\cdot\mathrm{VI}_l$
E_l:	$\mathrm{I}\cdot\mathrm{VI}_l$, $\mathrm{II}\cdot\mathrm{VI}_l$, $\mathrm{II}\cdot\mathrm{VII}_l$, $\mathrm{III}\cdot\mathrm{V}_l$, $\mathrm{III}\cdot\mathrm{VI}_l$, $\mathrm{III}\cdot\mathrm{VII}_l$, $\mathrm{IV}\cdot\mathrm{VI}_l$, $\mathrm{IV}\cdot\mathrm{VII}_l$
F_l:	$\mathrm{III}\cdot\mathrm{VI}_l$, $\mathrm{IV}\cdot\mathrm{VII}_l$

Table 10: *Product structure of the ring of the (80,0) model.*

Comparing the product structure of the chiral ring derived in Table 10 with the exactly solvable Yukawas of Table 7 shows that the simple renormalization

$$\begin{aligned}&\mathrm{II}\longrightarrow\kappa\mathrm{II},\quad \mathrm{IV}\longrightarrow\kappa\mathrm{IV},\quad \mathrm{VII}_l\longrightarrow\kappa\mathrm{VII}_l,\\ &\mathrm{A}\longrightarrow\kappa^{-1}\,\mathrm{A},\quad \mathrm{D}_l\longrightarrow\kappa^{-1}\,\mathrm{D}_l,\quad \mathrm{F}_l\longrightarrow\kappa\mathrm{F}_l\end{aligned}\tag{7.4.1.5}$$

transforms one set of couplings into the other. It should be noted that in contradistinction to the situation for the theories discussed in the (2,2) context [86, 96, 160] the transformation is not fixed uniquely but is determined only up to a two parameter families of rescalings. If, e.g., we define $\mathrm{A}\longrightarrow\kappa^a\ \mathrm{A}$ and $\mathrm{VII}\longrightarrow\kappa^x\ \mathrm{VII}$, then all the remaining normalizations are determined by the exponents a and x. In (7.4.1.5) we have set $a=-1$ and $x=1$ which seemed to be the simplest choice because it minimizes the number of renormalizations. The fact that the map between the Yukawa couplings is not determined uniquely in the present context will turn out to be a generic feature even though the degree of nonuniqueness depends on the specific models, in particular their gauge groups. We therefore find that knowledge of the underlying exactly solvable theory is a slightly less powerful tool than it is in the context of (2,2) theories.

7.4.2 The (0,2) linear σ–model

At this point we have shown that there exists a chiral ring whose product structure is isomorphic to the Yukawa coupling structure of an exactly solvable theory. It is then natural to ask whether this ideal can be derived from the superpotential of a (0,2) σ–model. We will show that indeed it can. Recall [179, 56], that the essential structure of a (0,2) linear σ–model is encoded in the superpotentials $(W_r(\Phi_i), F_a^l(\Phi_i)))$, where $W_r(\Phi_i)$ are polynomials of degree d_r which define the base space M of the vector bundle $V \longrightarrow M$ associated to the left–moving gauge fermions and the $F_a^l(\Phi_i)$ are polynomials of degree $\deg F_a^l = m_l - n_a$ which define the global structure of the bundle V. The appropriate constraints are imposed by introducing Lagrange multipliers Σ^r which are Fermi superfields with charge $(-d_r)$, as well as Lagrange multipliers Λ^a which are Fermi superfields with charges n_a. Finally, one introduces chiral superfields Θ_l with charges m_l such that $\sum_l m_l = \sum_a n_a$. The charges assigned to these fields read as follows

$$\begin{aligned} &Q(\Phi_i) = k_i, \quad i = 1, ..., N_i; && Q(\Sigma^r) = -d_l, \quad r = 1, ..., N_r \\ &Q(\Lambda^a) = n_a, \quad a = 1, ..., N_a = r + N_l; && Q(\Theta_l) = -m_l, \quad l = 1, ..., N_l. \end{aligned} \tag{7.4.2.1}$$

The action which summarizes the structure of the total bundle is then given by

$$\mathcal{A} = \int d^2 z d\theta \left[\Sigma^r W_r(\Phi_i) + \Theta_l \Lambda^a F_a^l(\Phi_i)\right]. \tag{7.4.2.2}$$

The first term leads to constraints on the Φ_i to the effect that they take values on the hypersurface $W_r = 0$, whereas the second term ensures that the gauge fermions λ^a (i.e. the lowest components of the Λ^a) are sections of the bundle V.

The structure of the bundle V can be summarized concisely by the short exact sequence

$$0 \longrightarrow V \longrightarrow \bigoplus_{a=1}^{r+N_l} \mathcal{O}(n_a) \xrightarrow{F} \bigoplus_{l=1}^{N_l} \mathcal{O}(m_l) \longrightarrow 0 \tag{7.4.2.3}$$

which allows to compute the Chern class

$$c(V) = \frac{c\left(\bigoplus_{a=1}^{r+N_l} \mathcal{O}(n_a)\right)}{c\left(\bigoplus_{l=1}^{N_l} \mathcal{O}(m_l)\right)} \tag{7.4.2.4}$$

which, with

$$c\left(\bigoplus_{a=1}^{r+N_l} \mathcal{O}(n_a)\right) = \prod_a c(\mathcal{O}(n_a)) = \prod_a (1 + c_1(\mathcal{O}(n_a)) = \prod_a (1 + n_a h) \tag{7.4.2.5}$$

and

$$c(\bigoplus_{l=1}^{N_l} \mathcal{O}(m_l)) = \prod_l c(\mathcal{O}(m_l)) = \prod_l (1 + m_l h), \tag{7.4.2.6}$$

leads to

$$c(V) = \frac{\prod_a (1 + n_a h)}{\prod_l (1 + m_l h)}. \tag{7.4.2.7}$$

Expanding this expression leads to the individual Chern classes

$$\begin{aligned} c_1(V) &= \left[\sum_a n_a - \sum_l m_l\right] h \\ c_2(V) &= \frac{1}{2}\left[\sum_l m_l^2 - \sum_a n_a^2\right] h^2 \\ c_3(V) &= -\frac{1}{3}\left[\sum_l m_l^3 - \sum_a n_a^3\right] h^3. \end{aligned} \tag{7.4.2.8}$$

Recalling that for a complete intersection of the form $\mathbb{P}_{(k_1,\ldots,k_{N_i})}[d_1 \ \cdots d_{N_r}]$ the second Chern class is given by $c_2(TM) = \frac{1}{2}\left[\sum_r d_r^2 - \sum_i k_i^2\right]$, the anomaly matching condition can be written as

$$\sum_l m_l^2 - \sum_a n_a^2 = \sum_r d_r^2 - \sum_i k_i^2. \tag{7.4.2.9}$$

Now, the structure of the ring $\mathcal{R}$ given by (7.4.1.3) and (7.4.1.4) tells us that we should look for potentials (W_r, F_a^l) defining a bundle V of rank four which are of degree four. As derived in the previous Subsection the base space is spanned by four coordinates $x_i,\ i = 1,..,4$ of unit weight and two coordinates $y_l,\ l = 1,2$ of weight two. Since we wish to consider a 3–fold it follows that we need to impose two polynomials W_1, W_2 of degree four, i.e. the bundle V will live on the base configuration $M = \mathbb{P}_{(1,1,1,1,2,2)}[4\ 4]$. This manifold is indeed a Calabi–Yau manifold, i.e. $c_1(M) = 0$, the Chern classes of which are $c_2(M) = 10h^2$ and $c_3(M) = -36h^3$, where h is the pullback of the generator of $\mathrm{H}^2(M)$.

The simplest way to define a rank four bundle with potentials F_a of degree four is via the short sequence

$$0 \longrightarrow V \longrightarrow \bigoplus_{a=1}^{5} \mathcal{O}(n_a = 1) \xrightarrow{F} \mathcal{O}(5) \longrightarrow 0, \tag{7.4.2.10}$$

the resulting bundle of which we denote by $V_{(1,1,1,1,1;5)}$, adopting the convention of [111]. The second Chern class of this bundle is equal to the second Chern class of the manifold and hence the anomalies cancel. Furthermore $c_1(V) = 0$ and therefore the bundle

$$V_{(1,1,1,1,1;5)} \longrightarrow \mathbb{P}_{(1,1,1,1,2,2)}[4\ 4] \tag{7.4.2.11}$$

describes the Calabi–Yau phase of the linear σ–model defined by the chiral primary fields with the charges $Q(\Phi_i) = 1,\ i = 1,..,4,\ Q(\Phi_i) = 2,\ i = 5,6$ and $Q(\Lambda^a) = 1,\ a = 1,..,5$. With $c_3(V) = -160h^3$ one finds for the Euler number

$$\chi(V_{(1,1,1,1,1;5)}) = -160 \tag{7.4.2.12}$$

and because the manifold is smooth we expect that this theory has no antigenerations. It follows from the structure of the chiral ring that this is indeed the case and therefore this σ–model leads to 80 generations, in agreement with the exact model discussed in Section 3.

The problem then boils down to finding a choice for the defining polynomials of the base and the defining data for the bundle $V_{(1,1,1,1,1;5)}$ which leads to the ideal (7.4.1.4). First one has to pick the defining data of the manifold, i.e. the polynomials W_r. A transverse choice is given by

$$\begin{aligned} W_1 &= \sum_i x_i^4 + \sum_l y_l^2 \\ W_2 &= \sum_i i x_i^4 + \sum_l l y_l^2. \end{aligned} \tag{7.4.2.13}$$

Next we need to pick the bundle data, i.e. the F_as, which we do as

$$(F_a) = (x_1^4, x_2^4, x_3^4, x_4^4, y_1 y_2). \tag{7.4.2.14}$$

One easily checks that the ideal generated by (W_r, F_a) then is given by

$$\mathcal{I}[x_1^4, x_2^4, x_3^4, x_4^4, y_1^2, y_2^2, y_1 y_2]. \tag{7.4.2.15}$$

This is precisely the ideal we have derived previously from the structure of the generations in the exactly solvable theory.

Having succeeded in identifying the (0,2) Calabi–Yau manifold, the conformal fixed point of which is described by the exactly solvable (80,0) theory, we realize that the product structure we have computed in Section 4.1 for the chiral ring derived from the exact model is nothing but the product structure of the cohomology ring $\mathrm{H}^p(M, \Lambda^q V)$. In particular the Yukawa couplings are determined by the product

$$\mathrm{H}^1(M, V) \otimes \mathrm{H}^1(M, V) \longrightarrow \mathrm{H}^1(M, \Lambda^2 V), \tag{7.4.2.16}$$

where we have used the isomorphism $\mathrm{H}^p(M, \Lambda^q V) = \mathrm{H}^{D-p}(M, \Lambda^{r-q} V)$ for a bundle V of rank r on a D–dimensional variety.

The linear σ–model that we have thus derived from the exactly solvable model of Section 3 has been shown [56] to contain both a (0,2) Calabi–Yau as well as a Landau–Ginzburg phase. We therefore have provided for the SO(10)–(80,0) (0,2)–theory all the ingredients familiar from the context of (2,2) exactly solvable Calabi–Yau σ–models. In particular we have established the existence of exactly solvable theories describing the conformal fixed points of (0,2) Calabi–Yau linear σ–models and have described its precise nature.

7.5 An SU(5) (64,0) (0,2)–model derived from $3^{\otimes 5}$

7.5.1 The exactly solvable theory

The exact model is derived from the parent tensor theory $3^{\otimes 5}$ by inserting the two simple currents

$$\begin{aligned} \Upsilon_1 &= [3\ 0\ -1] \otimes [0\ 0\ 0] \otimes [0\ 0\ 0]^{\otimes 3} \otimes [1] \otimes [0] \otimes [0] \\ \Upsilon_2 &= [3\ 0\ -1] \otimes [3\ 0\ -1] \otimes [0\ 0\ 0]^{\otimes 3} \otimes [1] \otimes [1] \otimes [0]. \end{aligned} \tag{7.5.1.1}$$

In this model the E_6 of the Gepner model is broken down to SU(5) and hence the generation/antigeneration structure is determined by the **10** and $\overline{\mathbf{10}}$ representations of SU(5). The massless spectrum of this model contains (among others) the modes of Table 11.

Representation	**0**	**10**	$\overline{\mathbf{10}}$	**5**	$\bar{\mathbf{5}}$
Spin 0	338	64	0	55	119
Spin 1	10	0	0	0	0

Table 11: *Massless spectrum of the (64,0) SU(5) daughter of the* $3^{\otimes 5}$ *model.*

Now, recall that the **10** decomposes with respect to the maximal subgroup SO(6)×U(1) as

$$\mathbf{10} = \mathbf{6}_{-1} \oplus \mathbf{4}_{3/2}. \tag{7.5.1.2}$$

The generations that result are enumerated in Table 12, where we again use the abbreviations g_i of (7.3.1.3) for the two states appearing in those two minimal N=2 factors at level three that are affected by the simple current as well as the notation (7.3.1.4) for the U(1) factors.

Type	Field	Number
I	$\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[2\,\begin{matrix}2&0\\2&0\end{matrix}\right]\left[3\,\begin{matrix}3&0\\3&0\end{matrix}\right]\ u_0\ u_0\ \begin{bmatrix}v\\v\end{bmatrix}$	6
II	$\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[3\,\begin{matrix}3&0\\3&0\end{matrix}\right]\ u_0\ u_0\ \begin{bmatrix}v\\v\end{bmatrix}$	3
III	$\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[2\,\begin{matrix}2&0\\2&0\end{matrix}\right]\left[2\,\begin{matrix}2&0\\2&0\end{matrix}\right]\ u_0\ u_0\ \begin{bmatrix}v\\v\end{bmatrix}$	3
IV$_i$	$g_i\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[3\,\begin{matrix}3&0\\3&0\end{matrix}\right]\ u_i\ u_0\ \begin{bmatrix}v\\v\end{bmatrix}$	6
V$_i$	$g_i\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[2\,\begin{matrix}2&0\\2&0\end{matrix}\right]\ u_i\ u_0\ \begin{bmatrix}v\\v\end{bmatrix}$	12
VI$_i$	$g_i\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\ u_i\ u_0\ \begin{bmatrix}v\\v\end{bmatrix}$	2
VII$_i$	$\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right] g_i \left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[3\,\begin{matrix}3&0\\3&0\end{matrix}\right]\ u_0\ u_i\ \begin{bmatrix}v\\v\end{bmatrix}$	6
VIII$_i$	$\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right] g_i \left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[2\,\begin{matrix}2&0\\2&0\end{matrix}\right]\ u_0\ u_i\ \begin{bmatrix}v\\v\end{bmatrix}$	12
IX$_i$	$\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right] g_i \left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\ u_0\ u_i\ \begin{bmatrix}v\\v\end{bmatrix}$	2
X$_{ij}$	$g_i\ g_j\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[0\,\begin{matrix}0&0\\0&0\end{matrix}\right]\left[1\,\begin{matrix}1&0\\1&0\end{matrix}\right]\ u_i\ u_j\ \begin{bmatrix}v\\v\end{bmatrix}$	12

Table 12: *Generations of the SU(5) (64,0)–daughter of* $3^{\otimes 5}$.

The **5**s of the SU(5) decompose with respect to the maximal subgroup SO(6)×U(1) as

$$\mathbf{5} = \mathbf{4}_{-1/2} \oplus \mathbf{1}_2. \tag{7.5.1.3}$$

Counting the singlet part and recalling the abbreviations (7.3.1.5) the 55 representations of the **5** then take the form presented in Table 13.

Type	Field	Number
A$_i$	$v_i\left[0\,\begin{matrix}0&0\\-2&-2\end{matrix}\right]\left[2\,\begin{matrix}-2&0\\-4&-2\end{matrix}\right]\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]\ u_i^+\ u_0\ \begin{bmatrix}0\\v\end{bmatrix}$	6
B$_i$	$\left[0\,\begin{matrix}0&0\\-2&-2\end{matrix}\right]\ v_i\left[2\,\begin{matrix}-2&0\\-4&-2\end{matrix}\right]\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]\ u_0\ u_i^+\ \begin{bmatrix}0\\v\end{bmatrix}$	6
C$_{ij}$	$v_i\ v_j\left[0\,\begin{matrix}0&0\\-2&-2\end{matrix}\right]\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]u_i^+\ u_j^+\ \begin{bmatrix}0\\v\end{bmatrix}$	12
D$_{ij}$	$v_i\ v_j\left[3\,\begin{matrix}-3&0\\-5&-2\end{matrix}\right]\left[2\,\begin{matrix}-2&0\\-4&-2\end{matrix}\right]\left[1\,\begin{matrix}-1&0\\-3&-2\end{matrix}\right]u_i^+\ u_j^+\ \begin{bmatrix}0\\v\end{bmatrix}$	24
E$_{ij}$	$v_i\ v_j\left[2\,\begin{matrix}-2&0\\-4&-2\end{matrix}\right]\left[2\,\begin{matrix}-2&0\\-4&-2\end{matrix}\right]\left[2\,\begin{matrix}-2&0\\-4&-2\end{matrix}\right]u_i^+\ u_j^+\ \begin{bmatrix}0\\v\end{bmatrix}$	4
F	$\left[3\,\begin{matrix}-4&-1\\-5&-2\end{matrix}\right]\left[3\,\begin{matrix}-4&-1\\-5&-2\end{matrix}\right]\left[1\,\begin{matrix}3&2\\-3&-2\end{matrix}\right]\left[1\,\begin{matrix}3&2\\-3&-2\end{matrix}\right]\left[2\,\begin{matrix}2&0\\-4&-2\end{matrix}\right]\ u_1^+\ u_1^+\ \begin{bmatrix}0\\v\end{bmatrix}$	3

Table 13: *The* **5***s of the* SU(5) *(64,0) (0,2) model of* 3^5.

We now proceed to the couplings.

7.5.2 The exact Yukawa couplings

Recalling the decomposition $\mathbf{10} = \mathbf{6}_{-1} \oplus \mathbf{4}_{3/2}$ one notes that the allowed nonvanishing combination for the Yukawa couplings can be taken to be of the form

$$< \mathbf{5} \cdot \mathbf{10} \cdot \mathbf{10} > \;=\; < \mathbf{1}_2 \cdot \mathbf{6}_{-1} \cdot \mathbf{6}_{-1} > . \qquad (7.5.2.1)$$

Getting the pictures aligned again involves $\Sigma = \begin{bmatrix} 0 & {0 \atop 1} & {0 \atop 1} \end{bmatrix}^{\otimes 5}$, and following our discussion of the first example we find that the only nonvanishing Yukawa couplings involving, say, the field A, are given by

$$< \mathrm{A}_1 \cdot \mathrm{I} \cdot \mathrm{IV}_1 >, \quad < \mathrm{A}_1 \cdot \mathrm{I} \cdot \mathrm{V}_1 >, \quad < \mathrm{A}_1 \cdot \mathrm{II} \cdot \mathrm{V}_1 >, \quad < \mathrm{A}_1 \cdot \mathrm{III} \cdot \mathrm{V}_1 >, \quad < \mathrm{A}_1 \cdot \mathrm{III} \cdot \mathrm{VI}_1 > . \quad (7.5.2.2)$$

For the explicit evaluation of these couplings it again suffices to consider the basic couplings in the individual N=2, $k = 3$ minimal theory collected in (7.3.2.17). Proceeding in this manner one finds again all possibly nonvanishing Yukawas. The result is summarized in Table 14.

5	Generations
$\mathrm{A}_i \cdot$	$\mathrm{I{\cdot}IV}_i = 1$, $\mathrm{I{\cdot}V}_i = 1$, $\mathrm{II{\cdot}V}_i = \kappa$, $\mathrm{III{\cdot}V}_i = 1$, $\mathrm{III{\cdot}VI}_i = \kappa$
$\mathrm{B}_i \cdot$	$\mathrm{I{\cdot}VII}_i = 1$, $\mathrm{I{\cdot}VIII}_i = 1$, $\mathrm{II{\cdot}VIII}_i = \kappa$, $\mathrm{III{\cdot}VIII}_i = 1$, $\mathrm{III{\cdot}IX}_i = \kappa$
$\mathrm{C}_{ij} \cdot$	$\mathrm{I{\cdot}X}_{ij} = 1$, $\mathrm{IV}_i \cdot \mathrm{VII}_j = 1$, $\mathrm{V}_i \cdot \mathrm{VIII}_j = 1$
$\mathrm{D}_{ij} \cdot$	$\mathrm{I{\cdot}X}_{ij} = 1$, $\mathrm{II{\cdot}X}_{ij} = \kappa$, $\mathrm{III{\cdot}X}_{ij} = 1$, $\mathrm{IV}_i \cdot \mathrm{VIII}_j = 1$, $\mathrm{V}_i \cdot \mathrm{VII}_j = 1$, $\mathrm{V}_i \cdot \mathrm{IX}_j = \kappa$, $\mathrm{VI}_i \cdot \mathrm{VIII}_j = \kappa$
$\mathrm{E}_{ij} \cdot$	$\mathrm{III{\cdot}X}_{ij} = \kappa$, $\mathrm{V}_i \cdot \mathrm{VIII}_j = \kappa$, $\mathrm{VI}_i \cdot \mathrm{IX}_j = \kappa^3$

Table 14: *Yukawa couplings of the (64,0) SU(5) theory.*

As indicated in Table 14 there are no couplings involving the modes F_i.

7.5.3 The chiral ring of the (64,0) SU(5) model: states and product structure

The structure of the chiral ring for the present model can be derived in a way completely analogous to our discussion of the first example. Consider again the structure of those generations that do not contain contributions from the minimal factors affected by the simple current. These suggest that again we should introduce coordinates x_i related to the exact state

$$x_i \sim \begin{bmatrix} 0 & {0 \atop 0} & {0 \atop 0} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 & {1 \atop 1} & {0 \atop 0} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & {0 \atop 0} & {0 \atop 0} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (7.5.3.1)$$

where the nonzero state is in the $(i+2)st$ factor of the n individual factors. Again we can read off from the structure of the generations that the weight of the corresponding fields is unity. From the remaining generations we can then read off that the variables

$$\begin{aligned}
y_1 &\sim \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\end{bmatrix} \\
y_2 &\sim \begin{bmatrix}2 & \begin{matrix}-3 & -1\\ 2 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}-1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\end{bmatrix} \\
z_1 &\sim \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0\\0\end{bmatrix} \otimes \begin{bmatrix}0\\0\end{bmatrix} \\
z_2 &\sim \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}2 & \begin{matrix}-3 & -1\\ 2 & 0\end{matrix}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \otimes \begin{bmatrix}0\\0\end{bmatrix} \otimes \begin{bmatrix}-1\\0\end{bmatrix}
\end{aligned} \tag{7.5.3.2}$$

must have degree $\deg(y_i) = 2 = \deg(z_i)$. With these coordinates the generations described in Subsection 5.1 are mapped into the monomials of Table 15.

Family	Monomial Representative	Degeneracy
I	$x_i^2 x_j^3$	6
II	$x_i x_j x_k^3$	3
III	$x_i x_j^2 x_k^2$	3
IV_l	$y_l x_i^3$	6
V_l	$y_l x_i x_j^2$	12
VI_l	$y_l x_i x_j x_k$	2
VII_m	$z_m x_i^3$	6
VIII_m	$z_m x_i x_j^2$	12
IX_m	$z_m x_i x_j x_k$	2
$\mathrm{X}_{l,m}$	$y_l z_m x_i$	12

Table 15: *Monomial representation of the generations of the (64,0) model.*

It follows from the form of the generations that the ideal $\mathcal{I}$ which generates the chiral ring $\mathcal{R} = \frac{\mathbb{C}[x_i,y_j,z_k]}{\mathcal{I}}$ from the free ring $\mathbb{C}[x_i, y_j, z_k]$ is generated by

$$\begin{aligned}
x_i^4 &= 0, \quad i = 1,2,3 \\
y_l y_{l'} &= 0, \quad l,l' = 1,2 \\
z_m z_{m'} &= 0, \quad m,m' = 1,2.
\end{aligned} \tag{7.5.3.3}$$

Next we consider the **5** representation, i.e. the monomials of degree ten, which take the form summarized in Table 16.

Family	Monomial Representative	Degeneracy
A_l	$y_l x_i^2 x_j^3 x_k^3$	6
B_m	$z_m x_i^2 x_j^3 x_k^3$	6
$\mathrm{C}_{l,m}$	$y_l z_m x_i^3 x_j^3$	12
$\mathrm{D}_{l,m}$	$y_l z_m x_i x_j^2 x_k^3$	24
$\mathrm{E}_{l,m}$	$y_l z_m x_i^2 x_j^2 x_k^2$	4

Table 16: *Monomial representation of the vectors of the (64,0) model.*

Thus there are a total of 52 vectors which can be represented by monomials. It is clear from their structure that for all generations $\gamma_i^2 \in \mathcal{I}$, where $\mathcal{I}$ is the ideal described above. Thus there exists no nonvanishing Yukawa 'selfcoupling' of any generation. The Yukawas follow from

$$\gamma_i \cdot \gamma_j \sim v_{ij}, \tag{7.5.3.4}$$

where v_{ij} is any of the **5**s. With this one finds the nonvanishing couplings of Table 17.

Vector	Generations
A_l:	$\mathrm{I}\cdot\mathrm{IV}_l,\ \ \mathrm{I}\cdot\mathrm{V}_l,\ \ \mathrm{II}\cdot\mathrm{V}_l,\ \ \mathrm{III}\cdot\mathrm{V}_l,\ \ \mathrm{III}\cdot\mathrm{VI}_l$
B_l:	$\mathrm{I}\cdot\mathrm{VII}_l,\ \ \mathrm{I}\cdot\mathrm{VIII}_l,\ \ \mathrm{II}\cdot\mathrm{VIII}_l,\ \ \mathrm{III}\cdot\mathrm{VIII}_l,\ \ \mathrm{III}\cdot\mathrm{IX}_l$
C_{lm}:	$\mathrm{I}\cdot\mathrm{X}_{l,m},\ \ \mathrm{IV}_l\cdot\mathrm{VII}_m,\ \ \mathrm{V}_l\cdot\mathrm{VIII}_m$
D_{lm}:	$\mathrm{I}\cdot\mathrm{X}_{lm},\ \ \mathrm{II}\cdot\mathrm{X}_{lm},\ \ \mathrm{III}\cdot\mathrm{X}_{lm},\ \ \mathrm{IV}_l\cdot\mathrm{VIII}_m,\ \ \mathrm{V}_l\cdot\mathrm{VII}_m,\ \ \mathrm{V}_l\cdot\mathrm{IX}_m,\ \ \mathrm{VI}_l\cdot\mathrm{VIII}_m$
E_{lm}:	$\mathrm{III}\cdot\mathrm{X}_{lm},\ \ \mathrm{V}_l\cdot\mathrm{VIII}_m,\ \ \mathrm{VI}_l\cdot\mathrm{IX}_m$

Table 17: *Product structure of the chiral ring of the (64,0) SU(5) theory.*

Renormalizing the fields as

$$\mathrm{II} \longrightarrow \kappa\mathrm{II}, \quad \mathrm{VI}_l \longrightarrow \kappa\mathrm{VI}_l, \quad \mathrm{IX}_l \longrightarrow \kappa\mathrm{IX}_l, \quad \mathrm{E}_{lm} \longrightarrow \kappa\mathrm{E}_{lm} \tag{7.5.3.5}$$

then transforms the sigma model couplings into the exactly solvable couplings. As in the (80,0) model this transformation is not unique, the difference being that now we have three scales at our disposal.

7.5.4 Remarks concerning the linear σ–model

As in the previous Section we wish to identify superpotentials $(W_r(\Phi_i), F_a^l(\Phi_i))$ which in the present case define a rank five bundle over a base space spanned by coordinates (x_i, y_l, z_m), $i = 1,2,3;\ l = 1,2;\ m = 1,2$. The structure of the chiral ring suggests that we again consider potentials W_r of degree $d_r = 4$. This does not seem to work however because we have seven coordinates of weights k_i such that the sum of the weights is eleven. Therefore it appears that in order to define a Calabi–Yau threefold we have to introduce three polynomials W_r of degree d_r such that $\sum d_r = 11$. This leads one to introduce a further coordinate of weight five and to consider the configuration

$$\mathbb{P}_{(1,1,1,2,2,2,2,5)}[4\ 4\ 4\ 4]. \tag{7.5.4.1}$$

The virtue of this space is that it has a number of pleasant properties. First, it satisfies the anomaly condition

$$\sum_i k_i = \sum_r d_r, \tag{7.5.4.2}$$

thereby defining a complete intersection Calabi–Yau manifold of codimension 4. Second, it satisfies a condition familiar from the context of Landau–Ginzburg theories [140]. Namely the total charge of the theory is the codimension of the corresponding σ–model. Third, it does not have any orbifold singularities. This is an expected feature if the Kähler sector is to be absent for a (0,2) Calabi–Yau manifold. The disadvantage is that this space does not allow for a transverse choice of polynomials. It has a hypersurface singularity at one point. We hope that future insight into the resolution of this singularity will in fact turn this apparent difficulty into a virtue which will resolve a second puzzle which this space generates.

As in the theory discussed in Sections 3 and 4 we have been lead to a ring which is generated by an ideal spanned by elements of degree four. All that has happened is that the number of generators has increased, a change which we have already incorporated in our choice of the base space. It thus would appear natural to consider a bundle with the same quantum numbers of the gauge fermions as before. This leads us to

$$V_{(1,1,1,1,1;5)} \longrightarrow \mathbb{P}_{(1,1,1,2,2,2,2,5)}[4\ 4\ 4\ 4]. \tag{7.5.4.3}$$

Furthermore, the condition involving the second Chern classes of the gauge vector bundle and the tangent bundle is met by this choice of a bundle and therefore the linear σ–model defined by this structure satisfies all of the anomaly conditions. If the bundle would be defined over a smooth manifold however, it would be of rank four and not of rank five. We have to leave for future work whether a better understanding of the resolution does in fact cancel these two difficulties and make this bundle into a proper (0,2) Calabi–Yau manifold or whether an altogether different bundle can be found.

7.6 The (50,0) SU(3)×SU(2) (0,2)–model derived from $3^{\otimes 5}$

7.6.1 The exact theory

Consider the quintic tensor model $3^{\otimes 5}$ enhanced with the three simple currents

$$\begin{aligned}
\Upsilon_1 &= [3\ 0\ -1] \otimes [0\ 0\ 0] \otimes [0\ 0\ 0] \otimes [0\ 0\ 0]^{\otimes 2} \otimes [1] \otimes [0] \otimes [0] \otimes [0] \\
\Upsilon_2 &= [3\ 0\ -1] \otimes [3\ 0\ -1] \otimes [0\ 0\ 0] \otimes [0\ 0\ 0]^{\otimes 2} \otimes [1] \otimes [1] \otimes [0] \otimes [0] \\
\Upsilon_3 &= [3\ 0\ -1] \otimes [3\ 0\ -1] \otimes [3\ 0\ -1] \otimes [0\ 0\ 0]^{\otimes 2} \otimes [1] \otimes [1] \otimes [1] \otimes [0]. \qquad (7.6.1.1)
\end{aligned}$$

In this model the E_6 of the parent Gepner model is broken to SU(3)× SU(2), hence the modes are arranged in representations of SU(3)× SU(2) as shown for the massless sector in Table 18.

Representation:	**0**	**2**	**3**	$\bar{\mathbf{3}}$	**6**	$\overline{\mathbf{6}}$
Spin 0:	370	134	54	154	50	0
Spin 1:	13	0	0	0	0	0

Table 18: *Massless spectrum of the (50,0) SU(3)×SU(2) model.*

The form of the generations can be found in Table 19.

Type		Field	Number
I	$\begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}3 & \begin{matrix}3 & 0\\ 3 & 0\end{matrix}\end{bmatrix}\ u_0^3$	$\begin{bmatrix}v\\ v\end{bmatrix}$	2
II_i	$g_i \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}3 & \begin{matrix}3 & 0\\ 3 & 0\end{matrix}\end{bmatrix}\ u_i\ u_0^2$	$\begin{bmatrix}v\\ v\end{bmatrix}$	4
III_i	$g_i \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix}\ u_i\ u_0^2$	$\begin{bmatrix}v\\ v\end{bmatrix}$	4
IV_i	$\begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix}\ g_i \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}3 & \begin{matrix}3 & 0\\ 3 & 0\end{matrix}\end{bmatrix}\ u_0\ u_i\ u_0$	$\begin{bmatrix}v\\ v\end{bmatrix}$	4
V_i	$\begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix}\ g_i \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix}\ u_0\ u_i\ u_0$	$\begin{bmatrix}v\\ v\end{bmatrix}$	4
VI_i	$\begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix}\ g_i \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}3 & \begin{matrix}3 & 0\\ 3 & 0\end{matrix}\end{bmatrix}\ u_0^2\ u_i$	$\begin{bmatrix}v\\ v\end{bmatrix}$	4
VII_i	$\begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix}\ g_i \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}2 & \begin{matrix}2 & 0\\ 2 & 0\end{matrix}\end{bmatrix}\ u_0^2\ u_i$	$\begin{bmatrix}v\\ v\end{bmatrix}$	4
VIII_{ij}	$g_i\ g_j \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix}\ u_i\ u_j\ u_0$	$\begin{bmatrix}v\\ v\end{bmatrix}$	8
IX_{ij}	$g_i \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix}\ g_j \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix}\ u_i\ u_0\ u_j$	$\begin{bmatrix}v\\ v\end{bmatrix}$	8
X_{ij}	$\begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix}\ g_i\ g_j \begin{bmatrix}0 & \begin{matrix}0 & 0\\ 0 & 0\end{matrix}\end{bmatrix} \begin{bmatrix}1 & \begin{matrix}1 & 0\\ 1 & 0\end{matrix}\end{bmatrix}\ u_0\ u_i\ u_j$	$\begin{bmatrix}v\\ v\end{bmatrix}$	8

Table 19: *Generations of the (50,0) SU(3)×SU(2) daughter of the* $3^{\otimes 5}$ *model.*

whereas the **3**s can be found in Table 20 in which we have used the abbreviations

$$s_0 = \left[2\ \begin{matrix} 2 & 0 \\ -4 & -2 \end{matrix}\right], \qquad s_1 = \left[2\ \begin{matrix} -3 & -1 \\ -4 & -2 \end{matrix}\right],$$
$$t_0 = \left[1\ \begin{matrix} 3 & 2 \\ -3 & -2 \end{matrix}\right], \qquad t_1 = \left[1\ \begin{matrix} -2 & -1 \\ -3 & -2 \end{matrix}\right]. \tag{7.6.1.2}$$

Note that the s_is are just the v_is with conjugate charge on the left side.

Type		Field	Number
A_{ij}	$v_i\ v_j\ \left[0\ \begin{smallmatrix} 0 & 0 \\ -2 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right]\ u_i^+\ u_j^+\ u_0$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	4
B_{ij}	$v_i\ \left[0\ \begin{smallmatrix} 0 & 0 \\ -2 & -2 \end{smallmatrix}\right]\ v_j\ \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right]\ u_i^+\ u_0\ u_j^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	4
C_{ij}	$\left[0\ \begin{smallmatrix} 0 & 0 \\ -2 & -2 \end{smallmatrix}\right]\ v_i\ v_j\ \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right]\ u_0\ u_i^+\ u_j^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	4
D_{ijk}	$v_i\ v_j\ v_k\ \left[1\ \begin{smallmatrix} -1 & 0 \\ -3 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -3 & 0 \\ -5 & -2 \end{smallmatrix}\right]\ u_i^+\ u_j^+\ u_k^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	16
E_{ijk}	$v_i\ v_j\ v_k\ \left[2\ \begin{smallmatrix} -2 & 0 \\ -4 & -2 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ -4 & -2 \end{smallmatrix}\right]\ u_i^+\ u_j^+\ u_k^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	8
F_i	$t_i\ \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[1\ \begin{smallmatrix} -3 & -2 \\ 1 & 0 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right]\ u_i^+\ u_1^{+2}$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	4
G_i	$\left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right]\ t_i\ \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[1\ \begin{smallmatrix} -3 & -2 \\ 1 & 0 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right]\ u_1^+\ u_i^+\ u_1^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	4
H_i	$\left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right]\ t_i\ \left[1\ \begin{smallmatrix} -3 & -2 \\ 1 & 0 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right]\ u_1^{+2}\ u_i^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	4
I_i	$s_i\ \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right]\ u_i^+\ u_1^{+2}$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	2
J_i	$\left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right]\ s_i\ \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right]\ u_1^+\ u_i^+\ u_1^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	2
K_i	$\left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right] \left[3\ \begin{smallmatrix} -4 & -1 \\ -5 & -2 \end{smallmatrix}\right]\ s_i\ \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right] \left[2\ \begin{smallmatrix} -2 & 0 \\ 2 & 0 \end{smallmatrix}\right]\ u_1^{+2}\ u_i^+$	$\left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right]$	2

Table 20: *The* **3***s of the (50,0) SU(3)×SU(2) daughter of the* $3^{\otimes 5}$ *model*

7.6.2 Exact Yukawa couplings

Due to the following decomposition of the representations of E_3 in those of SO(4)× U(1)

$$\begin{aligned} \mathbf{6} = (\mathbf{3},\mathbf{2}) &= \mathbf{4}^v_{-1} \oplus \mathbf{2}^c_2 \\ \mathbf{3} = (\mathbf{3},\mathbf{0}) &= \mathbf{1}_2 \oplus \mathbf{2}^s_{-1} \end{aligned} \tag{7.6.2.1}$$

we can calculate the Yukawa coupling in the following way

$$< \mathbf{3}\ \mathbf{6}\ \mathbf{6} > = < \mathbf{1}_2\ \mathbf{4}^v_{-1}\ \mathbf{4}^v_{-1} > . \tag{7.6.2.2}$$

Proceeding in a completely analogous way as in the two former examples leads to the nonvanishing Yukawas of Table 21.

3	Generations
$\mathrm{A}_{ij}\cdot$	$\mathrm{I}\cdot\mathrm{VIII}_{ij}=1,\ \mathrm{II}_i\cdot\mathrm{IV}_j=1,\ \ \mathrm{III}_i\cdot\mathrm{V}_j=1$
$\mathrm{B}_{ij}\cdot$	$\mathrm{I}\cdot\mathrm{IX}_{ij}=1,\ \mathrm{II}_i\cdot\mathrm{VI}_j=1,\ \ \mathrm{III}_i\cdot\mathrm{VII}_j=1$
$\mathrm{C}_{ij}\cdot$	$\mathrm{I}\cdot\mathrm{X}_{ij}=1,\ \mathrm{IV}_i\cdot\mathrm{VI}_j=1,\ \ \mathrm{V}_i\cdot\mathrm{VII}_j=1$
$\mathrm{D}_{ijk}\cdot$	$\mathrm{II}_i\cdot\mathrm{X}_{jk}=1,\ \mathrm{III}_i\cdot\mathrm{X}_{jk}=1,\ \ \mathrm{IV}_j\cdot\mathrm{IX}_{ik}=1,\ \mathrm{V}_j\cdot\mathrm{IX}_{ik}=1,\ \ \mathrm{VI}_k\cdot\mathrm{VIII}_{ij}=1,\ \mathrm{VII}_k\cdot\mathrm{VIII}_{ij}=1$
$\mathrm{E}_{ijk}\cdot$	$\mathrm{III}_i\cdot\mathrm{X}_{jk}=\kappa,\ \mathrm{V}_j\cdot\mathrm{IX}_{ik}=\kappa,\ \ \mathrm{VII}_k\cdot\mathrm{VIII}_{ij}=\kappa,$

Table 21: *Yukawa couplings of the generations of the SU(3)×SU(2) (50,0) daughter of* $3^{\otimes 5}$*.*

7.6.3 Extracting the chiral ring

As is familiar by now, the structure of the chiral primary ring can be read off from the form of the generations top down. It is clear from the structure of the simple current that the weights of the chiral fields of the last two minimal factors in the tensor model are not changed. Thus we are led to introduce the coordinates of Table 22.

Coordinate	Weight
$x_1=\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[1\;\begin{matrix}1&0\\1&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	1
$x_2=\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[1\;\begin{matrix}1&0\\1&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	1
$y_1=\left[2\;\begin{matrix}2&0\\2&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	2
$y_2=\left[2\;\begin{matrix}-3&-1\\2&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}-1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	2
$z_1=\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[2\;\begin{matrix}2&0\\2&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	2
$z_2=\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[2\;\begin{matrix}-3&-1\\2&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}-1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	2
$w_1=\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[2\;\begin{matrix}2&0\\2&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}$	2
$w_2=\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[2\;\begin{matrix}-3&-1\\2&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\left[0\;\begin{matrix}0&0\\0&0\end{matrix}\right]\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\end{bmatrix}\otimes\begin{bmatrix}-1\\0\end{bmatrix}$	2

Table 22: *Coordinates of the monomial ring for the (50,0) model.*

The generations then take the form shown in Table 23.

Family	Monomial Representative	Degeneracy
I	$x_i^2 x_j^3$	2
II_l	$u_l x_i^3$	4
III_l	$u_l x_i x_j^2$	4
IV_m	$v_m x_i^3$	4
V_m	$v_m x_i x_j^2$	4
VI_n	$w_n x_i^3$	4
VII_n	$w_n x_i x_j^2$	4
VIII_{lm}	$u_l v_m x_i$	8
IX_{ln}	$u_l w_n x_i$	8
X_{mn}	$v_m w_n x_i$	8

Table 23: *Monomial representation of the generations of the (50,0) model.*

From the generations we read off the ideal

$$\mathcal{I}[x_i^4, u_a u_b, v_a v_b, w_a w_b], \tag{7.6.3.1}$$

from which we see in turn that the monomials of degree ten decompose into the types listed in Table 24.

Family	Monomial Representative	Degeneracy
A_{lm}	$u_l v_m x_1^3 x_2^3$	4
B_{ln}	$u_l w_n x_1^3 x_2^3$	4
C_{mn}	$v_m w_n x_1^3 x_2^3$	4
D_{lmn}	$u_l v_m w_n x_i x_j^3$	16
E_{lmn}	$u_l v_m w_n x_i^2 x_j^2$	8

Table 24: *Monomial representation of the* **3***s of the (50,0) model.*

Thus we find that in the present case only 36 of the total of 54 **3**s admit a monomial representation. Comparing the product structure of the chiral ring with the exact Yukawa couplings we again find complete agreement using the renormalization

$$\mathrm{E} \longrightarrow \kappa \mathrm{E}. \tag{7.6.3.2}$$

For this model we have the least predictive power of the three exactly solvable theory we have discussed. As in the first two examples the number of undetermined scalings is given by $(\mathrm{rk}(V)-2)$ and therefore we find that we can adjust four different renormalizations.

7.6.4 Remarks concerning a possible σ–model representation

The story for the σ–model of the (50,0) model follows the pattern developed in the previous Sections for the (80,0) and the (64,0) models. The additional simple current has led to a further pair of coordinates of weight two. Again the ideal is generated by monomials of weight four and therefore we are led to consider the base space

$$\mathbb{P}_{(1,1,2,2,2,2,2,2,5,5)}[4\ 4\ 4\ 4\ 4\ 4], \tag{7.6.4.1}$$

where we have added two coordinates of weight five for reasons described already in our discussion of the (64,0) model in Section 5. Again this space does not admit transverse polynomials and will have hypersurface singularities, which in the present example is a complex curve, the projective line. Computing the second Chern class, we are led to consider yet again the same vector bundle

$$V_{(1,1,1,1,1;5)} \longrightarrow \mathbb{P}_{(1,1,2,2,2,2,2,2,5,5)}[4\ 4\ 4\ 4\ 4\ 4], \tag{7.6.4.2}$$

which presumably is modified by the resolution of the real–dimension two singular set in such a way as to lead to a rank six bundle.

Comparing the σ–models of all three models discussed in this chapter the emerging pattern indicates that a hypersurface singular set of real–dimension d should increase the rank of the bundle to $(4+d)$.

7.7 Conclusion

In this chapter we have established by construction the existence of conformal fixed points of (0,2) Calabi–Yau σ–models. Along the way we have identified the exactly solvable nature of such a conformal fixed point for the particular framework of (0,2) σ–models discussed in [56]. This generalizes to the framework of (0,2) compactification the work of [85, 144, 176, 179] on the triality of exactly solvable models, Landau–Ginzburg theories and Calabi–Yau manifolds in the context of (2,2) compactifications. Our result thus unifies the different description of (0,2) vacua in a similar manner and allows us to call on many different techniques which are available in conformal field theory, the framework of (non)linear σ–models and the theory of stable vector bundles over algebraic varieties, in order to attack some of the outstanding questions mentioned in the introduction, in particular possible generalizations of mirror symmetry and the structure of the (0,2) moduli space.

There are a number of avenues that present themselves for further exploration. First, it is important to understand the map between the chiral primary fields of the exact theory and the chiral rings in a more systematic fashion than we have described here. Simple currents behave like orbifolds in many respects, which raises the hope that the analysis of [141] can be generalized to the framework of (0,2) theories. Even without such an understanding it would be of interest to establish more (0,2) triality relations even in the context of the type of exactly solvable (0,2) theories which we have discussed. Instead of modifying Gepner models with simple currents, one can also start with more general classes of N=2 superconformal field theories, such as Kazama–Suzuki models and construct (0,2) models by modifying their modular invariants with appropriate simple currents. It is to these questions that we turn in the next and final chapter.

Chapter 8

(0,2) Mirror Symmetry

8.1 Introduction

By establishing triality for (0,2) theories we have in the previous chapter described a solution to a longstanding problem in the context of (0,2) compactification of the heterotic string. (0,2) triality answers in the positive the question whether (0,2) $\sigma-$models renormalize to nontrivial conformally invariant fixed points and if so, whether these fixed points are exactly solvable. A related question is whether such (0,2) theories are in fact consistent or whether they are destabilized by instanton corrections [53, 54, 55].

The examples described above indicate that for our class of (0,2) models the underlying exactly solvable theory directly determines the bundle structure. The structure of the base space is less apparent, however. In Sections 2–7 of the present chapter we generalize the considerations of the previous chapter to several classes of models. It will become apparent that the application of certain types of simple currents amounts to defining maps which result in (0,2) daughter theories for particular types of (2,2) vacua. These maps can be abstracted from the exactly solvable framework and applied to more general Landau–Ginzburg compactifications. Proceeding in this way we construct the first large class of (0,2) Landau–Ginzburg theories.

Armed with (0,2) triality we turn to the discussion of mirror symmetry. In the context of (2,2) Calabi–Yau compactification mirror symmetry usually is formulated as an isomorphism between conformal field theories which, in the large radius limit, leads to an exchange of the Kähler deformation of $\mathrm{H}^{(1,1)}(M)$ and the complex deformations of $\mathrm{H}^{(2,1)}(M)$. Using the Kähler class and the holomorphic 3-form of the Calabi–Yau manifold these cohomology groups are seen to be in 1–1 correspondence with the antigenerations $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM^*)$ $=\mathrm{H}^{(1,1)}(M)$ and generations

$\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM)$ $=\mathrm{H}^{(2,1)}(M)$. The mirror hypothesis thus states that for every ground state M there exists another vacuum $\widetilde{M}$ such that $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM^*)$ $=\mathrm{H}_{\bar\partial}^{(0,1)}(\widetilde{M},T\widetilde{M})$ and $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM)$ $=\mathrm{H}_{\bar\partial}^{(0,1)}(M,T\widetilde{M}^*)$.

In (0,2) compactifications the gauge connection is not embedded in the spin connection, hence the gauge bundle is not identified with the tangent bundle of the manifold. We can therefore ask whether for every consistent (0,2) vacuum described by a stable bundle ($V \longrightarrow M$) over a Calabi–Yau manifold M, with generations in $\mathrm{H}_{\bar\partial}^{(0,1)}(M,V)$ and antigenerations in $\mathrm{H}_{\bar\partial}^{(0,1)}(M,V^*)$, there exists another stable bundle $\widetilde{V} \longrightarrow \widetilde{M}$ such that

$$\begin{aligned} \mathrm{H}_{\bar\partial}^{(0,1)}(M,V^*) &= \mathrm{H}_{\bar\partial}^{(0,1)}(\widetilde{M},\widetilde{V}) \\ \mathrm{H}_{\bar\partial}^{(0,1)}(M,V) &= \mathrm{H}_{\bar\partial}^{(0,1)}(\widetilde{M},\widetilde{V}^*). \end{aligned} \tag{8.1.0.1}$$

As in the (2,2) framework the implication here is that the underlying conformal field theories are isomorphic. If this is the case we will call the two bundles ($V \longrightarrow M, \widetilde{V} \longrightarrow \widetilde{M}$) a (0,2) mirror pair.

In the exactly solvable context mirror symmetry can be seen trivially in terms of the order–disorder duality in each of the $N=2$ minimal factors, leading to the usual reversal of the charges. The implementation of this operation does not act on the string vacuum per se and therefore does not lead, in itself, to any insight. Since our construction of (0,2) models derives from (2,2) theories it is natural to ask whether the known mirror construction in the (2,2) context induce mirror pairs for the resulting (0,2) framework. This is indeed the case. Our strategy to obtain mirror pairs of (0,2) models can thus be summarized independently of any particular framework via the diagram

$$\begin{array}{ccc} (0,2) & \longrightarrow & (2,2) \\ \downarrow & & \downarrow \\ \text{mirror } (0,2) & \longleftarrow & \text{mirror } (2,2). \end{array} \tag{8.1.0.2}$$

We review in Section 8 those of the known (2,2) mirror constructions which we generalize in Section 9 to the framework of (0,2) vacua. We will see that (0,2) theories derived from (2,2) mirror pairs induce (0,2) mirror pairs. In the last Section we discuss some open problems and directions for future work. In an appendix we collect the three generation models that result from our constructions.

8.2 (0,2) Triality

In [23] a Gepner like construction of string models with (0,2) world sheet supersymmetry was introduced. This construction was based on the so-called simple current method to obtain heterotic modular invariant partition functions. We obtained a variety of models with different kinds of gauge groups and massless spectra.

(0,2) vacua have been constructed in a number of different contexts but what has been missing was the analog of (2,2) triality: the identification of the underlying exactly solvable models of (0,2) Calabi–Yau manifolds defined by stable bundles with a Landau–Ginzburg phase. In [25] we described such an identification for a (0,2) daughter of the $(3^5)_{A^5}$ Gepner model based on the simple current[1]

$$J = (0\ 5\ 1)(0\ 0\ 0)^4(1)(0). \tag{8.2.0.3}$$

The (0,2) Calabi–Yau theory is defined by the bundle

$$V_{(1,1,1,1,1;5)} \to \mathbb{P}_{(1,1,1,1,2,2)}[4\ \ 4]\Big|_{(80,0)} \tag{8.2.0.4}$$

constructed from the exact sequence

$$0 \to\ V \to \bigoplus_{a=1}^{5} \mathcal{O}(1) \to \mathcal{O}(5) \to 0, \tag{8.2.0.5}$$

over the threefold configuration $\mathbb{P}_{(1,1,1,1,2,2)}[4\ \ 4]$. Here the subscripts denote the number of generations and antigenerations. Not only did the massless spectra agree but also the $\langle 10 \cdot 16 \cdot 16\rangle$ Yukawa couplings and the spacetime R charges. In [24] the moduli space of this model has been investigated in some detail.

In [25] we confined our analysis to stable bundles defined over Calabi–Yau manifolds which are not only quasismooth but smooth. This left open the problem of establishing (0,2) triality in a more systematic fashion. Our first step will be to remedy this situation. In the process we will construct special classes of $(0,2)$ models with gauge group $SO(10)$, $SU(5)$ and E_3, which will turn out to provide a good testing ground for a number of aspects of $(0,2)$ models, such as mirror symmetry. Before describing a particular class of models we analyze a quasismooth model in some detail.

[1] The notation means $J = \prod_{i=1}^{5}(l_i\ m_i\ s_i)_{N=2}\,(n)_{U(1)_2}(\phi)_{SO(8)}$, where $(l_i\ m_i\ s_i)_{N=2}$ are the quantum numbers of the $i^{\rm th}$ $N=2$ minimal superconformal factor.

8.2.1 A quasismooth $(0,2)$ model

Consider the (0,2) daughter based on the diagonal Gepner model $(3\cdot 8^3)_{A^4}$ and the simple current as before

$$J = (0\ 5\ 1)(0\ 0\ 0)^3(1)(0). \tag{8.2.1.1}$$

The gauge group is $SO(10)\times SU(2)\times U(1)^3$ with $N_{16} = 113$ generations and $N_{\overline{16}} = 5$ antigenerations. More details of the massless spectrum of this model are summarized in Table 1.

$SO(10)$ rep.	**1**	**10**	**16**	$\overline{\mathbf{16}}$
spin 0	561	108	113	5
spin 1	6	0	0	0

Table 1: *Spectrum of the* $(3\cdot 8^3)_{A^4}$ *daughter.*

By carrying out the same analysis of the 113 untwisted states as in [25] one finds that the $(0,2)$ chiral ring is generated by the fields and constraints encoded in the following bundle and threefold data

$$V_{(1,1,1,2,5;10)} \to \mathbb{P}_{(1,1,1,4,4,5)}[8\ \ 8]\Big|_{(113,5)}. \tag{8.2.1.2}$$

These bundle data satisfy $c_1(V_{(1,1,1,2,5;10)}) = 0$ as well as the anomaly cancellation condition $c_2(V_{(1,1,1,2,5;10)}) = c_2(T)$ and by following [179] one finds that the linear $\sigma-$model leads to a well behaved Landau–Ginzburg phase for $r \ll 0$. For $r \gg 0$ one still obtains a compact parameter space but the threefold has both an orbifold and a hypersurface singularity at $(0,0,0,0,0,1)$. An interesting question is, whether one can generalize the algorithm for resolving $(0,2)$ orbifold singularities presented in [57] in such a way as to obtain agreement with the spectrum in the Landau–Ginzburg phase. The spectrum of the latter phase can be obtained by the methods in [113, 56]. In Table 2 we collect the left and right $U(1)$ charges of the chiral fields and Fermi fields in the linear $\sigma-$model.

Field	$\Phi^{1,2,3}$	$\Phi^{4,5}$	Φ^6	$\lambda^{1,2,3}$	λ^4	λ^5	$\sigma^{1,2}$
q_l	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$-\frac{9}{10}$	$-\frac{4}{5}$	$-\frac{1}{2}$	$-\frac{4}{5}$
q_r	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$

Table 2: *Left and right charges of the fields in the linear* $\sigma-$*model.*

Furthermore we need the ground state energies and charges of the twisted sectors of the

Landau–Ginzburg orbifold. These are contained in Table 3.

l	0	1	2	3	4	5	6	7	8	9	10
q_l	-2	0	2	$-\frac{11}{10}$	$\frac{9}{10}$	$-\frac{7}{5}$	$\frac{3}{5}$	$-\frac{1}{10}$	$-\frac{1}{2}$	$-\frac{6}{5}$	$-\frac{4}{5}$
q_r	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{5}$	$-\frac{3}{5}$	$-\frac{9}{10}$	$-\frac{9}{10}$	$-\frac{3}{5}$	0	$\frac{3}{10}$	$-\frac{3}{10}$
E	0	-1	0	$-\frac{13}{20}$	$-\frac{1}{5}$	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{7}{20}$	0	$-\frac{2}{5}$	0

Table 3: *Ground state energies and charges of the twisted sectors. The sectors $l = 11, \ldots, 19$ are determined by CPT invariance.*

Now, we choose the hypersurfaces W_j and bundle constraints F_a as

$$\begin{array}{lcccll} F_1 = \Phi_1^9 \qquad F_2 = \Phi_2^9 & F_3 & = & \Phi_3^9 & F_4 = \Phi_4^2 & F_5 = \Phi_6 \\ \text{and} & W_1 & = & \Phi_5^2 & W_2 = \Phi_4\Phi_5. & \end{array} \tag{8.2.1.3}$$

The left handed **16**s in the untwisted sector are given by polynomials of degree ten modulo

$$P_{10}(\Phi_0^i) \sim P_{10}(\Phi_0^i) + c^a\, F_a(\Phi_0^i) + s^j\, W_j(\Phi_0^i). \tag{8.2.1.4}$$

This leaves exactly 113 generations. Right handed **10**s are given by degree twenty polynomials modulo

$$P_{20}(\Phi_0^i) \sim P_{20}(\Phi_0^i) + c^a\, F_a(\Phi_0^i) + s^j\, W_j(\Phi_0^i), \tag{8.2.1.5}$$

yielding $N_{10} = 105$ vectors from the untwisted sector. The $l = 4$ sector contains 3 left handed **10**s, $\lambda^{1,2,3}_{-\frac{1}{5}}|0\rangle$ and 3 left handed $\overline{\mathbf{16}}$s, $\Phi^{1,2,3}_{-\frac{1}{5}}|0\rangle$. The remaining 2 antigenerations appear in the $l = 6$ sector, $\Phi^{4,5}_{-\frac{1}{5}}|0\rangle$. Thus the R–R sectors of the SCFT and LG model are in complete agreement. With a little more effort one can show that also the NS–R sectors, in particular the number of singlets, agree. The numbers of left moving singlets ($(q_l, q_r) = (0, -\frac{1}{2})$) are listed in Table 4.

l	1	3	5	7
number	$465 + n_g$	48	30	12

Table 4: *Left moving singlets in different sectors. n_g denotes the dimension of the special enhanced gauge group.*

For the choice of the W_j and F_a above we get $n_g = 6$ and thus the desired $N_1 = 561$ massless singlets.

We conclude that the above SCFT based on the $(3\cdot 8^3)_{A^4}$ Gepner parent describes a certain point in the LG phase of the linear $\sigma-$model with the data (8.2.1.2). Inspection of the untwisted $\overline{\mathbf{16}}$s reveals that the introduction of the simple current in the partition function corresponds to the following modification of the geometric data. First, choose the bundle data of the would-be $(0,2)$ model to be the Calabi–Yau data of the $(2,2)$ model corresponding to the parent Gepner model. In the case at hand the $(3\cdot 8^3)_{A^4}$ Gepner model lives on the CY manifold $\mathbb{P}_{(1,1,1,2,5)}[10]$, where the $K=3$ tensor factor corresponds to the coordinate of weight $k=2$ in the ambient space and we have introduced a trivial factor. The CY base manifold of the $(0,2)$ model then is obtained by replacing the coordinate corresponding to the $K=3$ tensor factor by two fields whose weight is twice that of the original coordinate. Finally, the two hypersurfaces have equal weight. This operation which defines a (0,2) theory in terms of the data of certain (2,2) theories can be generalized to the class of Gepner models and, more generally, to all Landau–Ginzburg theories with an appropriate scaling field.

8.2.2 Odd Theories

The application of the supersymmetry breaking simple current to the exactly solvable model can be encoded directly on the Landau–Ginzburg theory by a generalization of the above operation. We will abbreviate our construction by calling it a 'move'. For the class of Gepner models the prescription is as follows.

Choose a Gepner model which contains at least one factor with odd $K=2\ell-1$. Let us assume this factor to be the first one. Let d be the lowest common multiple of the numbers $\{K_1+2, K_2+2, K_3+2, K_4+2, K_5+2\}$. For models with only four factors set $K_5=0$. Then the analysis of the chiral ring reveals that a model obtained by using the following simple currents in the diagonal Gepner parent model

$$J=(0\ K{+}2\ 1)(0\ 0\ 0)^4(1)(0) \tag{8.2.2.1}$$

corresponds to a linear $\sigma-$model with the following data

$$\begin{aligned} &V_{\left(\frac{d}{2\ell+1},\frac{d}{K_2+2},\frac{d}{K_3+2},\frac{d}{K_4+2},\frac{d}{K_5+2};d\right)} \longrightarrow \\ &\qquad\qquad \mathbb{P}_{\left(\frac{2d}{2\ell+1},\frac{\ell d}{2\ell+1},\frac{d}{K_2+2},\frac{d}{K_3+2},\frac{d}{K_4+2},\frac{d}{K_5+2}\right)}\left[\frac{(\ell+2)d}{2\ell+1},\frac{2\ell d}{2\ell+1}\right]. \end{aligned} \tag{8.2.2.2}$$

This set of exactly solvable $(0,2)$ theories can be regarded as the $SO(10)$ analog of the Gepner class of models in the E_6 case. The generalization to Gepner parents with more than five factors or $D-$ and $E-$invariants is straightforward. For instance, the Gepner parent $(1^3\cdot 3^2\cdot 8)_{A^6}$ with one simple current acting on one of the $K = 3$ tensor factors, gives rise to a $(0,2)$ model with input data

$$V_{(3,6,6,10,10,10,15;30)} \rightarrow \mathbb{C}_{(3,6,10,10,10,12,12,15)}[24 \quad 24]. \tag{8.2.2.3}$$

We have calculated the massless spectra of all SCFTs in this class. Figure 1 shows $(N_{16} - N_{\overline{16}})$ and $(N_{16} + N_{\overline{16}})$ for all the resulting 455 models which lead to 140 distinct spectra.

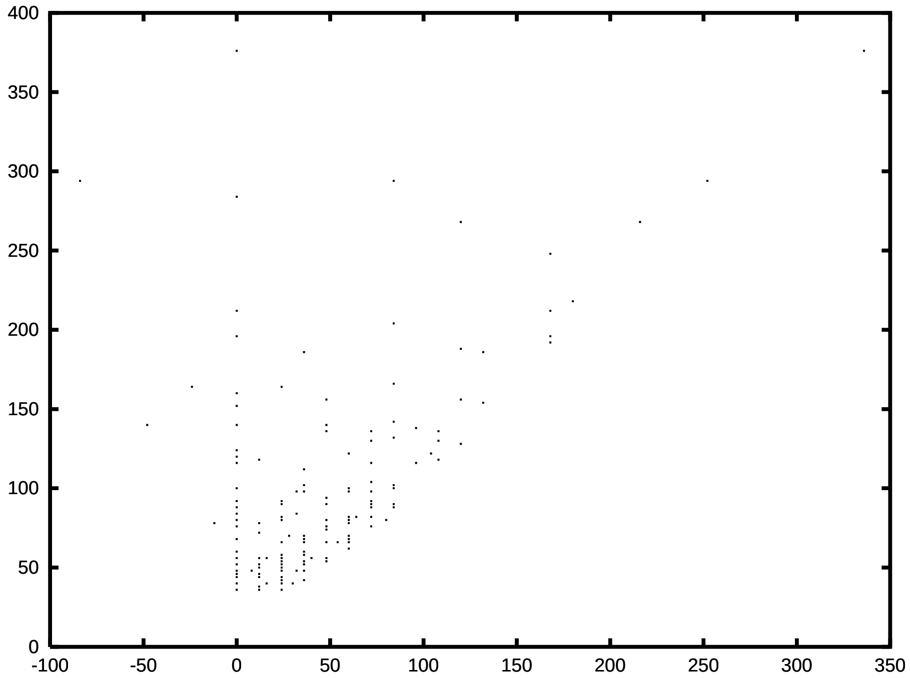


Figure: *140 CFT spectra, $n_{16} + n_{\overline{16}}$ is plotted vs. $n_{16} - n_{\overline{16}}$.*

Similar to the $(2,2)$ case the class of all Gepner models is not mirror symmetric [138, 139, 140, 82, 83]. To get a mirror symmetric plot one had to consider all phase orbifolds including those with discrete torsion [132].

Streamlining the notation, the SCFT analysis has lead us to consider Calabi–Yau hypersurface configurations $\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d]$ in weighted projective fourspace with degree $d = \sum_i k_i$, such that at least one of the weights k_i divides the degree as $d/k_i = (2\ell+1)$. We will characterize such coordinates and the configurations which contain them, as being 'odd'. For any odd configuration $\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d]$, for which we can arrange the odd coordinate to be z_1, we define a (0,2) daughter theory by considering the following vector bundle configuration

$$V_{(k_1,k_2,k_3,k_4,k_5;d)} \rightarrow \mathbb{P}_{(2k_1,\ell k_1,k_2,k_3,k_4,k_5)}[(\ell+2)k_1 \quad 2\ell k_1], \tag{8.2.2.4}$$

where $V_{(k_1,k_2,k_3,k_4,k_5;d)}$ describes a stable bundle over the Calabi–Yau threefold with weights k_i describing the charges of the gauge fermions defined as sections of this bundle. It can be checked that the anomalies of these (0,2) models cancel.

8.3 Pairing of generations and antigenerations

The identification of the model (8.2.1.2) was fairly straightforward. The following example shows that more involved things may happen. Consider the Gepner parent model $(1 \cdot 7 \cdot 25 \cdot 52)_{A^3D}$, where the index D indicates that for the level $K = 52$ minimal $N = 2$ theory we have chosen the nondiagonal affine $D-$invariant. Applying the move to the $K = 25$ tensor factor should correspond to the following model

$$V_{(1,1,3,9,13;27)} \longrightarrow \mathbb{P}_{(1,2,3,9,13,13)}[15\ \ 26]. \tag{8.3.0.5}$$

The spectrum of the superconformal field theory is $(N_{16}, N_{\overline{16}}) = (119, 11)$. The Landau–Ginzburg computation reveals that in the $l = 18$ twisted sector something new appears. The ground state energy is $E = 0$ with charges $(q_l, q_r) = (-\frac{1}{9}, -\frac{11}{18})$. The zero modes are

$$\Phi_0^3,\ \Phi_0^4,\ \overline{\lambda}_0^3,\ \overline{\lambda}_0^4,\ \overline{\sigma}_0^2. \tag{8.3.0.6}$$

In the $q_l = 1$ sector we obtain the following sequence

$$0 \longrightarrow \begin{array}{c} (\Phi_0^3)^2\ \left(\overline{\lambda}_0^3\right) \\ (\Phi_0^3)^4\ \left(\overline{\lambda}_0^4\right) \\ (\Phi_0^3)\ (\Phi_0^4)\ \left(\overline{\lambda}_0^4\right) \\ (\Phi_0^3)^5\ (\overline{\sigma}_0^2) \\ (\Phi_0^3)^2\ (\Phi_0^4)\ (\overline{\sigma}_0^2) \end{array} \longrightarrow \begin{array}{c} (\Phi_0^3)^1\ 0 \\ (\Phi_0^3)^7\ (\Phi_0^4) \\ (\Phi_0^3)^4\ (\Phi_0^4)^2 \\ (\Phi_0^3)\ (\Phi_0^4)^3 \end{array} \longrightarrow 0 \tag{8.3.0.7}$$

with right moving charges

$$0 \longrightarrow \left(1, -\frac{1}{2}\right) \longrightarrow \left(1, \frac{1}{2}\right) \longrightarrow 0. \tag{8.3.0.8}$$

Now, choosing the BRST operator relevant in this sector as

$$\overline{Q}_+ = \lambda_0^3\ \left(\Phi_0^3\right)^8 + \lambda_0^4\ \left(\Phi_0^4\right)^2 + \sigma_0^2\ \left(\Phi_0^3\right)^2\ \left(\Phi_0^4\right) \tag{8.3.0.9}$$

one gets in this sector exactly one massless field with $(q_l, q_r) = (1, -\frac{1}{2})$. Taking also all the other sectors into account one ends up with the spectrum $(N_{16}, N_{\overline{16}}) = (118, 10)$, which is not expected from the SCFT. However, analogously to the model discussed at the beginning, if we choose the BRST operator to be

$$\overline{Q}_+ = \lambda_0^3\ \left(\Phi_0^3\right)^8 + \lambda_0^4\ \left(\Phi_0^4\right)^2 + \sigma_0^2\ \left(\left(\Phi_{-\frac{2}{3}}^2\right)\ \left(\Phi_{\frac{2}{3}}^6\right) + \left(\Phi_{\frac{2}{3}}^2\right)\ \left(\Phi_{-\frac{2}{3}}^6\right)\right) \tag{8.3.0.10}$$

the field $(\Phi_0^3)^7\ (\Phi_0^4)$ is not any longer in the image of $\overline{Q}_+$ and the overall massless spectrum becomes $(N_{16}, N_{\overline{16}}) = (119, 11)$. Thus, this example reveals for the first time an effect for $(0, 2)$ models which was long believed to exist. The number of generations and antigenerations is not necessarily constant over the moduli space. This is not an exception, for approximately 13% of all SCFT models have higher numbers (up to 3) of generations than those models obtained by minimal choices of the constraints in the LG models. The number of net generations is constant however.

8.4 (0,2) Theories from (2,2) Orbifolds

So far we have focused on $(2,2)$ parent models given by Gepner models. We can, however, also establish (0,2) triality for more general models, such as orbifolds. Consider, e. g., the $(2,2)$ orbifold

$$\left(3^5\right)_{A^5}/\mathbb{Z}_5. \tag{8.4.0.11}$$

We will see in Section 7 that this orbifold can alternatively be described, via the fractional transform, as the weighted configuration

$$\mathbb{P}_{(3,4,4,4,5)}[20] = \mathrm{FT}\left(\mathbb{P}_4[5]/\mathbb{Z}_5 : [0\ 4\ 1\ 0\ 0]\right). \tag{8.4.0.12}$$

Applying the move to one of the $K=4$ coordinates leads to

$$V_{(3,4,4,4,5;20)} \longrightarrow \mathbb{P}_{(3,4,4,5,8,8)}[16\ \ 16]\Big|_{(42,2)}. \tag{8.4.0.13}$$

We can obtain a superconformal field theory in the moduli space of this model by first implementing the simple current which generates the orbifolds and then break the (2,2) supersymmetry as in the previous models. This leads to the modification of the partition function with

$$\begin{aligned} J_1 &= (0\ 0\ 0)(0\ \ {-2}\ 0)(0\ 2\ 0)(0\ 0\ 0)^2(0)(0) \\ J_2 &= (0\ 5\ 1)(0\ 0\ 0)^4(1)(0). \end{aligned} \tag{8.4.0.14}$$

Indeed, this SCFT has $(N_{16}, N_{\overline{16}}) = (42,2)$. This model will be reconsidered in the discussion about mirror symmetry.

Another example is provided by the $(1\cdot 5\cdot 82^2)_{A^4}$ parent model describing a point in the moduli space of the threefold configuration $\mathbb{P}_{(1,1,12,28,42)}[84]$. Via fractional transformations one obtains the weighted representation of the orbifold as

$$\mathbb{P}_{(2,3,24,55,84)}[168] = \mathrm{FT}\left(\mathbb{P}_{(1,1,12,28,42)}[84]/\mathbb{Z}_3 : [2\ 0\ 0\ 1\ 0]\right), \tag{8.4.0.15}$$

leading to the $(0,2)$ model

$$V_{(2,3,24,55,84;168)} \longrightarrow \mathbb{P}_{(2,3,48,55,72,84)}[120\ \ 144]\Big|_{(128,32)}. \tag{8.4.0.16}$$

The superconformal theory of this model is given by the above Gepner parent modified by the simple currents

$$\begin{aligned} J_1 &= (0\ \ {-2}\ 0)(0\ 0\ 0)(0\ 56\ 0)(0\ 0\ 0)(0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 7\ 1)(0\ 0\ 0)(0\ 0\ 0)(1)(0). \end{aligned} \tag{8.4.0.17}$$

There exist many other examples. Thus, we have found a large class of $(0,2)$ models featuring the desired CFT/LG/CY triality of description. Further work has to be done to gain a better understanding of the Calabi–Yau phase when the resulting models have hypersurface singularities.

8.4.1 SCFTs with gauge group SU(5) and E_3

In [25] we have generalized our $SO(10)$ analysis to models with gauge group $SU(5)$ by considering the (0,2) theory derived from the 3^5 tensor model via the simple currents

$$\begin{aligned} J_1 &= (0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)^3(1\ 0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 5\ 1)(0\ 0\ 0)^3(0\ 1)(0). \end{aligned} \tag{8.4.1.1}$$

The $(0,2)$ LG theory associated to this exactly solvable model is defined by the following data

$$V_{(0,1,1,1,1,1;5)} \to \mathbb{P}_{(1,1,1,2,2,2,2,5)}[4\ \ 4\ \ 4\ \ 4]\Big|_{(N_{10}=64, N_{\overline{10}}=0)}. \tag{8.4.1.2}$$

Here we introduce a new constraint F_1 of weight $k=5$ which resolves the nontransversality of the coordinate of weight $k=5$, implying that the quadratic term $\lambda_1\,\phi_8$ in the superpotential gives no contributions to the cohomology. This constraint furthermore guarantees that the central charge of the internal CFT sector is $(c_l, c_r) = (11, 9)$, as needed for a rank 5 gauge bundle. The generalization of the move to these models is obvious, one simply applies it twice on each of the coordinates separately.

Consider the $(1 \cdot 3^3 \cdot 13)_{A^5}$ parent model. The simple currents in (8.4.1.1 then yield a SCFT with $(N_{10}, N_{\overline{10}}) = (49, 1)$. The same spectrum can be found for the LG model

$$V_{(0,1,3,3,3,5;15)} \longrightarrow \mathbb{P}_{(1,3,5,6,6,6,6,15)}[12\ \ 12\ \ 12\ \ 12]\Big|_{(N_{10}=49, N_{\overline{10}}=1)}. \tag{8.4.1.3}$$

Another example is provided by the $(1^2 \cdot 3 \cdot 7 \cdot 43)_{A^5}$ parent with the simple currents

$$\begin{aligned} J_1 &= (0\ 0\ 0)^2(0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)(1\ 0)(0) \\ J_2 &= (0\ 0\ 0)^2(0\ 0\ 0)(0\ 9\ 1)(0\ 0\ 0)(0\ 1)(0), \end{aligned} \tag{8.4.1.4}$$

which corresponds to the LG model

$$V_{(0,1,5,9,15,15;45)} \longrightarrow \mathbb{P}_{(1,10,15,15,18,18,20,45)}[30\ \ 36\ \ 36\ \ 40]\Big|_{(N_{10}=36, N_{\overline{10}}=12)}. \tag{8.4.1.5}$$

Finally, the generalization to $E_3 = SU(3) \times SU(2)$ is straightforward. The following three simple currents

$$\begin{aligned} J_1 &= (0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)^2(1\ 0\ 0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)^2(0\ 1\ 0)(0) \\ J_3 &= (0\ 0\ 0)(0\ 0\ 0)(0\ 5\ 1)(0\ 0\ 0)^2(0\ 0\ 1)(0), \end{aligned} \tag{8.4.1.6}$$

applied to the $(3)^5$ Gepner parent yield the spectrum $(N_6, N_{\overline{6}}) = (50, 0)$. This corresponds to the LG model

$$V_{(0,0,1,1,1,1,1;5)} \longrightarrow \mathbb{P}_{(1,1,2,2,2,2,2,2,5,5)}[4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4]\Big|_{(N_6=50, N_{\overline{6}}=0)}, \tag{8.4.1.7}$$

in which the move has been applied three times. Since one needs three odd levels in a Gepner model to get such an E_3 model, there do not exist so many of them, in comparison.

8.5 (0,2) Landau Ginzburg Theories

In this Section we abstract the move map formulated above from its exactly solvable origin and generalize it to the class to (2,2) LG models. A list of all possible weight combinations which allow the choice of a superpotential with an isolated singularity, defining quasismooth projective varieties, was presented in [122, 130]. Given one (2,2) LG theory it is in general possible to define a number of different (0,2) models with different gauge groups, depending on the number of implemented moves. We discuss these possibilities in turn.

8.5.1 The $SO(10)$ Move

The simplest case generalizes the case in which there is only one supersymmetry breaking simple current. The resulting map can be formulated independent of any exactly solvable considerations in the LG framework as follows:

Move: Suppose one has a $(2,2)$ LG model, denoted as

$$\mathbb{C}_{(k_1,\ldots,k_{\max})}[d] \qquad \text{with } d = \tfrac{2}{\max-3}\textstyle\sum_{i=1}^{\max} k_i$$

for which one of the weights, e. g. k_1, satisfies $\frac{d}{k_1} = (2\ell+1)$ with $\ell = 1, 2, \ldots$. Then the choice of the data

$$V_{(k_1,\ldots,k_{\max};d)} \longrightarrow \mathbb{C}_{(2k_1,\ell k_1,k_2,\ldots,k_{\max})}[(\ell+2)k_1 \quad 2\ell k_1]$$

guarantees anomaly cancellation.

Whether such a model really does yield a bona fide $(0,2)$ model with gauge group $SO(10)$ is a more subtle question, since depending on the weights and the constraints effects like nontransversality of the $(0,2)$ LG model or extra massless gauginos in twisted sectors may occur. The latter has been pointed out in [56] where it was considered as the LG analog of the destabilization of the vacuum by world sheet instantons in the CY phase. We will come back to this point below. Note that a move with $\ell = 1$ does not change the model, one simply gets the $(2,2)$ parent. The same can be observed in the SCFT, where the simple current (8.2.2.1) in a $K = 1$ tensor factor does not change the Gepner spectrum at all.

Concerning the transversality of the $(0,2)$ LG superpotential, one can make the following general statement. If the $(2,2)$ superpotential can be chosen as

$$W(x_i) = x_1^{2\ell+1} + P(x_2, \dots, x_{\max}) \qquad (8.5.1.1)$$

then the choice for the $(0,2)$ constraints

$$F_i = \frac{\partial P}{\partial x_i} \quad \text{for } i \in \{2, \dots, \max\}, \qquad F_1 = y_1^\ell,\; W_1 = y_1\, y_2,\; W_2 = y_2^2 \qquad (8.5.1.2)$$

clearly is transverse. Here the coordinates y_1, y_2 are the new ones of weight $\kappa_1 = 2k_1$ and $\kappa_2 = \ell k_1$, respectively. If the coordinate x_1 also appears in P

$$W(x_i) = x_1^{2\ell+1} + x_1 x_2^{a_2} + Q(x_2, \dots, x_{\max}) \qquad (8.5.1.3)$$

then the following choice

$$F_1 = x_2^{a_2},\; F_2 = \frac{\partial Q}{\partial x_2},\; F_i = \frac{\partial Q}{\partial x_i},\; i \in \{3, \dots, \max\}, \quad W_1 = y_1\, y_2, \quad W_2 = y_1^\ell + y_2^2 \qquad (8.5.1.4)$$

is transverse. It may happen that Q does not depend on x_2. In this case one can either choose another polynomial of the right degree or, if this is not possible, set $F_2 = 0$. One still has (max+1) nontrivial constraints for (max+1) variables and thus the $(0,2)$ LG phase makes perfect sense. Most, but not all, such rather rare examples are inconsistent due to the appearance of extra gauginos, however[2].

In order to calculate the massless spectrum in the Landau–Ginzburg phase in a systematic manner an algorithm is needed which can be translated into a computer program. The contribution from the individual twisted sectors to the cohomology however depends on the precise form of the constraints W_j and F_a defining the stable bundle

$$V_{(n_1,\dots,n_{r+1};\, m)} \longrightarrow \mathbb{P}_{(k_1,\dots,k_{N_i})}\,[d_1 \cdots d_{N_c}] \qquad (8.5.1.5)$$

of rank r over a complete intersection space of codimension N_c. More useful in this context is the elliptic genus whose contribution in the α^{th} twisted sector is given by

$$Z_{LG}^\alpha(q, y) = \mathrm{Tr}_{\mathcal{H}_\alpha}\, e^{-\pi i (J_0 - \overline{J}_0)} y^{J_0}\, q^{L_0} \sim \chi_y + O(q), \qquad (8.5.1.6)$$

[2]There are 26 such examples among 8027 models, 5 of which are consistent

its virtue being that it is easily computed. Using the notation of [119] the χ_y genus of a bundle of rank r can be written as

$$\chi_y^\alpha = (-1)^{r\alpha} \frac{\prod_a (-1)^{[\alpha\nu_a]} \left(y^{\nu_a} q^{\frac{\beta_a}{2}}\right)^{\{\alpha\nu_a\}} (1-y^{\nu_a}q^{\{\alpha\nu_a\}})(1-y^{-\nu_a}q^{-\beta_a})}{\prod_i (-1)^{[\alpha q_i]} \left(y^{q_i} q^{\frac{\beta_i}{2}}\right)^{\{\alpha q_i\}} (1-y^{q_i}q^{\{\alpha q_i\}})(1-y^{-q_i}q^{-\beta_i})}\Bigg|_* \tag{8.5.1.7}$$

where $|_*$ indicates the evaluation of the $q^0 y^n$, $n \in \{0, 1, \ldots, r\}$ terms only and

$$\{x\} := x - [x], \quad \beta_a := \{\alpha\nu_a\} - 1, \quad \beta_i := \{\alpha q_i\} - 1. \tag{8.5.1.8}$$

Here the charges of the fields are given by $q_i = k_i/m$ and $\nu_a = n_a/m$. Implementing this in a C–code yields for the example discussed in detail in Section 2.1 the result shown in Table 5.

α	χ^α
0	$y^0 + 113y^1 - 113y^3 - y^4$
1	y^4
2	$-3y^2 + 3y^3$
3	$2y^3$
4	0
5	0
6	0
7	$-2y^1$
8	$-3y^1 + 3y^2$
9	$-y^0$
χ	$108y^1 - 108y^3$

Table 5: *The χ_y genus of $(3 \cdot 8^3)_{A^4}$.*

Using $l = 2\alpha$ one recognizes this as exactly the massless R–R spectrum obtained in Section 2.1. For models which do not lead to a pairing of generations and antigenerations and for which there are no massless gauginos coming from twisted sectors one can read off the number of generations and antigenerations directly from the χ_y genus.

There are circumstances, however, in which extra gauginos do occur. Consider, for instance, the model

$$V_{(5,5,8,9,18;45)} \longrightarrow \mathbb{P}_{(5,5,8,18,18,18)}[36 \;\; 36] \tag{8.5.1.9}$$

which contains a state with $(q_l, q_r) = (1, \frac{3}{2})$ in the $\alpha = 17$ sector, yielding gauginos in the spinor representation of $SO(10)$. As pointed out in [56], for restricted choices for the constraints this theory can still make sense, but not as a model with gauge group $SO(10)$. Unfortunately, the χ_y genus is only sensible for the right moving $U(1)$ charge modulo two and consequently can not detect such gauge symmetry enhancement. What is needed is a convenient condition for the consistency of a $(0,2)$ model. One necessary condition is not hard to formulate. To this end, we also have to take explicitly into account the right moving $U(1)$ charge. Generalizing the χ_y genus, the generating function for all states in the LG model before calculating the cohomology is

$$\chi^\alpha = \frac{\prod_a \left(x^{-\rho_a} y^{\nu_a} q^{\frac{\beta_a}{2}}\right)^{\{\alpha\nu_a\}} (1 + x^{-\rho_a} y^{\nu_a} q^{\{\alpha\nu_a\}})(1 + x^{\rho_a} y^{-\nu_a} q^{-\beta_a})}{\prod_i \left(x^{-(1-q_i)} y^{q_i} q^{\frac{\beta_i}{2}}\right)^{\{\alpha q_i\}} (1 - x^{q_i} y^{q_i} q^{\{\alpha q_i\}})(1 - x^{-q_i} y^{-q_i} q^{-\beta_i})}\Bigg|_* \tag{8.5.1.10}$$

with the abbreviation

$$\rho_a = 1 - \nu_a. \tag{8.5.1.11}$$

For simplicity let us choose a bundle with rank 4 in the following. Since the BRST operator $\overline{Q}_+$ has $(q_l, q_r) = (0,1)$, one obtains sequences of the form

$$0 \xrightarrow{\overline{Q}_+} \left(q_l, -\frac{3}{2}\right) \xrightarrow{\overline{Q}_+} \left(q_l, -\frac{1}{2}\right) \xrightarrow{\overline{Q}_+} \left(q_l, \frac{1}{2}\right) \xrightarrow{\overline{Q}_+} \left(q_l, \frac{3}{2}\right) \xrightarrow{\overline{Q}_+} 0 \tag{8.5.1.12}$$

(or longer) with multiplicities

$$0 \to a_{-\frac{3}{2}} \to a_{-\frac{1}{2}} \to a_{\frac{1}{2}} \to a_{\frac{3}{2}} \to 0. \tag{8.5.1.13}$$

The BRST operator depends explicitly on the constraints

$$\overline{Q}_+ = \oint dz \sum_a \lambda_a F_a + \sum_j \sigma_j W_j. \tag{8.5.1.14}$$

However, if a sequence (8.5.1.12) in a sector contains gauginos $(q_l, q_r = \pm\frac{3}{2})$, for $q_l = 0, \pm 1$, such that $a_{q_r} > a_{q_r+1} + a_{q_r-1}$ then these gauginos are inevitable, independently of the detailed form of the constraints. If this inequality is not satisfied then it appears that one has to check the various possible choices in detail.

In Figure 2 we exhibit all $(0,2)$ models obtained by applying the move on all odd $(2,2)$ LG models in the table of [122, 130][3]. We have checked all model for 'inevitable' gauginos. We have further analyzed a number of models with 'noninevitable' gauginos and have found no examples where the gauginos could not be avoided by appropriate choices for the constraints.

[3]The list of (0,2) models leading to Figure 2 (and also those with smaller gauge groups, cf. Figure 3) can be accessed on the web at the European Calabi–Yau page http://thew02.physik.uni–bonn.de/˜netah/cy.html and its US mirror site http://www.math.okstate.edu/˜katz/CY.

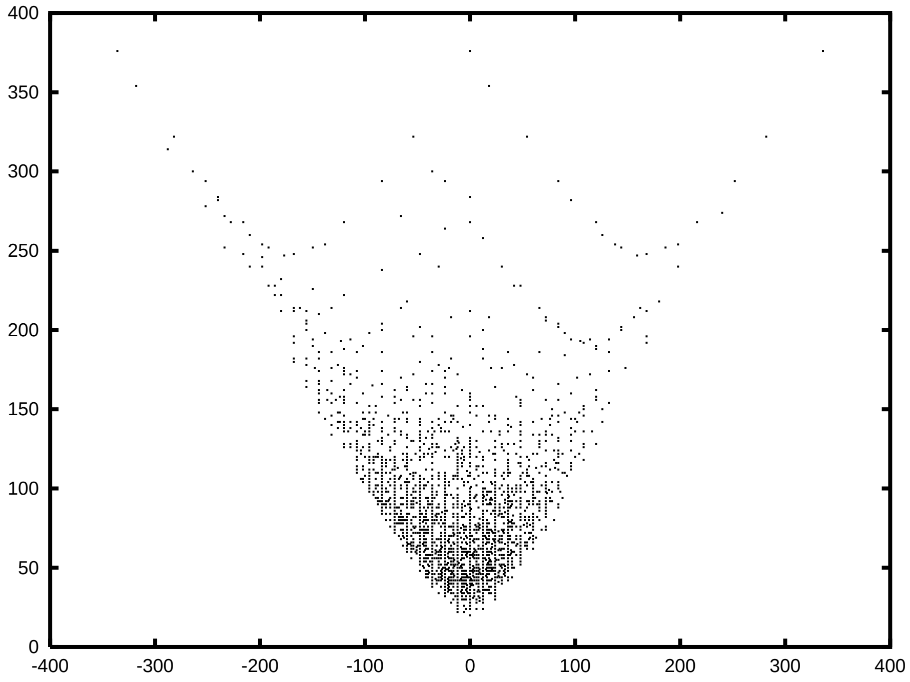


Figure 2:$(n_{16}+n_{\overline{16}})$ *vs.* $(n_{16}-n_{\overline{16}})$ *for all 1757* $SO(10)$ *Landau–Ginzburg spectra.*

Assuming that all these models are bona fide $(0,2)$ models with gauge group $SO(10)$ we have a total number of 6328 different models, which have 1757 different numbers of generations and antigenerations. For 1560 of these 6328 models there exists a mirror partner which results from applying the 'move' to a $(2,2)$ mirror pair. For further 3009 models there exist a model with flipped $(N_{16}, N_{\overline{16}})$. However, we are not claiming that these models really are mirror pairs for the matching of $(N_{16}, N_{\overline{16}})$ alone can also be a coincidence. Even though only 1759 models are not paired, the plot contains 845 unpaired points, which are 48%. As we will discuss in the last Section such an asymmetry is to be expected for our class of models for a number of reasons.

8.5.2 $SU(5)$ and E_3

The construction described above can be generalized to smaller gauge groups $SU(5)$ and $E_3 = SU(3)\times SU(2)$ by simply applying the above move iteratively. We obtain 1216 models and 499 distinct spectra with gauge group $SU(5)$ which are shown in Figure 3. For E_3 there are 81 models and 48 distinct spectra. In this case one needs two or three odd coordinates which have to be preserved by the $(2,2)$ mirror transformation, hence it is clear that we obtain far fewer mirror pairs than for the $SO(10)$ case. Therefore, in the following discussion we will focus on the latter case.

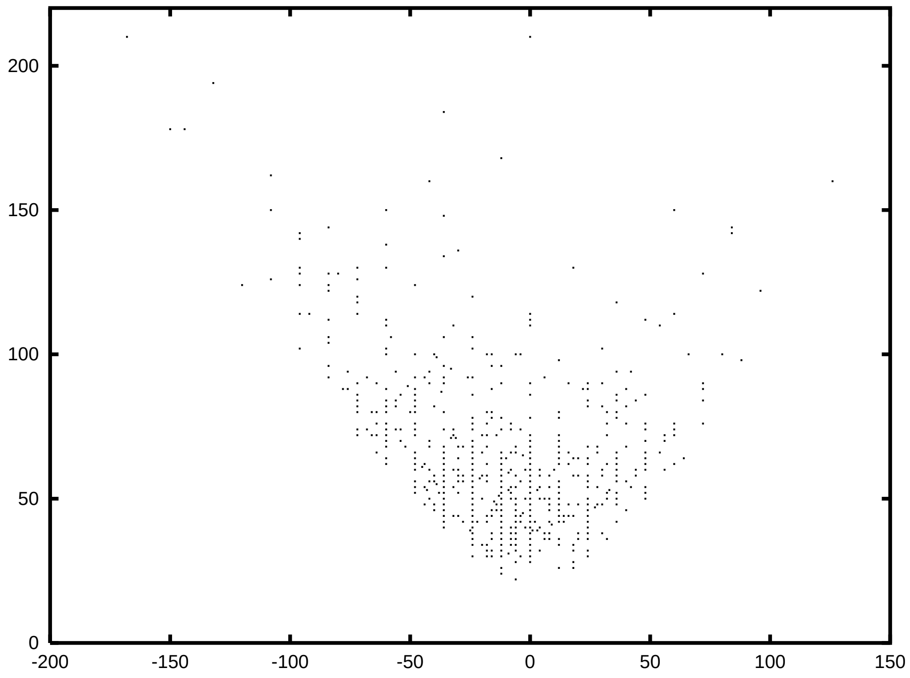


Figure 3:$(n_{10}+n_{\overline{10}})$ *vs.* $(n_{10}-n_{\overline{10}})$ *for the 499* $SU(5)$ *spectra.*

We finally list the 23 $SO(10)$ and 4 $SU(5)$ three generation models in Table A.1 of the appendix.

8.6 Codim>1 weighted (0,2) CICYs

The class of Landau–Ginzburg vacua constructed in [41, 122, 130] contains potentials with more than five scaling variables and many of these theories have an interpretation as weighted complete intersection Calabi–Yau manifolds described by more than one polynomial. The codimension of these more general weighted CICYs is obtained from the LG theory by the total charge $Q=\sum_i q_i$, where the q_i are the weights of the variables, normalized such that the weight of the potential is unity. This total charge precisely determines the codimension of the Calabi–Yau manifold (if it exists) [140].

A simple class is provided by

$$\mathbb{C}_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5)}[2K] \ni \left\{ \sum_{i=1}^{2} \left(x_i^{K/k} + x_i y_i^2 \right) + \sum_{i=3}^{5} x_i^{K/k_i} = 0 \right\} \tag{8.6.0.1}$$

with total charge $Q=2$. If z_5 is the odd coordinate with $K/k_5=(2\ell{+}1)$ the move leads to the

(0,2) theory

$$V_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5;2K)} \to \mathbb{C}_{(2k,K-k,2k,K-k,2k_3,2k_4,4k_5,2\ell k_5)}[2(\ell+2)k_5 \quad 4\ell k_5]. \qquad (8.6.0.2)$$

In manifold speak these (2,2) Landau–Ginzburg theories lead to five-dimensional generalized Calabi–Yau varieties [161, 162, 36]

$$\mathbb{P}_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5)}[2K] \qquad (8.6.0.3)$$

from which one can derive, via the construction of [161, 162], the codimension two Calabi–Yau manifold

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k,k,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 \\ k & K \end{bmatrix}, \qquad (8.6.0.4)$$

defined by polynomials

$$\begin{aligned} p_1 &= y_1^2 x_1 + y_2^2 x_2 \\ p_2 &= x_1^{K/k} + x_2^{K/k} + x_3^{K/k_3} + x_4^{K/k_4} + x_5^{K/k_5}. \end{aligned} \qquad (8.6.0.5)$$

8.7 (2,2) Mirror Symmetry

Our (0,2) mirror construction is based on mirror symmetry in the context of (2,2) theories. In the present Section we therefore briefly review the known mirror constructions before we proceed to generalize them to the (0,2) framework in the following Section.

There have been roughly four different types of constructions of mirror pairs of (2,2) symmetric string vacua. The most precise, if most narrow, framework is provided by considering orbifolds of exactly solvable theories [97]. More general, if less precise, are the Landau–Ginzburg mirror constructions of [141] and [16]. Finally, there exists a toric formulation [12, 19, 35] in terms of reflexive polyhedra. Our focus in the present chapter will be on the first three constructions.

Our first task will be to generalize the orbifolding procedure of the exactly solvable models. As in the (2,2) case we will see that this provides the most precise control over the structure of the mirror. In order to extend the orbifolding analysis of the exactly solvable framework to the more general class of Landau–Ginzburg theories we need some way to extract the weights of the fermions defining the stable bundle for the purported (0,2) mirror Landau–Ginzburg theory. There are two tools available which readily allow us to determine the structure of these gauge bundles. The first is the fractional transform [141] which maps orbifolds with respect to certain discrete group actions into weighted configurations. The second is the transposition construction of Berglund and Hübsch which in particularly simple situations provides the weighted mirror as well. In the present Section we review these (2,2) constructions and in the following Section we apply them to (0,2) mirrors.

8.7.1 Orbifold Mirrors of Gepner Models

The Gepner class [86] of exactly solvable models for which mirror symmetry has been established is the class of tensor products $\otimes_{i=1}^{r} K_i$ of $N=2$ superconformal minimal models constructed in [138, 139, 140, 82, 83]. It was observed in [97] that orbifolding with respect to the maximal discrete symmetry group defines the mirror theory.

The mirror of the quintic can be obtained by taking a $\mathbb{Z}_5^3$ orbifold

$$\left(\mathbb{P}_4[5]\Big/\mathbb{Z}_5^3 : \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix}\right). \tag{8.7.1.1}$$

It turns out that in the SCFT this orbifold can be obtained by using the following sequence of simple currents in the 3^5 Gepner model

$$\begin{aligned} J_1 &= (0\ -2\ 0)(0\ 2\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(0) \\ J_2 &= (0\ -2\ 0)(0\ -2\ 0)(0\ 4\ 0)(0\ 0\ 0)(0\ 0\ 0)(0) \\ J_3 &= (0\ -2\ 0)(0\ -2\ 0)(0\ -2\ 0)(0\ -4\ 0)(0\ 0\ 0)(0). \end{aligned} \tag{8.7.1.2}$$

It should be noted that in this case the simple currents can be chosen such that they are local with respect to the GSO projection.

For the tensor product

$$\left(3\cdot 8^3\right)_{A^4} \sim \mathbb{P}_{(1,1,1,2,5)}[10], \tag{8.7.1.3}$$

which we have been discussing repeatedly, the mirror orbifold is obtained by modding with respect to the group $\mathbb{Z}_{10}^2$

$$\left(\mathbb{P}_{(1,1,1,2,5)}[10]\Big/\mathbb{Z}_{10}^2 : \begin{bmatrix} 9 & 1 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \end{bmatrix}\right). \tag{8.7.1.4}$$

The direct generalization of the simple currents in (8.7.1.2) to this case leads to a model with spectrum $(99,3)$ instead of $(145,1)$. We have not found a way to implement this $\mathbb{Z}_{10}^2$ orbifold with only two $\mathbb{Z}_{10}$ simple currents which are local with respect to the GSO projection. If one does not insist on locality, then the following simple currents surely yield the mirror spectrum

$$\begin{aligned} J_1 &= (0\ 0\ 0)(0\ 2\ 0)(0\ 0\ 0)(0\ 0\ 0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 0\ 0)(0\ 2\ 0)(0\ 0\ 0)(0) \\ J_3 &= (0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 2\ 0)(0). \end{aligned} \tag{8.7.1.5}$$

In this case one has to use two GSO projections.

An example we will come back to is given by

$$(16 \cdot 7 \cdot 2^2 \cdot 1)_{A^5} \sim \mathbb{P}_{(2,4,9,9,12)}[36]^{(19,43)}. \tag{8.7.1.6}$$

The mirror of this model can be obtained by orbifolding with respect to the group

$$\mathbb{Z}_3: \ [2\ 0\ 0\ 0\ 1] \tag{8.7.1.7}$$

which translates into the simple current

$$J_{orb} = (0\ \ -12\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 2\ 0)(0). \tag{8.7.1.8}$$

8.7.2 Weighted $(2,2)$ Mirrors via Fractional Transformations

Mirror symmetry is not restricted to exactly solvable theories however. Contemporaneously with the orbifolding construction mirror symmetry was discovered in the context of Landau–Ginzburg theories [41]. Fractional transformations were introduced in [141] in order to put mirror symmetry among these vacua on firmer ground by establishing mirror pairs in the context of weighted Calabi–Yau configurations.

The simplest starting point again is the set of exactly solvable Gepner theories of products of $N=2$ minimal theories. It was shown in [87, 144, 176, 98, 180] that the Gepner class of models endowed with the diagonal affine invariant in each of the minimal factors describe the renormalization group fixed points of $N=2$ supersymmetric Landau–Ginzburg theories with superpotentials which are of Fermat type

$$W = \sum_{i=1}^{r} \Phi_i^{a_i+2}, \tag{8.7.2.1}$$

where $\Phi_i(z,\theta,\bar{\theta}) = \phi_i(z) + \cdots$ are chiral $N=2$ superfields. Associated to these superpotentials are affine varieties

$$\mathbb{C}_{(k_1,\ldots,k_r)}[d] \ni \{p(z_i) = \sum_i z_i^{a_i+2} = 0\} \tag{8.7.2.2}$$

defined as the zero locus of the polynomials obtained by considering constant lowest components $\phi_i = z_i$ of the superfields.

It was shown in [141] that the mirror symmetry among Landau–Ginzburg theories observed in [41] can be understood by mapping orbifold theories to weighted configurations via the iterated application of the basic isomorphism given by

$$\begin{aligned} &\mathbb{C}_{\left(\frac{b}{g_{ab}},\frac{a}{g_{ab}}\right)}\left[\frac{ab}{g_{ab}}\right] \ni \left\{z_1^a + z_2^b = 0\right\} \Big/ \mathbb{Z}_b : [\,(b-1) \quad 1\,] \quad \sim \\ &\mathbb{C}_{\left(\frac{b^2}{h_{ab}},\frac{(a(b-1)-b)}{h_{ab}}\right)}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \left\{y_1^{\frac{a(b-1)}{b}} + y_1 y_2^b = 0\right\} \Big/ \mathbb{Z}_{b-1} : [\,1 \quad (b-2)\,] \end{aligned} \tag{8.7.2.3}$$

induced by the fractional transformations[4]

$$\begin{aligned} z_1 &= y_1^{1-(\frac{1}{b})}, & y_1 &= z_1^{\frac{b}{(b-1)}} \\ z_2 &= y_1^{\frac{1}{b}} y_2, & y_2 &= z_2 z_1^{-\frac{1}{(b-1)}}. \end{aligned} \tag{8.7.2.4}$$

Here g_{ab} is the greatest common divisor of a and b and h_{ab} is the greatest common divisor of b^2 and $(ab-a-b)$. The action of a cyclic group $\mathbb{Z}_b$ of order b denoted by $[m \quad n]$ indicates that the symmetry acts like $(z_1, z_2) \mapsto (\alpha^m z_1, \alpha^n z_2)$ where α is the b^{th} root of unity.

The isomorphism (8.7.2.3) itself generates the mirror only for very few models. The simplest examples of this type are provided by manifolds for which a $\mathbb{Z}_2$ suffices to generate the mirror. In such a situation the fractional transform

$$z_1 = y_1^{1/2}, \qquad z_2 = y_1^{1/2} y_2 \tag{8.7.2.5}$$

immediately leads to the weighted representation of the orbifold mirror. In the exactly solvable framework such mirrors are obtained by replacing the diagonal invariant by a $D-$invariant in one of the factors of the tensor model. This is exemplified by the Gepner model

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A^4} \quad \sim \quad \mathbb{P}_{(3,8,33,88,132)}[264] \tag{8.7.2.6}$$

with (57,86) for which the replacement $86_A \to 86_D$ leads to the mirror configuration

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A^3 D} \quad \sim \quad \mathbb{P}_{(3,8,66,88,99)}[264]. \tag{8.7.2.7}$$

Much more useful is the iterative application of the relation (8.7.2.3). For the diagonal model (8.7.1.3 the orbifolding with respect to the action $\mathbb{Z}_{10}^2$ leads to the fractional transformations

$$y_1 = z_1^{9/10}, \quad y_2 = z_1^{1/10} z_2^{9/10}, \quad y_3 = z_2^{1/10} z_3, \quad y_4 = z_4, \quad y_5 = z_5 \tag{8.7.2.8}$$

which maps the orbifold into the weighted mirror configuration

$$\mathbb{P}_{(90,80,73,162,405)}[810] \;=\; \mathrm{FT}\left(\mathbb{P}_{(1,1,1,2,5)}[10] \big/ \mathbb{Z}_{10}^2 : \begin{bmatrix} 9 & 1 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \end{bmatrix}\right). \tag{8.7.2.9}$$

[4]This transformation has constant Jacobian and therefore preserves the measure of the path integral.

The fractional transform is not restricted to the Fermat type Landau–Ginzburg theories of exactly solvable minimal models however. As an illustration consider the configuration

$$\mathbb{P}_{(1,1,1,1,3)}[7]^{(2,122)} \ni \{z_1^7 + z_2^7 + z_3^7 + z_4^7 + z_4 z_5^2 = 0\}. \tag{8.7.2.10}$$

After orbifolding by $\mathbb{Z}_7^2$ and applying the fractional transform we find the mirror configuration

$$\mathbb{P}_{(31,35,36,42,108)}[252]^{(122,2)} = \mathrm{FT}\left(\mathbb{P}_{(1,1,1,1,3)}[7]\Big/\mathbb{Z}_7^2 : \begin{bmatrix} 6 & 1 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 \end{bmatrix}\right). \tag{8.7.2.11}$$

The relation (8.7.2.3) shows that in general we should expect some additional modding to be necessary on the fractional transform of the original orbifold in order to find the isomorphic representation. Even though the necessary modding of the FT image becomes part of the projective equivalence of the image in the majority of mirror theories it does happen that the orbifolding has to be performed. An example in case is the mirror orbifold (8.7.1.6) whose fractional transform leads to the weighted configuration $\mathbb{P}_{(3,4,9,9,11)}[36]$. This configuration has the cohomology $(h^{(1,1)}, h^{(2,1)}) = (20,32)$. We see that for the resulting weighted space the $\mathbb{Z}_2$ action on the weighted image is not trivial and must by modded out. Doing so results in the mirror spectrum, as it must. Thus we find for the mirror the relation

$$\begin{aligned}\left(\mathbb{P}_{(2,4,9,9,12)}[36]^{(19,43)}\Big/\mathbb{Z}_3 : \ [2\ 0\ 0\ 0\ 1]\right)^{(43,19)} \sim \\ \left(\mathbb{P}_{(3,4,9,9,11)}[36]^{(20,32)}\Big/\mathbb{Z}_2 : \ [1\ 0\ 0\ 0\ 1]\right)^{(43,19)}.\end{aligned} \tag{8.7.2.12}$$

8.7.3 The Berglund–Hübsch Construction

For manifolds defined by polynomials which are not of Fermat type there exists an alternative construction which sometimes allows to determine the weighted representation of the mirror configuration. This method [16] is based on the 'transposition of polynomials'. In general the transpose of a manifold does not suffice to construct the mirror of a hypersurface and must be accompanied by an additional orbifolding. It does happen for certain configurations, however, that this additional modding is not necessary in which case transposition itself already generates the mirror weights. In such circumstances we are not restricted to the use of fractional transformations in order to generate the weights of the mirror stable bundle but instead can use the transposition prescription or a combination thereof.

A simple class of models is described by the n-tadpole polynomials [41]

$$p = z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n}, \tag{8.7.3.1}$$

whose degree matrix can be written as

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \tag{8.7.3.2}$$

in which the rows contain the degrees for each of the monomials. The transposed polynomial [16] is defined by the transpose of its degree matrix[3]

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix}. \tag{8.7.3.3}$$

Since the dimension of the manifold is to be unchanged under transposition it is apparent that only those polynomials are amenable to transposition for which the number of monomials is equal to the number of variables, excluding the exceptional types discussed in [41].

As mentioned previously, the variety defined by the transposed polynomial is not, in general, the mirror of the original one defined by the polynomial p. Again the modding of an action is the missing ingredient. The orbifolding which produces mirror pairs for pairs of transposed manifolds is determined by splitting off the cyclic group $\mathbb{Z}_d$ whose order is the degree of the polynomial. The symmetry group $D_p = \mathbb{Z}_{a_1 \cdots a_n}$ thus can be decomposed into the quantum phase group [173] $\mathbb{Z}_d$ and the geometric phase group G_p as $D_p = \mathbb{Z}_d \times G_p$ and $\mathbb{Z}_{d_t} \times G_{p_t}$. The prescription for determining the additional orbifolding is that by going from a manifold to its mirror the role of the quantum and the geometric phase groups are exchanged [18]. Toric descriptions of this construction can be found in [19, 35].

We illustrate this with some examples. First consider the manifold

$$\mathbb{P}_{(1,4,16,19,40)}[80]^{(15,127)} \ni \{x_1^{61}x_4 + x_2^{20} + x_3^5 + x_1 x_4^4 + x_5^2 = 0\}. \tag{8.7.3.4}$$

[3]The resulting transformation does not have constant Jacobian.

The matrix of exponents is given by

$$\begin{bmatrix} 61 & 0 & 0 & 1 & 0 \\ 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{8.7.3.5}$$

Transposing it leads to the hypersurface $\mathbb{P}_{(10,23,122,150,305)}[610]^{(127,15)}$ with the correct mirror spectrum.

An even simpler example is provided by $\mathbb{P}_{(4,5,13,13,30)}[65]$ for which partial transposition

$$\begin{bmatrix} 13 & 0 & 1 & 0 & 0 \\ 0 & 13 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \oplus \begin{bmatrix} 13 & 1 \\ 0 & 2 \end{bmatrix} \tag{8.7.3.6}$$

leads to the mirror configuration $\mathbb{P}_{(5,5,12,13,30)}[65]$.

Finally, the transpose of the hypersurface

$$\mathbb{P}_{(35,43,48,84,126)}[336]^{(122,2)} \ni \{z_1^6 z_5 + z_1 z_2^7 + z_3^7 + z_4^4 + z_4 z_5^2 = 0\} \tag{8.7.3.7}$$

yields directly $\mathbb{P}_{(1,1,1,1,3)}[7]^{(2,122)}$. Another mirror configuration for the latter model was obtained in (8.7.2.11).

8.8 (0,2) Mirror Symmetry

In the class of $(2,2)$ vacua a mirror symmetric pair is given by two linear $\sigma-$models or two toric varieties, respectively, which lead to isomorphic superconformal fixed points, differing only by the relative sign of the $U(1)$ charges. In the Calabi–Yau phase this leads to an interchange of complex and Kähler deformations. In the $(0,2)$ setting the first definition still makes perfect sense. In the (0,2) Calabi–Yau phase we are lead to more general cohomology groups, such as $H_{\bar\partial}^{(0,1)}(M,V)$ and $H_{\bar\partial}^{(0,1)}(M,V^*)$, which are exchanged. Thus (0,2) mirror symmetric pairs differ both in the threefold and the vector bundle.

We now apply the (2,2) construction reviewed in the previous Section in order to derive (0,2) mirror symmetry. We will first consider the orbifolding procedure of the exactly solvable models and then determine the weighted (0,2) Landau–Ginzburg mirrors by applying fractional transformations and/or transposition in order to obtain the needed weights of the mirror gauge bundle.

8.8.1 $(0,2)$ Orbifold Mirrors

As in the (2,2) case we can orbifold the exactly solvable tensor model with respect to the cyclic groups $\mathbb{Z}_{K_i+2}$ in each of the minimal factors to generate the exact (2,2) mirror. By applying the supersymmetry breaking simple current (8.2.2.1) to this (2,2) mirror we obtain the (0,2) mirror theory. It is clear from this fact that the space of (0,2) theories which we consider is trivially mirror symmetric by virtue of the order–disorder duality of the individual minimal factors [90].

This observation, however, is not particularly useful because this orbifolding is not an operation on the conformal field theory describing the target space. In order to define the modding procedure on the string ground state proper we need to restrict the symmetry actions to leave invariant the holomorphic threeform of the base space. It can be shown that it is always possible to generate the exact mirror of Gepner models with actions of this type and therefore we arrive at the exact (0,2) mirror after implementing the supersymmetry breaking simple current.

As an example we continue our discussion of the model (8.7.1.6). The application of the supersymmetry breaking simple current

$$J_{\rm sb} = (0\ 0\ 0)(0\ 9\ 1)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(1)(0) \tag{8.8.1.1}$$

leads to a (0,2) theory whose LG/CY phase is described by

$$V_{(2,4,9,9,12;36)} \to \mathbb{P}_{(2,8,9,9,12,16)}[24\quad 32] \tag{8.8.1.2}$$

and whose spectrum can be computed to be $(N_{16}, N_{\overline{16}}) = (34, 10)$. Combining therefore the simple current (8.7.1.8) which produces the (2,2) mirror with supersymmetry breaking simple current (8.8.1.1) we obtain the (0,2) exactly solvable theory

$$\left(\left(16 \cdot 7 \cdot 2^2 \cdot 1\right)_{A^5} \Big/ J_{\rm orb} \otimes J_{\rm sb}\right)^{(10,34)}. \tag{8.8.1.3}$$

Thus we derive the mirror theory by applying the move to the mirror of the (2,2) theory.

8.8.2 $(0,2)$ LG Mirrors via Fractional Transformations

To generalize mirror symmetry to the (0,2) Landau–Ginzburg framework we proceed as follows. Given a (0,2) LG model we first apply the move in order to obtain the (2,2) Calabi–Yau manifold. For this CY space we construct the weighted mirror configuration. We then apply the move in reverse order to get the (0,2) mirror. We thus realize our general strategy described in the introduction in the present context via

$$\begin{array}{ccc} V_{(k_1,k_2,k_3,k_4,k_5;d)} \to \mathbb{P}_{(k_1,k_2,k_3,k_4,2k_5,\ell k_5)}[(\ell+2)k_5 \ \ 2\ell k_5] & \longrightarrow & \mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d] \\ \downarrow & & \downarrow \\ V_{(k'_1,k'_2,k'_3,k'_4,k'_5;d')} \to \mathbb{P}_{(k'_1,k'_2,k'_3,k'_4,2k'_5,\ell k'_5)}[(\ell+2)k'_5 \ \ 2\ell k'_5] & \longleftarrow & \mathbb{P}_{(k'_1,k'_2,k'_3,k'_4,k'_5)}[d']. \end{array} \tag{8.8.2.1}$$

There are a number of subtleties associated with this idea. Consider e.g. the quintic hypersurface. It can be seen via fractional transformations that the mirror of the quintic can be described by

$$\mathbb{P}_{(41,48,51,52,64)}[256] = \mathrm{FT}\left(\mathbb{P}_4[5]\big/\mathbb{Z}_5^4 : \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix}\right). \tag{8.8.2.2}$$

This weighted hypersurface representation of the mirror is not a configuration which is amenable to the application of the move – there are no coordinates such that $d/k_i = (2\ell+1)$. It can be shown, however, that only three of the cyclic groups $\mathbb{Z}_5$ are independent, the fourth one being part of the projective equivalence of the space

$$\mathbb{P}_{(51,64,60,80,65)}[320] = \mathrm{FT}\left(\mathbb{P}_4[5]\big/\mathbb{Z}_5^3 : \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix}\right), \tag{8.8.2.3}$$

obtained via fractional transformations from the $\mathbb{Z}_5^3$ orbifold. This space does contain an appropriate coordinate and we can apply the move map which leads us to the mirror candidate

$$V_{(51,64,60,80,65;320)} \to \mathbb{P}_{(51,60,80,65,128,128)}[256 \ \ 256]. \tag{8.8.2.4}$$

The first obstacle thus is that the weighted representation of the (2,2) mirror of an odd configuration must itself be odd.

Returning yet again to our example from Section 2 based on $(3\cdot 8^3)_{A^4}$ consider

$$V_{(1,1,1,2,5;10)} \to \mathbb{P}_{(1,1,1,4,4,5)}[8 \ \ 8] \tag{8.8.2.5}$$

with spectrum $(N_{16}, N_{\overline{16}}) = (113, 5)$. The fractional transform (8.7.2.9) of its (2,2) image $\mathbb{P}_{(1,1,1,2,5)}[10]$ leads to the (0,2) mirror configuration

$$V_{(90,80,73,162,405;810)} \to \mathbb{P}_{(90,80,73,324,324,405)}[648 \ \ 648] \tag{8.8.2.6}$$

with the reversed (0,2) spectrum.

Next consider

$$V_{(1,1,3,5,5;15)} \to \mathbb{P}_{(1,1,5,5,6,6)}[12 \ \ 12] \tag{8.8.2.7}$$

with spectrum $(N_{16}, N_{\overline{16}}) = (80, 8)$. Via a move and the fractional transform of the (2,2) orbifold mirror

$$\mathbb{P}_{(15,13,42,70,70)}[210] = \mathrm{FT}\left(\mathbb{P}_{(1,1,3,5,5)}[10]\big/\mathbb{Z}_{15} : [\,14 \quad 1 \quad 0 \quad 0 \quad 0\,]\right) \tag{8.8.2.8}$$

we arrive at the (0,2) mirror theory

$$V_{(15,13,42,70,70;210)} \to \mathbb{P}_{(15,13,70,70,84,84)}[168 \;\; 168] \tag{8.8.2.9}$$

with the correct mirror spectrum.

A second possible obstacle is that in general it can happen that one hypersurface gives rise to more than one (0,2) model and hence to more than one mirror relation. In such a situation it is necessary to be able to use different representations of the mirror by mapping different representations of the mirror orbifold via fractional transformations. As an example consider the manifold $\mathbb{P}_{(1,2,6,18,27)}[54]$ which gives rise to the two (0,2) models

$$\begin{array}{rcl} \ell = 4 & : & V_{(1,2,6,18,27;54)} \to \mathbb{P}_{(1,2,12,18,24,27)}[36 \;\; 48] \\ \ell = 13 & : & V_{(1,2,6,18,27;54)} \to \mathbb{P}_{(1,4,6,18,26,27)}[30 \;\; 52], \end{array} \tag{8.8.2.10}$$

each of which gives rise to a mirror diagram. For $\ell = 4$ we orbifold with respect to the group $\mathbb{Z}_{27} \times \mathbb{Z}_3$ to find the Calabi–Yau mirror $\mathbb{P}_{(1,2,6,18,27)}[54]$. Transforming this (2,2) model into a (0,2) theory leads to

$$\begin{array}{ccc} V_{(1,2,6,18,27;54)} \to \mathbb{P}_{(1,2,12,18,24,27)}[36 \; 48] & \longrightarrow & \mathbb{P}_{(1,2,6,18,27)}[54] \\ \downarrow & & \downarrow \\ V_{(18,51,104,295,468;936)} \to \mathbb{P}_{(18,51,208,295,416,468)}[624 \;\; 832] & \longleftarrow & \mathbb{P}_{(18,51,104,295,468)}[936]. \end{array} \tag{8.8.2.11}$$

For $\ell = 13$ on the other hand we orbifold with respect to $\mathbb{Z}_8 \times \mathbb{Z}_3$ and thereby are led to the (0,2) mirror pair

$$\begin{array}{ccc} V_{(1,2,6,18,27;54)} \to \mathbb{P}_{(1,4,6,18,26,27)}[30 \;\; 52] & \longrightarrow & \mathbb{P}_{(1,2,6,18,27)}[54] \\ \downarrow & & \downarrow \\ V_{(18,32,141,241,432;864)} \to \mathbb{P}_{(18,64,141,241,416,432)}[480 \;\; 832] & \longleftarrow & \mathbb{P}_{(18,32,141,241,432)}[864]. \end{array} \tag{8.8.2.12}$$

A third possible obstacle is that a weighted representation of the (2,2) mirror must be found such that the resulting (0,2) theory is not inconsistent because of gauginos. Consider the (0,2) image of the move with $\ell = 3$ of the $(2,2)$ mirror pair in (8.7.2.11). This leads to

$$V_{(1,1,1,1,3;7)} \to \mathbb{P}_{(1,1,1,2,3,3)}[5 \;\; 6] \tag{8.8.2.13}$$

with spectrum $(N_{16}, N_{\overline{16}}) = (86, 2)$ and

$$V_{(31,35,36,42,108;252)} \rightarrow \mathbb{P}_{(31,35,42,72,108,108)}[180 \;\; 216]. \tag{8.8.2.14}$$

The latter model is not consistent because there inevitably are gauginos in the untwisted sector. In the next Section we will see that the transposition of bundles provides for a (0,2) mirror configuration.

8.8.3 Transposition of Stable Bundles

We can carry out the transposition of Calabi–Yau manifold almost verbatim. In the context of the examples discussed above we obtain the following. First consider the (0,2) theory

$$V_{(1,4,16,19,40;80)} \rightarrow \mathbb{P}_{(1,4,19,32,32,40)}[64 \;\; 64] \tag{8.8.3.1}$$

with the move image $\mathbb{P}_{(1,4,16,19,40)}[80]^{(15,127)}$. As shown in (8.7.3.4) and (8.7.3.5) the transposition rule leads to the hypersurface $\mathbb{P}_{(10,23,122,150,305)}[610]^{(127,15)}$ to which we can again apply a move with $\ell = 2$ to obtain

$$V_{(10,23,122,150,305;610)} \rightarrow \mathbb{P}_{(10,23,150,244,244,305)}[488 \;\; 488]. \tag{8.8.3.2}$$

An even simpler example is provided by

$$V_{(4,5,13,13,30;65)} \rightarrow \mathbb{P}_{(4,5,13,26,26,30)}[52 \;\; 52] \tag{8.8.3.3}$$

with spectrum (21,41) and mirror

$$V_{(5,5,12,13,30;65)} \rightarrow \mathbb{P}_{(5,5,12,26,26,30)}[52 \;\; 52] \tag{8.8.3.4}$$

with spectrum (41,21), obtained by only partial transposition of

$$\begin{bmatrix} 13 & 0 & 1 & 0 & 0 \\ 0 & 13 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \oplus \begin{bmatrix} 13 & 1 \\ 0 & 2 \end{bmatrix} \tag{8.8.3.5}$$

in the fiber.

The last example to discuss in this Section is the $(0,2)$ image of the mirror pairs in (8.7.3.7). In contrast to (8.8.2.14), in this case the $(0,2)$ mirror of

$$V_{(1,1,1,1,3;7)} \rightarrow \mathbb{P}_{(1,1,1,2,3,3)}[5 \;\; 6] \tag{8.8.3.6}$$

with $(N_{16}, N_{\overline{16}}) = (86, 2)$ really is

$$V_{(35,43,48,84,126;336)} \rightarrow \mathbb{P}_{(35,43,84,96,126,144)}[240 \ \ 288] \qquad (8.8.3.7)$$

with $(N_{16}, N_{\overline{16}}) = (2, 86)$.

In general then the problem of constructing (0,2) Landau–Ginzburg mirror pairs splits into two parts. Given an odd configuration we first need to check whether it is possible to find complete intersection representation of the mirror orbifold. Both the fractional transform and the transposition operation in general need an additional orbifolding on the image theory and and it is a priori not clear whether the necessary orbifolding will be trivial by becoming part of the projective equivalence of the purported weighted mirror configuration. Second, even if one succeeds in finding a hypersurface fractional transform of the orbifold, in order for our construction to work, one still has to make certain that this complete intersection is odd in the way defined above.

It can be checked that for many Landau–Ginzburg theories both of these problems can be solved via the fractional transform, transposition or a combination thereof.

8.9 Discussion and open problems

By restricting ourselves to $(0, 2)$ models which have a LG description and originate from $(2, 2)$ models admitting a LG phase we should not expect to get a completely mirror symmetric class of theories. There are a number of reasons for the resulting asymmetry.

Firstly, our focus has been on odd configurations, thereby restricting our considerations to a subclass of Landau–Ginzburg theories. More importantly, however, the class of all $(2, 2)$ LG models itself is not mirror symmetric. The missing mirror partners have been shown [35] to be more general hypersurfaces in toric varieties which can formally be written as nontransversal hypersurfaces in weighted projective spaces. Consider, for instance, the manifold $\mathbb{P}_{(5,8,12,15,35)}[75]^{(30,27)}$ contained in the list of [122, 130]. This space has no mirror partner in the class of Landau–Ginzburg theories. It can be described in terms of a reflexive polyhedron as a hypersurface in a toric variety and therefore Batyrev's construction [12] shows that the mirror can be described by its dual polyhedron. It is therefore clear that the weights associated to the dual polyhedron cannot admit a transverse polynomial and must lead to a weighted manifold with hypersurface singularities [35]. It can be shown [16] by combining transposition with fractional transformations that the weighted configuration is given by $\mathbb{P}_{(2,4,5,5,9)}[25]$. All manifolds in this deformation class have a hypersurface singularity at the point $(0,0,0,0,1)$. Applying the move yields the possible $(0,2)$ mirror pair

$$\begin{aligned} &V_{(5,8,12,15,35;75)} \longrightarrow \mathbb{P}_{(5,8,12,30,30,35)}[60\ \ 60]\Big|_{(18,30)} \\ &V_{(2,4,5,5,9;25)} \longrightarrow \mathbb{P}_{(2,4,5,9,10,10)}[20\ \ 20]. \end{aligned} \tag{8.9.0.8}$$

The coordinate of weight $k = 9$ can not appear in the constraints in a transversal way hence this model does not have a $(0,2)$ LG phase and further analysis has to await a more general description of (0,2) models.

Even more strikingly, it can also happen that a nontransversal $(2,2)$ model yields a transversal $(0,2)$ model. For instance, applying the move to the $(2,2)$ toric variety $\mathbb{P}_{(2,2,2,3,5)}[14]$ gives

$$V_{(2,2,2,3,5;14)} \longrightarrow \mathbb{P}_{(2,2,3,4,5,6)}[10\ \ 12]\Big|_{(51,3)}. \tag{8.9.0.9}$$

Due to the new constraint of weight $d = 10$ the coordinate of weight $k = 5$ can now appear in a transversal way leading to a $(0,2)$ model which has a bona fide LG phase. This example indicates that even in the weighted framework the (0,2) class is much richer than the more restricted (2,2) vacua.

A further phenomenon specific to (0,2) models is that, as we have discussed, a $(2,2)$ LG model can lead to a $(0,2)$ model which is destabilized.

Finally, it can happen that given a $(2,2)$ mirror pair with the same ℓ the resulting $(0,2)$ models are not mirror partners. Consider, for instance, the following two models

$$\begin{aligned} &V_{(3,4,4,4,5;20)} \longrightarrow \mathbb{P}_{(3,4,4,5,8,8)}[16\ \ 16]\Big|_{(42,2)} \\ &V_{(13,15,16,16,20;80)} \longrightarrow \mathbb{P}_{(13,15,16,20,32,32)}[64\ \ 64]\Big|_{(8,36)} \end{aligned} \tag{8.9.0.10}$$

The $(2,2)$ parents are mirror symmetric, in particular as already discussed they can be realized as orbifolds of the quintic by the discrete subgroups $\mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$, respectively. The puzzle is resolved by realizing that the SCFT based on the $(K=3)^5$ Gepner parent and the following two simple currents

$$\begin{aligned} J_1 &= (0\ 5\ 1)(0\ 0\ 0)^4(1)(0) \\ J_2 &= (0\ 2\ 0)(0\ -2\ 0)(0\ 0\ 0)^3(0)(0) \end{aligned} \tag{8.9.0.11}$$

yields the $(0,2)$ spectrum $(N_{16}, N_{\overline{16}}) = (36,8)$. This observation naturally leads to the suggestion that the mirror of

$$V_{(13,15,16,16,20;80)} \longrightarrow \mathbb{P}_{(13,15,16,20,32,32)}[64\ \ 64]\Big|_{(8,36)} \tag{8.9.0.12}$$

is a further $(0,2)$ orbifold of

$$V_{(3,4,4,4,5;20)} \rightarrow \mathbb{P}_{(3,4,4,5,8,8)}[16\ \ 16]\Big|_{(42,2)} \tag{8.9.0.13}$$

and vice versa. In order to check this not only on the level of SCFT but on the level of LG models, it would be interesting to develop methods to deal with general $(0,2)$ LG orbifolds.

The same happens with the models

$$\begin{aligned} V_{(4,9,12,15,20;60)} &\longrightarrow \mathbb{P}_{(4,9,15,20,24,24)}[48\ \ 48]\Big|_{(39,3)} \\ V_{(5,11,12,12,20;60)} &\longrightarrow \mathbb{P}_{(5,11,12,20,24,24)}[48\ \ 48]\Big|_{(13,25)}, \end{aligned} \tag{8.9.0.14}$$

the $(2,2)$ parents of which are orbifolds of the $(1 \cdot 3^3 \cdot 15)_{A^5}$ Gepner model.

Even though in dealing with $(0,2)$ models one has to be very careful and prepared for surprises, our results show that the status of $(0,2)$ models is on a par with (2,2) compactifications. As in the latter class we have established mirror symmetry and shown that it can be understood by extending known $(2,2)$ mirror constructions to the $(0,2)$ case. Thus mirror symmetry does in fact generalize to (0,2) theories.

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