Universität Bonn 16.10.2009
Physikalisches Institut
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Condensed Matter Theory I — WS09/10

Exercise 1

(Please return your solutions before Mo. 26.10., 12:00 h)

1.1. Bravais Lattices and Symmetries(I) (10 points)

In the lecture the *Bravais lattice*, the *space group*, i.e. the set $\mathcal{S}$ of all symmetry transformations, which map the lattice onto itself, and its subgroup $\mathcal{P}$ the *point group*, whose elements additionally have at least one fixed point, were defined.

a) Prove that the two-dimensional honeycomb lattice (Fig. 1) is not a Bravais lattice.

b) Give an example, how the honeycomb lattice can be described as a Bravais lattice with basis.

c) Show that $\mathcal{P}$ can only contain rotations with 1-,2-,3-,4- or 6-fold symmetries.
Hint: Rotate a lattice point $\vec{v}$ by $\pm\phi$ and consider the sum of the two new vectors. Which condition must ϕ fulfil?

1.2. Reciprocal Lattice and Symmetries(II) (10 points)

Consider a Bravais lattice $\mathcal{B}$ spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$. The *reciprocal lattice* $\mathcal{R}$ is the Bravais lattice spanned by $\vec{w}_1, \vec{w}_2, \vec{w}_3$, which fulfil $\vec{v}_i \cdot \vec{w}_j = 2\pi\, \delta_{i,j}$

a) Give one possible realization of $\vec{w}_1, \vec{w}_2, \vec{w}_3$. What is the reciprocal lattice of $\mathcal{R}$?

b) Consider a function $f(\vec{x})$ which has the same symmetry as the Bravais lattice. Show that f is given by

$$f(\vec{x}) = \sum_{\vec{k}\in\mathcal{R}} \hat{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

In that sense, $\mathcal{R}$ is the "Fourier transform of $\mathcal{B}$ ".

c) Prove: $\mathcal{P}$ is the point group of $\mathcal{B} \Leftrightarrow \mathcal{P}$ is the point group $\mathcal{R}$.

1.3. Symmetries(III): Quasicrystals (10 points)

In exercise 1.1 you proved that lattices never have a 5-fold symmetry. Nevertheless such *quasicrystals* can be observed in nature. These objects have a local rotational symmetry but no translational symmetry and are built from at least two different unit cells. We will construct an example of such a one-dimensional quasicrystal in the following.

Consider a two-dimensional cubic lattice and choose a coordinate system $(e_{\|}, e_{\perp})$, which is rotated by an angle $\alpha = 30°$ (Fig. 3). We will project now all lattice points in a stripe around the $e_{\|}$ axis onto this axis. As we will see, the result will be a 1-dimensional quasicrystal.
To perform the projection we have to work a little bit more formally: For each lattice point $\vec{n} = (n_1, n_2) \in \mathbb{Z}^2$ its unit cell is given by $C(\vec{n}) := \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_i \in [n_i, n_i + 1)\,, i = 1, 2\}$ (Fig. 2). Denote further the $e_{\|}$ axis by l and define $S := \{\vec{n} \in \mathbb{Z}^2 \,|\, l \cap C(\vec{n}) \neq \emptyset\}$. S is the set of all lower left-handed vertices of all square cells cut by l (open dots in Fig. 3).

a) Project all points of S onto l and show that the result is a non-periodic intersection of l. What does happen for $\alpha = 45°$? Which condition must α fulfil to ensure non-periodicity?

b) Show that the intersection of l is built up by two different unit cells (Fig. 4), i.e. the distance between two projected, neighbouring points on l is either a or b. Express a and b as functions of α.

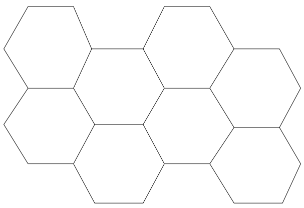

Figure 1: part of a honeycomb lattice

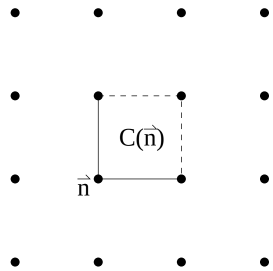

Figure 2: unit cell

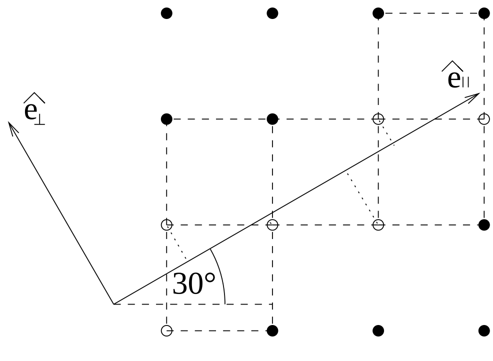

Figure 3: projection scheme

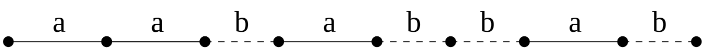

Figure 4: 1-d quasicrystal