Universität Bonn
Physikalisches Institut
Prof. Dr. J. Kroha, J. Martens, Z. Y. Lai
http://www.thp.uni-bonn.de/kroha

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Advanced Theoretical Condensed Matter Physics — SS11

Exercise 7

(Please return your solutions by Tues. 21.06.2011)

In the lecture you have heard about the Keldysh formalism and how it can be used to calculate nonequilibrium quantites. You have also heard how diagrammatics can be similarly done using the Keldysh formalism. In this exercise sheet we will first derive a "Keldysh" version of Wick's theorem, which is essential for the development of any form of diagrammatics. In the second part of the sheet we will write down the expression for the Fock self-energy term in the Keldysh formalism.

7.1. Wick's Theorem for Contour-Ordered Perturbation Theory

As you already know very well by now, Wick's theorem deals with the statement that a quadratically weighted trace (we work in the Matsubara finite-temperature formalism) of a (time-ordered) product of creation and annihilation operators can be decomposed into the sum of products of all possible pairwise contractions. In the case of the Keldysh formalism, the statement can be written down as follows:

$$\langle T_C \left(c(\tau_n) c(\tau_{n-1}) \dots c(\tau_2) c(\tau_1) \right) \rangle = \sum_P \prod_{q,q'} \langle T_{c_t} \left(c_q(\tau) c_{q'}(\tau') \right) \rangle \tag{1}$$

The operators $c(\tau)$ where the contour-ordering symbol, T_C is defined in the following way:

$$T_C \left(c(\tau) c(\tau') \right) = \begin{cases} c(\tau) c(\tau') & \text{for } \tau >_C \tau' \\ c(\tau') c(\tau) & \text{for } \tau' >_C \tau \end{cases}$$

The operators $c(\tau)$ are either creation or annihilation operators, the sum over P denote a sum over all possible ways of picking pairs of operators, and on the right-hand side the q-indices label states with good quantum numbers. We note that on the left hand side of (1) we have dropped the q labels as they are not that important for the purpose of stating Wick's theorem, but one should always keep in mind their existence. In addition, the thermal average $\langle \dots \rangle$ is defined in the usual way as

$$\langle \dots \rangle \equiv \mathrm{tr}(\rho_T \dots)$$

where ρ_T is the statistical operator for the equilibrium state of the *non-interacting* bosons or fermions

$$\rho_T = \frac{e^{-H^{(0)}_{b/f}/kT}}{\mathrm{Tr} e^{-H^{(0)}_{b/f}/kT}} \tag{2}$$

and the noninteracting Hamiltonian is given in the usual form

$$H^{(0)}_{b/f} = \sum_q h_q = \sum_q \epsilon_q \hat{c}^\dagger_q \hat{c}_q$$

where b/f denotes *bosons / fermions*. In our notation we differentiate between the actual creation and annihilation *operators* with a hat, $\hat{c}_q^\dagger$ and $\hat{c}_q$ respectively, while the notation c_q can mean either one, which will be specified in the problem.

We note that for a noninteracting Hamiltonian the different qs are good quantum numbers, and therefore we can write (2) as the decomposition

$$\rho_T = \prod_q \rho_q^T, \quad \rho_q^T = z_q^{-1} e^{-h_q/kT}$$

where $z_q = \frac{1}{1-e^{-\frac{\epsilon_q}{kT}}}$ is the partition function for a single mode q.

a) Show that given the information above that

$$\langle [c_q, A]_s \rangle = \left(1 + s e^{\lambda_c \epsilon_q/kT}\right) \langle c_q A \rangle \tag{3}$$

where $s = \mp$ signifies bose and fermi statistics, respectively, A is an arbitrary operator, and λ_c has the meaning

$$\lambda_c = \begin{cases} +1 & \text{for } c_q = \hat{c}_q^\dagger \\ -1 & \text{for } c_q = \hat{c}_q \end{cases}$$

Hint: It will be easier to show this for bose and fermi statistics separately, and then combine the results. Also it is easier to first compute the quantity $[c_q, \rho_T]$

b) Now we can use the fact that (3) holds to rewrite the product of operators on the left-hand side of (1). First show that

$$\left\langle \prod_{n=1}^{2N} c(\tau_n) \right\rangle = \left(1 + s e^{\lambda_{c(\tau_{2N})}\epsilon_q/kT}\right)^{-1} \left\langle \left[c(\tau_{2N}), \prod_{n=1}^{2N-1} c(\tau_n)) \right]_s \right\rangle \tag{4}$$

Note that we have a a total of $2N$ operators since the expectation value is only nonvanishing for an even number of operators.

c) Work out the commutator in the angled brackets in (4). Show that you obtain

$$\begin{aligned} \left\langle \left[c(\tau_{2N}), \prod_{n=1}^{2N-1} c(\tau_n)) \right]_s \right\rangle = (-s) \left\langle c(\tau_{2N-1}) c(\tau_{2N}) \prod_{n=1}^{2N-2} c(\tau_n) \right\rangle \ + \\ + [c(\tau_{2N}), c(\tau_{2N-1})]_s \left\langle \prod_{n=1}^{2N-2} c(\tau_n) \right\rangle \ + \\ + s \left\langle \left(\prod_{n=1}^{2N-1} c(\tau_n) \right) c(\tau_{2N}) \right\rangle \end{aligned}$$

d) Continue commuting the operator $c(\tau_{2N})$ in the first term repeatedly, and show that you finally obtain

$$\left[c(\tau_{2N}), \prod_{n=1}^{2N-1} c(\tau_n) \right]_s = \sum_{n=1}^{2N-1} (-s)^{n-1} [c(\tau_{2N}), c(\tau_n)]_s \prod_{m=1, m\neq n}^{2N-1} c(\tau_m) \tag{5}$$

e) Finally, putting steps a) - d) together, show that by repeating the steps N number of times, one obtains Wick's theorem as stated in (1).

6.2. Diagrammatics in the Kelydsh Formalism.

In this exercise we would like to see, at least in principle, how the extra complexity of the Keldysh formalism due to the matrix structure of the Green functions can be handled in a perturbative calculation. Specifically, we will write down the expression for the Fock diagram, shown in Fig. 1 below. For purposes of consistency we use the Larkin-Ovchinnikov representation of the tridiagonal matrices[1] and the following dictionary:

Diagram Component	**Convention**
Total Self Energy	$-i\Sigma_{ij}$
Electron Propagator	$iG_{\mathbf{q}}^{aa'}(0^-)$
Photon Propagator	$iD_{\mathbf{q}}^{kk'}$
Vertices	$-i\lambda\gamma_{ia}^{k}$

The meanings of the symbols and indices on the vertices and propagators has been given in the lecture, or you can read[1] for further information.

a) Write down the expression for Fig. 1 in terms of its respective components and vertices without first performing any contraction of the indices. Recall that due to the validity of Wick's theorem the usual diagrammatic rules apply.

b) From the expression in a), perform all the necessary contractions of the indices, and write down the individual components (i, j) of the self-energy matrix $\mathbf{\Sigma}$ in terms of the propagators. In particular, show that $\Sigma^A = \left(\Sigma^R\right)^*$. Note: you don't need to evaluate the propagators!

c) Justify the fact that $\Sigma_{21} = 0$.

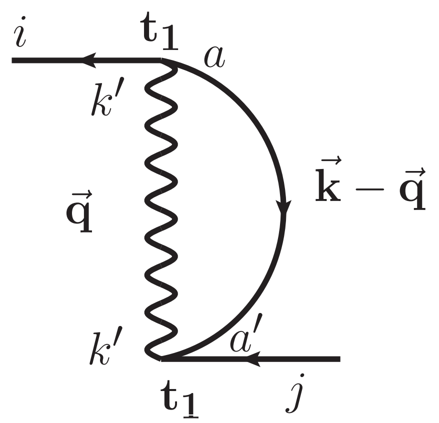

Figure 1: The Fock diagram with the indices labeled.

[1]See equation 2.27 in J. Rammer, H. Smith, *Rev. Mod. Phys., Vol. 58, No. 2 (1986)*; you can find this paper from the course website.