Universität Bonn 30.11.2009
Physikalisches Institut
Dr. A. Lubatsch, K. Ballmann, T. Stollenwerk
http://www.thp.uni-bonn.de/kroha/

Condensed Matter Field Theory — WS09/10

Exercise 4

(Please return your solutions before Fr. 11.12. 12:00h)

4.1 The free causal Green's function (5 points)

Because of the cyclic invariance of the trace for the causal Green's function

$$\begin{aligned} G^c_{\mathbf{k}\sigma}(t,t') &= -\,\mathrm{i}\langle T_\epsilon\{a_{\mathbf{k}\sigma}(t)a^\dagger_{\mathbf{k}\sigma}(t')\}\rangle \\ &= -\,\mathrm{i}\langle T_\epsilon\{a_{\mathbf{k}\sigma}(t-t')a^\dagger_{\mathbf{k}\sigma}(0)\}\rangle \\ &= G^c_{\mathbf{k}\sigma}(t-t') \end{aligned}$$

holds.

(a) Show that

$$\begin{aligned} G^c_{\mathbf{k}\sigma}(0^+) &= -\mathrm{i}\langle 1-n_{\mathbf{k}\sigma}\rangle \\ G^c_{\mathbf{k}\sigma}(0^-) &= +\mathrm{i}\langle n_{\mathbf{k}\sigma}\rangle \end{aligned}$$

holds.

As an ansatz for the *free* causal Green's function as a solution to the equation of motion (see lecture) we use

$$G^{c,0}_{\mathbf{k}\sigma}(E) = \frac{\hbar c_1}{E-(\epsilon(\mathbf{k})-\mu)+i0^+} + \frac{\hbar c_2}{E-(\epsilon(\mathbf{k})-\mu)-i0^+}\,,$$

(b) Calculate $G^{c,0}_{\mathbf{k}\sigma}(t-t')$ by Fourier transform and use the boundary conditions from (a) to fix the constants to $c_1 = 1-\langle n_{\mathbf{k}\sigma}\rangle$ and $c_2 = \langle n_{\mathbf{k}\sigma}\rangle$.
Hint: See exercise 2.1.

4.2 Contractions (5 points)

For the ground state $|\eta_0\rangle$ all energy levels in the Fermi sphere are occupied. We therefore define construction operators for particles and holes as

$$\gamma^\dagger_{\mathbf{k}\sigma} = \begin{cases} a^\dagger_{\mathbf{k}\sigma} & , \quad k > k_F \\ a_{\mathbf{k}\sigma} & , \quad k \le k_F \end{cases}$$

and

$$\gamma_{\mathbf{k}\sigma} = \begin{cases} a_{\mathbf{k}\sigma} & , \quad k > k_F \\ a^\dagger_{\mathbf{k}\sigma} & , \quad k \le k_F \end{cases}\,.$$

The *contraction* is defined by

$$\underbrace{A(t)B(t')} \equiv T_\varepsilon\{A(t)B(t')\} - N\{A(t)B(t')\}\,,$$

where N is the normal product, which arranges the operators $\gamma_{\mathbf{k}\sigma}$ and $\gamma^\dagger_{\mathbf{k}\sigma}$ in the way, that all creation operators are to the left of all annihilation operators. Each commutation gives a factor (-1).

(a) Calculate all possible contractions for two construction operators γ_k and $\gamma_k^\dagger$. There are only two nonvanishing cases.

The time dependence of the original operators $a_{\mathbf{k}\sigma}(t)$ and $a^\dagger_{\mathbf{k}\sigma}(t)$ is given by

$$\begin{aligned} a_{\mathbf{k}\sigma}(t) &= e^{-\frac{\mathrm{i}}{\hbar}(\epsilon(\mathbf{k})-\mu)t} a_{\mathbf{k}\sigma} \\ a^\dagger_{\mathbf{k}\sigma}(t) &= e^{\frac{\mathrm{i}}{\hbar}(\epsilon(\mathbf{k})-\mu)t} a^\dagger_{\mathbf{k}\sigma} \, . \end{aligned}$$

(b) Show that

$$\begin{aligned} \underbracket{a_{\mathbf{k}\sigma}(t)a^\dagger_{\mathbf{k}'\sigma'}(t')} &= \mathrm{i}G^{c,0}_{\mathbf{k}\sigma}(t-t')\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \\ \underbracket{a^\dagger_{\mathbf{k}'\sigma'}(t')a_{\mathbf{k}\sigma}(t)} &= -\,\mathrm{i}G^{c,0}_{\mathbf{k}\sigma}(t-t')\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \end{aligned}$$

holds. Note that the argument of the Green's function always contains the annihilation time minus creation time. Therefore one defines for the contraction of two operators at the same time:

$$t_{annihil.\,op.} - t_{creat.\,op.} = 0^-$$

(c) Use this convention to derive

$$\begin{aligned} \underbracket{a_{\mathbf{k}\sigma}(t)a^\dagger_{\mathbf{k}'\sigma'}(t)} &= -\langle n_{\mathbf{k}\sigma}\rangle\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} \\ \underbracket{a^\dagger_{\mathbf{k}\sigma}(t)a_{\mathbf{k}'\sigma'}(t)} &= \langle n_{\mathbf{k}\sigma}\rangle\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}, . \end{aligned}$$

4.3 Diagrammatic expansion of the causal Green's function (20 points)

Consider a pair interaction of the form

$$V(t_1) = \frac{1}{2}\sum_{klmn} v(kl;nm) \int_{-\infty}^{\infty} dt_1' a_k^\dagger(t_1)a_l^\dagger(t_1')a_m(t_1')a_n(t_1)\delta(t_1-t_1')$$

where we have introduced an additional time t_1' artificially. The indices $k \equiv (\mathbf{k},\sigma)$ denote momentum and spin.

(a) Use equation (8) of exercise 3.2 to calculate the contribution to the causal single electron Green's function

$$\begin{aligned} \mathrm{i}G^c_{\mathbf{k}\sigma}(t,t') =& \langle E_0|T_\epsilon\{a_{\mathbf{k}\sigma}(t)a^\dagger_{\mathbf{k}\sigma}(t')\}|E_0\rangle \\ =& \lim_{\alpha\to 0}\frac{1}{\langle\eta_0|S_\alpha|\eta_0\rangle}\Bigg(\mathrm{i}G^{c,0}_{\mathbf{k}\sigma}(t,t') \\ &+ \frac{1}{2}\left(\frac{-\mathrm{i}}{\hbar}\right)\sum_{plmn} v(pl;nm)\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_1'\delta(t_1-t_1')e^{-\alpha|t_1|} \\ &\cdot \langle\eta_0|T_\epsilon\left\{a_p^\dagger(t_1)a_l^\dagger(t_1')a_m(t_1')a_n(t_1)a_k(t)a_k^\dagger(t')\right\}|\eta_0\rangle\Bigg) \end{aligned} \quad (1)$$

in 1st order perturbation theory in V

In the lecture it was shown, that the vacuum amplitude in the denominator cancels out all the unlinked diagrams in the expansion of the numerator (*linked cluster theorem*). The evaluation of the expectation value of the time ordered product in the numerator of (1) is done using Wick's theorem (see lecture), i.e. the sum of all possible contractions.

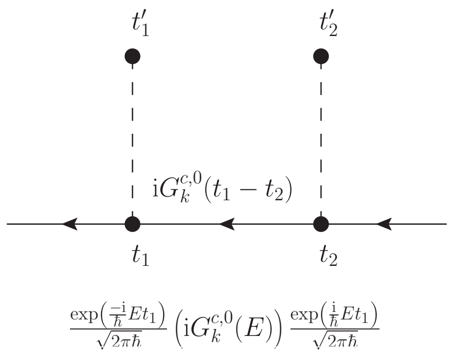
(b) Use Wick's theorem to find out all possible contractions in the numerator of (1). Draw the corresponding Feynman diagrams and label them. Identify the diagrams which are canceled out due to the linked cluster theorem. Calculate the contributions of the relevant diagrams and summarize the Feynman diagram rules, i.e. the rules one has to apply to calculate the contribution of a given diagram.

Hint: Commute the operators in such a way, that two contracted operators stand next to each other.

Energy representation
Since the free causal Green's function $G^{c,0}_{\mathbf{k}\sigma}(t,t')$ has an arkward time dependence (4.1 (b)) it is more convenient to use the energy dependent Fourier transform $G^{c,0}_{\mathbf{k}\sigma}(E)$.

$$G^{c,0}_{\mathbf{k}\sigma}(t,t') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dE\; G^{c,0}_{\mathbf{k}\sigma}(E)\, e^{-\frac{\mathrm{i}}{\hbar}E(t-t')}\,.$$

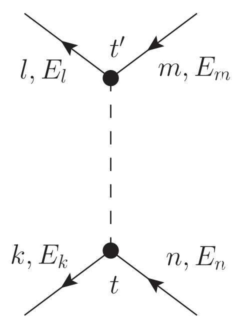
In the diagrams this transition is done in the following way:



We have to integrate now over all internal times *and* energies. An energy dependent exp-factor is now attached to each incoming and outgoing line. Beside $v(kl;nm)$ we attribute the following factor (comprised of four exp-factors) to the vertex

$$\frac{1}{(2\pi\hbar)^2}\exp\left\{\frac{\mathrm{i}}{\hbar}(E_k-E_n)t+\frac{\mathrm{i}}{\hbar}(E_l-E_m)t'-\alpha|t|\right\}\delta(t-t')\,. \tag{2}$$

Therefore also the lines attached to a vertex are denoted by energies:



Now the integration over all internal times can be performed and one is left with the integration over all internal energies.

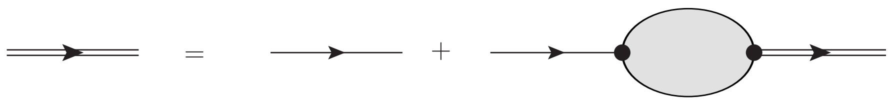
(c) Show that the time integration over t and t' of (2) leads to an energy conserving term at each vertex and that the incoming and outgoing lines of a given diagram correspond to Green's functions with the same energy.

(d) Conclude that the Feynman rules for diagrams in energy representation read

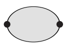
1. Vertex $\Leftrightarrow \frac{1}{2\pi\hbar} v(kl;nm)\delta\left[(E_k + E_l) - (E_m + E_n)\right]$.
2. Propagating and non-propagating line $\Leftrightarrow \mathrm{i}G^{c,0}_{k_\nu}(E_{k_\nu})\delta_{k_\nu k_\mu}$.
3. Factor $(-1)^S \left(\frac{-\mathrm{i}}{\hbar}\right)^n$, with S number of loops.
4. Summation/Integration over all internal wavenumbers, spins and energies.
5. External lines: $\mathrm{i}G^{c,0}_k(E)$.

4.4 Selfenergy in 1st order perturbation theory (10 points)

The Green's function can be written as



$\mathrm{i}G^c_{\mathbf{k}\sigma}(E)$ $\quad$ $\mathrm{i}G^{c,0}_{\mathbf{k}\sigma}(E)$ $\quad$ $\mathrm{i}G^{c,0}_{\mathbf{k}\sigma}(E)\left(\frac{-\mathrm{i}}{\hbar}\Sigma_{\mathbf{k}\sigma}(E)\right)\mathrm{i}G^c_{\mathbf{k}\sigma}(E)$

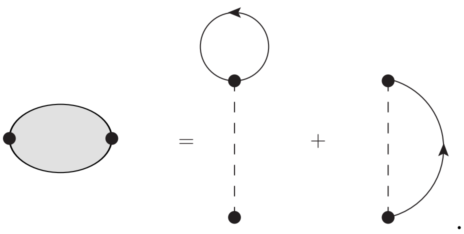
where the selfenergy $\frac{-\mathrm{i}}{\hbar}\Sigma_{\mathbf{k}\sigma}(E) =$  contains all the diagrams which emerge by cutting the incoming and outgoing lines of a $G^c_{\mathbf{k}\sigma}(E)$ diagram and can not be split up into two independent selfenergy diagrams by cutting a $G^{c,0}_{\mathbf{k}\sigma}(E)$ line (see lecture).

(a) Show that $\Sigma_{\mathbf{k}\sigma}(E)$ completely determines $G^c_{\mathbf{k}\sigma}(E)$ by solving Dyson's equation.

(b) Draw all selfenergy diagrams up to second order.

It is convenient to do perturbation theory of the selfenergy instead of the causal Green's function.

(c) Use the Feynman rules to calculate the selfenergy in 1st order perturbation theory.



(d) Consider a pair interaction $V(\mathbf{x},\mathbf{y}) = V(|\mathbf{x}-\mathbf{y}|)$ for free electrons ($|k\rangle = |\mathbf{k}\sigma\rangle$ plane waves). Show that

$$v(kl;nm) = \langle k,l|V|n,m\rangle = \delta_{\mathbf{k}+\mathbf{l},\mathbf{n}+\mathbf{m}}\delta_{\sigma_k\sigma_n}\delta_{\sigma_l\sigma_m}\tilde{v}(\mathbf{k}-\mathbf{n})$$

where $\tilde{v}(\mathbf{k}-\mathbf{n}) = \tilde{v}(\mathbf{q})$ is the Fourier transform of V. Use the results of (c) to calculate the selfenergy in 1st order perturbation theory.