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Physikalisches Institut
Dr. A. Lubatsch, K. Ballmann, T. Stollenwerk
http://www.thp.uni-bonn.de/kroha/

Condensed Matter Field Theory — WS09/10

Exercise 6

(Please return your solutions before Fr. 22.1., 12:00h)

6.1 Keldysh Green's function (30 points)

In class we derived the non-equilibrium Green's function as a matrix in the so-called Keldysh space according to

$$\hat{G}(r,r';t,t') = \begin{pmatrix} 0 & G^A(r,r';t,t') \\ G^R(r,r';t,t') & G^K(r,r';t,t') \end{pmatrix}. \tag{1}$$

a) Use the definitions of $G^{\alpha\beta}(r,r';t,t')$, with $\alpha,\beta = \pm$, as given in class, to convince yourself of

$$\begin{aligned} G^K(r,r';t,t') &= G^{++}(r,r';t,t') + G^{--}(r,r';t,t') \\ &= G^{+-}(r,r';t,t') + G^{-+}(r,r';t,t'). \end{aligned}$$

The retarded and advanced Green's functions $G^{R/A}$ appearing in Eq. (1), contain information about the spectral weight and therefore characterize the states of the system. This is in complete analogy to the already discussed equilibrium situation. The Keldysh component G^K, however, contains information about the occupation of those states. To illuminate this point, let us chose a particularly simple non-equilibrium system, described by the Hamiltonian

$$H = \sum_p \epsilon_p a_p^\dagger a_p, \tag{2}$$

and calculate G^K for this system in $d = 1$ spatial dimensions.

b) Write down the operators $\Psi_H(x,t)$ and $\Psi_H^\dagger(x',t')$ appearing in $G^{+-}(x,x';t,t') = i\langle\Psi_H(x,t)\Psi_H^\dagger(x',t')\rangle$.
Hint: Use an expansion into spatial Fourier modes and separate the time-dependence of the creation/annihilation operators $a_p^\dagger, a_p$ as given by H, Eq. (2).

c) Derive the real-time expression
$\langle\Psi_H(x,t)\Psi_H^\dagger(x',t')\rangle = \frac{1}{2\pi}\sum_q e^{iq(x-x')}e^{-i\epsilon_q(t-t')}\left[1 - \langle a_q^\dagger(0)a_q(0)\rangle\right]$

d) Find the spectral function $S(E)$ of this system Eq. (2). Then use $S(E)$ and the spectral theorem to explicitely show $\langle a_q^\dagger(0)a_q(0)\rangle = f_p$ with f_p the electron distribution function.
Note: In an actual non-equilibrium system $\langle a_q^\dagger(0)a_q(0)\rangle = f_p$ still holds, except that now f_p is the non-equilibrium distribution function.

e) From c) and d) find the Fourier representation $G^{+-}(p,\omega)$. Why is there only one frequency (energy) and one momentum argument?

f) Calculate $G^{-+}(p,\omega)$ as well as $G^{K}(p,\omega)$ and discuss the result.

g) Use f) to find a general expression for the distribution function and discuss the actual non-equilibrium situation.
Hint: Introduce relative time $T = t - t'$ and center of mass time $\tau = (t + t')/2$ and find $f_p(\tau)$ and discuss its meaning.