Mirror Symmetry on Kummer Type $K3$ Surfaces

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Abstract: We investigate both geometric and conformal field theoretic aspects of mirror symmetry on $N = (4,4)$ superconformal field theories with central charge $c = 6$. Our approach enables us to determine the action of mirror symmetry on (non-stable) singular fibers in elliptic fibrations of $\mathbb{Z}_N$ orbifold limits of $K3$. The resulting map gives an automorphism of order $4, 8$, or 12, respectively, on the universal cover of the moduli space. We explicitly derive the geometric counterparts of the twist fields in our orbifold conformal field theories. The classical McKay correspondence allows for a natural interpretation of our results.

1. Introduction

We investigate the version of mirror symmetry [Di2, L-V-W, G-P] which was found by Vafa and Witten for orbifolds of toroidal theories [V-W], and which was generalized to the celebrated Strominger/Yau/Zaslow conjecture [S-Y-Z]. Since the conceptual issues of mirror symmetry for $N = (4,4)$ superconformal field theories on first sight are different and more controversial than for strict $N = (2,2)$ theories, we first discuss the latter, as a preparation.

An $N = (2,2)$ superconformal field theory is a fermionic conformal field theory (CFT) together with a marking, i.e. a map from the standard super–Virasoro algebra into the operator product expansion (OPE) of this theory. Due to the marking the theory has well defined left and right handed $U(1)$ charges Q_l, Q_r. Markings which differ by Q_l, Q_r gauge transformations are identified. Results on $N = (2,2)$ deformation theory [Di2] show that for given central charge c a moduli space of $N = (2,2)$ superconformal field theories can be defined. Its irreducible components at generic points are Riemannian manifolds under the Zamolodchikov metric [Za] and have at most orbifold singularities. Note that the completion of the moduli space may contain points of extremal transitions which do not have CFT descriptions and will not be of relevance for our discussion. We shall restrict considerations to a connected part of the moduli space. By the above it has a unique smooth, simply connected covering space $\widetilde{\mathcal{M}}$ [Th2]. We also assume that all theories in our moduli space include the spectral flow operators in their Hilbert spaces.

Let us now consider strict $N = (2,2)$ theories, i.e. those for which the marking has no continuous deformations. Then with respect to the left and right $U(1)$ action the tangent bundle $T\widetilde{\mathcal{M}}$ canonically splits into two subbundles. The cover of the moduli space has a corresponding canonical product realization $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_2$ [Di2, D-G1]. Let $\Gamma^0 \subset \Gamma$ be the subgroup of elements which admit a factorization $\gamma = \gamma_1\gamma_2$ such that $\gamma_i \in \Gamma$ only acts on $\widetilde{\mathcal{M}}_i$ (other automorphisms may exist that are related to the effect of monodromy [G-H-L, K-M-P]). The factorization of $\widetilde{\mathcal{M}}$ induces a factorization $\mathcal{M} := \widetilde{\mathcal{M}}/\Gamma^0 = \mathcal{M}_1 \times \mathcal{M}_2$. The corresponding two subbundles of $T\mathcal{M}$ are distinguished by the marking. The standard mirror automorphism of the super–Virasoro OPE which inverts the sign of one of the $U(1)$ generators interchanges these subbundles.

One expects that near some boundary component of the moduli space any of our theories has a geometric interpretation as supersymmetric sigma model on a space X with Ricci flat Kähler metric of large radius. Though there may be several boundary components of this kind, we assume uniqueness of X for simplicity, also since no counter example is known to us, and we write $\mathcal{M} = \mathcal{M}(X), \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(X)$. One possible deformation of X is given by the scale transformation of the metric. It turns out to belong to the tangent space of one of the factors, say $\mathcal{M}_1$. Then $\mathcal{M}_2$ becomes the space of complex structures on X, and close to the boundary $\mathcal{M}_1$ corresponds to variations of the (instanton corrected) complexified Kähler structure. Under mirror symmetry the roles of the two factors are interchanged, such that $\mathcal{M}_1$ becomes the moduli space of complex structures on some other space X'. This induces a duality between geometrically different Calabi Yau manifolds, an observation that has had a striking impact on both mathematics and physics (see [C-O-G-P, Mo1], and [C-K] for a more complete list of references).

Let $\Sigma(X)$ be the space of sigma model Lagrangians on X (possibly with a marking of the homology of X), and $\Sigma(X)_b \subset \Sigma(X)$ a connected and simply connected boundary region whose points define a conformal field theory by some quantization scheme. Let $\sigma_b(X) : \Sigma(X)_b \to \mathcal{M}(X)$ be the corresponding continuous map. Locally, $\sigma_b(X)$ is a homeomorphism. By deformation, a mirror symmetry $\Sigma(X)_b \cong \Sigma(X')_b$ induces an isomorphism $\mathcal{M}(X) \cong \mathcal{M}(X')$. Since isomorphic CFTs yield the same point in $\mathcal{M}(X)$, such an isomorphism cannot depend on the choice of the boundary region. When $X = X'$ the induced automorphism of $\mathcal{M}(X)$ corresponds to an automorphism of the CFT which changes the sign of one of the two $U(1)$ currents and exchanges the corresponding supercharges. This automorphism changes the marking and therefore acts non-trivially on $\mathcal{M}(X)$.

When a base point has been chosen, the local isomorphism $\sigma_b(X)$ lifts to an inclusion $\widetilde{\sigma}_b(X) : \Sigma(X)_b \hookrightarrow \widetilde{\mathcal{M}}(X)$ and analogously for X'. When $X = X'$ and $\Sigma(X)_b \cap \Sigma(X')_b$ is non-empty, we choose a base point in the intersection. Then $\sigma_b'(X') \circ \sigma_b(X)^{-1}$ lifts to a mirror isomorphism $\gamma_{ms}^b : \widetilde{\mathcal{M}}(X) \to \widetilde{\mathcal{M}}(X')$. Sometimes there are canonical choices for the base point, otherwise one obtains equivalent choices which correspond to multiplication of γ_{ms}^b by an element of some subgroup of Γ.

Given a Calabi-Yau manifold X one can construct the corresponding family of sigma models, the moduli space $\mathcal{M}$, and the mirror Calabi-Yau manifold X'. Since X, X' determine (families of) classical geometric objects, it should be possible to transform one into the other by purely classical methods. Such constructions for mirror pairs have been proposed in the context of toric geometry [Ba] as well as T–duality [V-W,S-Y-Z,K-S].

The basic case is given by theories with central charge $c = 3$, which correspond to sigma models on a two-dimensional torus. Here one has $\widetilde{\mathcal{M}} = \mathcal{H} \times \mathcal{H}$, where we shall use coordinates ρ, τ for the two copies of the upper complex half-plane $\mathcal{H} \cong \widetilde{\mathcal{M}}_1 \cong \widetilde{\mathcal{M}}_2$. In our conventions on automorphisms, the fundamental group is given by the standard $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ action on $\widetilde{\mathcal{M}}$. Orientation change, given by $\rho \mapsto -\bar{\rho}$, $\tau \mapsto -\bar{\tau}$, and space parity change, given by $\rho \mapsto -\bar{\rho}$, $\tau \mapsto \tau$, are not considered as automorphisms, since they change the marking. For purely imaginary ρ, τ the theory is the (fermionic) product of circle models with squared radii $r_1^2 = \rho/\tau$, $r_2^2 = -\rho\tau$. Hence our theory has a nonlinear sigma model description given by two Abelian $U(1)$ currents j_1, j_2, and their $N = 2$ superpartners ψ_1, ψ_2, together with analogous right handed fields $\overline{\jmath}_1, \overline{\jmath}_2, \overline{\psi}_1, \overline{\psi}_2$, all compactified on a real torus. The $\psi_i, \overline{\psi}_i$ are Majorana fermions. The eigenvalues of the four currents $j_i, \overline{\jmath}_i$ lie in a four-dimensional vector space with natural $O(2)_{left} \times O(2)_{right}$ and $O(2,2)$ actions. Due to the compactification they form a lattice of rank 4. The $U(1)$ current of the left-handed $N = 2$ superconformal algebra is $J = i{:}\psi_2\psi_1{:}$. Hence mirror symmetry can be induced by the OPE preserving map which leaves right handed fields unchanged and transforms left handed ones by $(\psi_1, \psi_2) \mapsto (-\psi_1, \psi_2)$, $(j_1, j_2) \mapsto (-j_1, j_2)$. For the fermionic sigma model on the first circle this is the T–duality map, i.e. $r_1 \mapsto (r_1)^{-1}$. The existence of this OPE preserving map implies that $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$, as stated above. Summarizing, close to the "boundary point" $\tau = i\infty, \rho = i\infty$ of $\widetilde{\mathcal{M}}$ mirror symmetry is given by the exchange of ρ and τ. This is fiberwise T-duality in an $\mathbb{S}^1$ fibration (with section) on the underlying torus and motivates the construction [V-W,S-Y-Z]. The cusp $\tau = i\infty, \rho = i\infty$ corresponds to a limit where the basis volume r_2 becomes infinite, whereas the relative size of the fiber is arbitrarily small, e.g. for constant r_1. Since base space and fiber are flat, semiclassical considerations are applicable for arbitrary r_1, such that mirror symmetry yields a relation between two classical spaces. Conjecturally the idea carries over to suitable torus fibrations over more complicated base spaces of infinite volume. Mathematically, the expected map is given by a Fourier–Mukai type functor [Ko2,Mo2].

The construction of mirror symmetry by fiberwise T-duality also makes sense when X is a hyperkähler manifold and the corresponding sigma model has a superconformal symmetry which is extended beyond $N = (2,2)$, though the relationship with CFT is somewhat different. The moduli space $\mathcal{M}$ no longer splits canonically. Moreover, there are no quantum corrections to the Kähler structure, such that each point of $\widetilde{\mathcal{M}}$ corresponds to a classical sigma model, with well defined Ricci flat metric and B-field. According to the previous arguments, mirror symmetry must yield an element γ_{ms} of Γ, which depends on the geometric interpretation.

The picture developed so far is conjectural, but in some cases it can be verified, since the moduli space is entirely known. This is true for toroidal theories and for those theories with $c = 6$ whose Hilbert spaces include the spectral flow operators [Na4, A-M, N-W]. Every such theory admits geometric interpretations in terms of nonlinear sigma models either on tori or on $K3$, depending on the CFT. Thus any mirror symmetry relates two different geometric interpretations within the same moduli space $\mathcal{M}$.

In this note, we explore a version of mirror symmetry on Kummer type $K3$ surfaces that was proven in a much more general context in [V-W] and actually led to the Strominger/Yau/Zaslow conjecture, see [S-Y-Z, G-W1, Mo2, vEn]. Namely, for a T^2 fibred $K3$ surface $p : X \longrightarrow \mathbb{P}^1$ with elliptic fibers and a section, mirror symmetry is induced by T–duality on each regular fiber of p. Among various maps known as mirror symmetry, this is the only one with general applicability. We show that it generalizes to the singular fibers and determine the induced map. It turns out to be of finite order $4, 8$, or 12 in the different cases we discuss. Our approach enables us to read off the exact identification of twist fields in the relevant orbifold conformal field theory with geometric data on the corresponding Kummer type $K3$ surface. The role of "geometric" versus "quantum" symmetries is thereby clarified. The correct identification is also of major importance for the discussion of orbifold cohomology and resolves the objection of [F-G] to Ruan's conjecture [Ru1] on the orbifold cohomology for hyperkähler surfaces[1]. For those Kummer type $K3$ surfaces discussed in this note, our results in fact prove part of Ruan's conjecture.

To make the paper more accessible to mathematicians, we do not use the language of branes for the geometric data, but a translation is not hard.

This work is organized as follows: In Sect. 2 we discuss our mirror map on four-tori. In Sect. 3 we show how this map induces a mirror map on Kummer $K3$ surfaces X. In particular, we determine the induced map on the (non-stable) singular fibers of our elliptic fibration $p : X \longrightarrow \mathbb{P}^1$. In Sect. 4 we give its generalization to other Kummer type surfaces, i.e. other non-stable singular fibers. Sect. 5 deals with the CFT side of the picture: As explained above, the mirror map is an automorphism on a given superconformal field theory. Its action on the Hilbert space of $\mathbb{Z}_N$ orbifold conformal field theories on $K3$, $N \in \{2, 3, 4, 6\}$, is determined independently from the results of Sects. 3 and 4. In Sect. 6 we use the previous results to read off an explicit formula that maps twist fields to cohomology classes on $K3$. This is interpreted in terms of the classical McKay correspondence. We close with a summary and discussion in Sect. 7.

Acknowledgments:. It is a pleasure to thank Paul Aspinwall, David R. Morrison, M. Ronen Plesser, Miles Reid, Daniel Roggenkamp, Yongbin Ruan, Eric Sharpe, and Bernardo Uribe for helpful discussions. W.N. thanks the LPTENS in Paris for an invitation, which led to the start of part of this work. K.W. was supported by U.S. DOE grant DE-FG05-85 ER 40219, TASK A.

2. The Mirror Map for Tori

As pointed out in the Introduction, for an $N = (2, 2)$ superconformal field theory on a two–dimensional orthogonal real torus with radii r, r' and vanishing B–field,

[1] The fact that our transformation resolves this objection was explained to us by Yongbin Ruan [Ru2] and goes back to an earlier observation by Edward Witten.

at large r', mirror symmetry is just the T–duality map $r \mapsto r^{-1}$ for one radius, whereas r' remains unchanged. This map is naturally continued to arbitrary values of r, r' [V-W].

Now consider a toroidal theory on the Cartesian product T of two two-dimensional orthogonal tori with radii r_1, r_3, and r_2, r_4, respectively. Since the $U(1)$ currents of the $N = 2$ superconformal algebras in the lower dimensional theories add up to give the $U(1)$ current of the full theory, mirror symmetry is induced by $r_1 \mapsto (r_1)^{-1}, r_2 \mapsto (r_2)^{-1}$ [V-W]. After a suitable choice of complex structure this is fiberwise T–duality on a special Lagrangian fibration (with section) of our four–torus. Hence we are discussing the version of mirror symmetry that was generalized in [S-Y-Z], see also [Mo2, vEn]. Alternatively, the fibration can be understood in terms of a Gromov-Hausdorff collapse [K-S].

We wish to determine the corresponding map on the cover of the moduli space. Recall [E-T] that in the present case of $N = (2, 2)$ superconformal field theories with central charge $c = 6$ we actually have extended, i.e. $N = (4, 4)$ supersymmetry. By [Na4, Se, C, A-M, N-W], a theory in the corresponding moduli space is specified by the relative position of an even self-dual lattice L and a positive definite[2] four-plane x in $\mathbb{R}^{4,4+\delta} \cong H^{even}(Y, \mathbb{R})$ with $\delta = 0$ or 16, depending on whether the theory is associated to a torus or a $K3$ surface Y. In terms of parameters (g, B) of nonlinear sigma models on Y, g an Einstein metric on Y and B a B-field, and for vanishing B–field, x is the positive eigenspace of the Hodge star operator $*$ in $H^{even}(Y, \mathbb{R})$, and $L = H^{even}(Y, \mathbb{Z})$. The symmetry group of x is $SO(4) \times O(4 + \delta)$, such that

$$\widetilde{\mathcal{M}} = O^+(4, 4 + \delta; \mathbb{R}) / \left(SO(4) \times O(4 + \delta)\right). \qquad (1)$$

For the torus we have $\delta = 0$, and the denominator in (1) contains $SO(4)_{left} \times SO(4)_{right}$, the elements of which act as rotations on the left and right handed charges Q_l, Q_r, analogously to the case of the two-dimensional torus discussed in the Introduction. The fundamental group of the moduli space is given by

$$\Gamma = Aut(L(Y_0)) \cong O^+(4, 4 + \delta; \mathbb{Z}),$$

where $L(Y_0)$ describes the base point.

To determine the element of Γ that acts as mirror symmetry, it is crucial to gain a detailed understanding of the map that associates a point in moduli space to given nonlinear sigma model data. In d dimensions, it is customary to specify a toroidal theory by a lattice with generators $\lambda_1, \dots, \lambda_d \in \mathbb{R}^d$, i.e. $T = \mathbb{R}^d / \mathrm{span}_{\mathbb{Z}}\{\lambda_1, \dots, \lambda_d\}$, and a B-field $B \in Skew(d)$. Here $\mathbb{R}^d$ carries the standard metric. We choose a reference torus T_0 given by the lattice $\mathbb{Z}^d$ with standard orthonormal generators $e_1, \dots, e_d$, and $\Lambda \in Gl^+(d)$ such that $\lambda_i = \Lambda e_i$, $i \in \{1, \dots, d\}$.

The group $Gl^+(d)$ has a natural representation on the dual space $\check{\mathbb{R}}^d$ which is identified with $\mathbb{R}^d$ by the standard metric. The corresponding image of Λ is

$$M := (\Lambda^T)^{-1}.$$

The vectors $\mu_i := M e_i, i \in \{1, \dots, d\}$, form a dual basis with respect to $\lambda_1, \dots$ $\dots, \lambda_d$. Similarly, we have a natural representation on $\Lambda^n(\check{\mathbb{R}}^d)$. The image of

[2] On cohomology, we generally use the scalar product (4) that is induced by the intersection form on the respective surface.

Λ under this representation will be denoted $\Lambda^n(M)$. Note that $\Lambda^d(M)$ acts by multiplication with V^{-1}, where $V = det(\Lambda)$ is the volume of the torus.

In the standard description, the charge lattice of the theory is given by pairs $\frac{1}{\sqrt{2}}(Q_l + Q_r, Q_l - Q_r) \in \mathbb{R}^{d,d}$, where it is natural to take $\mathbb{R}^{d,d} = \check{\mathbb{R}}^d \oplus \mathbb{R}^d \cong H^1(T_0, \mathbb{R}) \oplus H_1(T_0, \mathbb{R})$ with bilinear form

$$(\alpha, \beta) \cdot (\alpha', \beta') = \alpha\beta' + \alpha'\beta. \tag{2}$$

The charge lattice is even and integral and is obtained as image of the standard lattice $\mathbb{Z}^{d,d} \cong H^1(T_0, \mathbb{Z}) \oplus H_1(T_0, \mathbb{Z})$ under

$$v(\Lambda, B) = \begin{pmatrix} M & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} \mathbb{1} & -B \\ 0 & \mathbb{1} \end{pmatrix} \in O^+(d,d). \tag{3}$$

Here B appears as a skew symmetric linear transformation from $\mathbb{R}^d$ to $\check{\mathbb{R}}^d$. The corresponding element of $\Lambda^2(\check{\mathbb{R}}^d)$ will be denoted b, such that b is a vector with components $b_{ij} = B_{ij}$ with respect to the basis $e_i \wedge e_j$. We will also use its dual $\check{b}$ with components $\check{b}_{ij} = \sum_{k,l} \epsilon_{ijkl} B_{kl}/2$.

As mentioned below (1), the rotations of the left and right charge lattices form a subgroup $SO(d)_{left} \times SO(d)_{right} \subset O^+(d,d)$. Rotations which act on Q_l and Q_r in the same way are generated by

$$\begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}, \qquad \Omega \in so(d).$$

Rotations for which the respective actions on Q_l, Q_r are inverse to each other are generated by

$$\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}, \qquad \Omega \in so(d).$$

To describe torus orbifolds we have to work with the lattice $H^{even}(T_0, \mathbb{Z})$ instead of $H^1(T_0, \mathbb{Z}) \oplus H_1(T_0, \mathbb{Z})$. The vector space $H^{even}(T_0, \mathbb{R})$ carries a bilinear form $\langle \cdot, \cdot \rangle$ that we obtain from the intersection form upon Poincaré duality, i.e.

$$\langle a, b \rangle = \int_{T_0} a \wedge b. \tag{4}$$

Accordingly, we have to use a half spinor representation s of $O^+(d,d)$. We now specialize to the case $d = 4$, where $H^1(T_0, \mathbb{R}) \oplus H_1(T_0, \mathbb{R}) \cong H^{even}(T_0, \mathbb{R})$. Hence the representation s can be obtained from v by triality [Di1,N-W,K-O-P,O-P]. In other words,

$$s(\Lambda, B) = V^{1/2} \begin{pmatrix} V^{-1} & 0 & 0 \\ 0 & \Lambda^2(M) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \check{b} & -\frac{||B||^2}{2} \\ 0 & \mathbb{1} & -b \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Here $||B||^2 = \langle B, B \rangle$ as given in (4), and by $\check{b}$ we denote the dual of b as introduced above. The matrix $s(\Lambda, B)$ acts on $L(T_0) := H^{even}(T_0, \mathbb{Z}) \cong \mathbb{Z}^{4,4}$ and is given with respect to the basis $e_1 \wedge e_2 \wedge e_3 \wedge e_4$, $e_i \wedge e_j$, 1. Note that in [N-W] we have used a normalization of the scalar product on $H^2(T_0, \mathbb{Z})$ which differs by a factor of V from the above.

For later use we note that analogously to (5) one determines

$$s(C) = \begin{pmatrix} 1 & 0 & 0 \\ -\check{c} & \mathbb{1} & 0 \\ -\frac{\|C\|^2}{2} & c & 1 \end{pmatrix} \qquad (6)$$

as the triality conjugate of

$$v(C) = \begin{pmatrix} \mathbb{1} & 0 \\ -C & \mathbb{1} \end{pmatrix} \in O^+(4,4),$$

where c is the row vector with components $c_{ij} = C_{ij}$ with respect to $e_i \wedge e_j$.

The sigma model on T_0 with $B = 0$ is described by the lattice $L(T_0)$ and the positive definite four-plane $x_0 \subset H^{even}(T_0, \mathbb{R})$ which is left invariant by the Hodge star operator $*$. The latter is given by

$$x_0 = \mathrm{span}_{\mathbb{R}} \left\{ \begin{array}{ll} 1 + e_1 \wedge e_2 \wedge e_3 \wedge e_4, & e_1 \wedge e_3 + e_4 \wedge e_2, \\ e_1 \wedge e_2 + e_3 \wedge e_4, & e_1 \wedge e_4 + e_2 \wedge e_3 \end{array} \right\}. \qquad (7)$$

For arbitrary sigma model parameters (Λ, B), x_0 as in (7) remains the $+1$ eigenspace of $*$, but we have $H^{even}(T, \mathbb{Z}) \cong s(\Lambda, B)L(T_0) =: L$. A point in the cover $\widetilde{\mathcal{M}}$ of the moduli space is described by the relative position of L with respect to x_0, i.e. the pair $x_0, s(\Lambda, B)$. Since only the relative position counts, we can use Rx_0 and $Rs(\Lambda, B)$ with arbitrary $R \in O^+(4,4)$. To avoid confusion one should note that, as mentioned above, in [N-W] we have used $Rs(\Lambda, B) = s(V^{-1/4}\Lambda, B)$. For the description of mirror symmetry on orthogonal tori, however, it is more convenient to choose $R = \mathbb{1}$.

Let us now determine the lattice automorphism that acts as mirror symmetry. It suffices to consider $B = 0$ and a torus T with defining matrix $\Lambda = \mathrm{diag}(r_1, \dots, r_4)$, $r_i > 0$. The generators of its even cohomology group $H^{even}(T, \mathbb{Z})$ will be denoted $\upsilon = \mu_1 \wedge \dots \wedge \mu_4$, $\mu_i \wedge \mu_j$, $\upsilon^0 = 1$. By the above we need to find a map that leaves $H^{even}(T, \mathbb{Z})$ and the four–plane (7) invariant and induces $r_1 \mapsto (r_1)^{-1}, r_2 \mapsto (r_2)^{-1}$ on the torus parameters. To this end, substituting $e_i = r_i \mu_i$ into (7) one in particular finds

$$\gamma_{MS}(T_0) : \left\{ \begin{array}{ll} \pm\, \upsilon^0 \longleftrightarrow \pm\, \mu_1 \wedge \mu_2, & \pm\, \mu_1 \wedge \mu_3 \longleftrightarrow \pm\, \mu_2 \wedge \mu_3, \\ \pm\, \upsilon \;\; \longleftrightarrow \pm\, \mu_3 \wedge \mu_4, & \pm\, \mu_4 \wedge \mu_2 \longleftrightarrow \pm\, \mu_1 \wedge \mu_4 \end{array} \right.$$

for the base point of the moduli space given by T_0 and $B = 0$ [Na3]. To fix the signs, recall that T-duality in the x_1, x_2 fiber of our T^2 fibration of T, as automorphism of the Grassmannian of four-planes in $H^1(T_0, \mathbb{R}) \oplus H_1(T_0, \mathbb{R})$, acts by conjugation with the element $\sigma = (\mathrm{diag}(-1,-1,1,1), \mathbb{1}) \in SO(4)_{left} \times SO(4)_{right} \subset O^+(4,4)$. Correspondingly, its action on $H^{even}(T_0, \mathbb{R})$ is given by the spinor representation $s(\sigma)$ of this group element. Since σ is a rotation by π, the square of $s(\sigma)$ is $-\mathbb{1}$. We will argue that (up to an irrelevant overall sign)

$$\gamma_{MS}(T_0) : \left\{ \begin{array}{r} \upsilon^0 \longmapsto \mu_1 \wedge \mu_2 \longmapsto -\upsilon^0, \\ \upsilon \longmapsto \mu_3 \wedge \mu_4 \longmapsto -\upsilon, \\ \mu_1 \wedge \mu_3 \longmapsto \mu_2 \wedge \mu_3 \longmapsto -\mu_1 \wedge \mu_3, \\ \mu_4 \wedge \mu_2 \longmapsto \mu_1 \wedge \mu_4 \longmapsto -\mu_4 \wedge \mu_2, \end{array} \right. \qquad (8)$$

where the lower two lines of (8) are induced by $\mu_1 \mapsto \mu_2 \mapsto -\mu_1$. This can be explained as follows. Above, we have described mirror symmetry by the sign change of two left handed current components $(j_1, j_2) \mapsto (-j_1, -j_2)$, whereas the right handed components are unchanged. Since only the relative rotation by π between the two chiralities is important, one may as well consider the maps $(j_1, j_2) \mapsto (j_2, -j_1)$ for the left handed components and $(\bar{j}_1, \bar{j}_2) \mapsto (-\bar{j}_2, \bar{j}_1)$ for the right handed ones. By the above, this rotation is described by

$$v_{ms} = \exp(\pi\Omega/2), \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \Omega_{12} \\ \Omega_{12} & 0 \end{pmatrix} \in o^+(4,4),\ \Omega_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We use (5) and (6) to translate this into the half spinor representation and find

$$s_{ms} = \exp(\pi\widetilde{\Omega}_{12}/2), \quad \text{where} \quad \widetilde{\Omega}_{12} = \begin{pmatrix} 0 & -\check{\omega}_{12}^T & 0 \\ \check{\omega}_{12} & 0 & \omega_{12} \\ 0 & -\omega_{12}^T & 0 \end{pmatrix},$$

and $\omega_{12}, \check{\omega}_{12}$ are obtained from Ω_{12} as explained above for $b, \check{b}, B$. Since this gives the transformation with respect to the basis $\upsilon, \mu_i \wedge \mu_j, \upsilon^0$, the mirror symmetry transformation given by s_{ms} can be written as

$$\gamma_{MS}(T_0') : \begin{cases} \upsilon^0 \longmapsto \mu_1 \wedge \mu_2, & \mu_1 \wedge \mu_2 \longmapsto -\upsilon^0, \\ \upsilon \longmapsto \mu_3 \wedge \mu_4, & \mu_3 \wedge \mu_4 \longmapsto -\upsilon, \\ \mu_1 \wedge \mu_3 \longmapsto \mu_1 \wedge \mu_3, & \mu_2 \wedge \mu_3 \longmapsto \mu_2 \wedge \mu_3, \\ \mu_4 \wedge \mu_2 \longmapsto \mu_4 \wedge \mu_2, & \mu_1 \wedge \mu_4 \longmapsto \mu_1 \wedge \mu_4. \end{cases} \tag{9}$$

To obtain the mirror map $\gamma_{MS}(T_0)$ corresponding to the original transformation $(j_1, j_2) \mapsto (-j_1, -j_2)$, $(\bar{j}_1, \bar{j}_2) \mapsto (\bar{j}_1, \bar{j}_2)$ one composes $\gamma_{MS}(T_0')$ with the classical symmetry $(j_1, j_2) \mapsto (j_2, -j_1)$, $(\bar{j}_1, \bar{j}_2) \mapsto (\bar{j}_2, -\bar{j}_1)$ and finds (8).

When we discuss orbifolds with respect to threefold rotations in fiber and basis, the transformation (8) is not always applicable. For such rotations, the forms υ, υ^0, $\mu_1 \wedge \mu_2$, and $\mu_3 \wedge \mu_4$ are invariant, but not necessarily the others. In particular, the four plane (7) is not invariant. The transformation $\gamma_{MS}(T_0')$ is well behaved in all cases, however. When we work with orbifolds of the corresponding fibred tori we always keep μ_1, μ_2 as generators of periods in the fiber and μ_3, μ_4 as generators of periods in the basis, such that $\gamma_{MS}(T_0')$ lifts to a symmetry of the conformal field theory which commutes with the symmetries used for orbifolding.

We stress that $\gamma_{MS}(T_0), \gamma_{MS}(T_0')$ are lattice automorphisms of order 4, and $\gamma_{MS}(T_0) \circ \gamma_{MS}(T_0) = -\mathbb{1}$. It has also been observed before that $\gamma_{MS}(T_0)$ can be understood as hyperkähler rotation in $\widetilde{\mathcal{M}}$ [B-B-R-P, Di1, B-S, A-C-R+1, A-C-R+2], in full agreement with the above.

With respect to the complex structure $\mathcal{I}$ given by $(e_1 - ie_3) \wedge (e_2 + ie_4)$, the second line in (7) corresponds to the complex structure and the first to its orthogonal complement. $\gamma_{MS}(\Lambda, B)$ exchanges the two, in accord with the general notion of mirror symmetry discussed in the Introduction. Moreover, the T^2 fibration with fiber coordinates x_1, x_2 is elliptic with respect to the complex structure $\mathcal{J}$ given by $(e_1 + ie_2) \wedge (e_3 + ie_4)$ (with a holomorphic section), and therefore it is special Lagrangian with respect to $\mathcal{I}$ (c.f. [H-L]). Hence (8) also describes a version of mirror symmetry in the sense of [S-Y-Z, Mo2], obtained from $r_1 \mapsto (r_1)^{-1}, r_2 \mapsto (r_2)^{-1}$, as in [V-W].

3. The Mirror Map for Kummer Surfaces

Recall the classical Kummer construction of $K3$: Given a four-torus T, we have a $\mathbb{Z}_2$ symmetry induced by multiplication with -1 on $\mathbb{R}^4$. By blowing up the 16 singularities of the corresponding $\mathbb{Z}_2$ orbifold of T and assigning volume zero to all exceptional divisors in the blow up we obtain an orbifold limit of $K3$, a Kummer surface X. In particular, there is a rational map $\pi : T \longrightarrow X$ of degree 2 which is defined outside the fixed points. The T^2 fibration of T used in Sect. 2 induces a T^2 fibration $p : X \longrightarrow \mathbb{P}^1$ which is elliptic with respect to $\pi_* \mathcal{J}$ and therefore special Lagrangian with respect to $\pi_* \mathcal{I}$. Note that the holomorphic section is not the π_* image of the section in our fibration of T. We rather have to make sure that the fibration can be written in the Weierstraß form, such that each singular fiber can be labeled by its Kodaira type [Ko1]. The Poincaré duals of the generic fiber and generic section are given in (11) below. Apart from the behavior at the four singular fibers, mirror symmetry as discussed above is induced by mirror symmetry on the torus. This was in fact proven more generally in [V-W] and generalized to the Strominger/Yau/Zaslow conjecture in [S-Y-Z].

It again suffices to specialize to the standard torus T_0, consider the corresponding Kummer surface X_0, and determine the automorphism of the lattice $L(X_0)$ that acts as mirror symmetry at this base point. To this end, let us recall the description of $L(X_0)$ as found in [Ni, N-W]. The orbifolding map π induces an injective map π_* on cohomology such that[3] $\pi_* H^{even}(T_0, \mathbb{Z}) \cong H^{even}(T_0, \mathbb{Z})(2)$. We embed $H^{even}(T_0, \mathbb{Z})(2)$ in $H^{even}(X_0, \mathbb{Z})$ by rescaling with $\sqrt{2}$. With this convention we have $H^{even}(X_0, \mathbb{R}) = \pi_* H^{even}(T_0, \mathbb{R}) \perp \mathrm{span}_{\mathbb{R}}\{E_i | i \in I\}$, where I labels the 16 fixed points of the $\mathbb{Z}_2$ orbifolding, and the E_i project to the Poincaré duals of the exceptional divisors in the blow up of these fixed points (see below). Since each divisor is a rational curve of self-intersection number -2 on X_0, the E_i generate a lattice $\mathbb{Z}^{16}(-2) \subset H^{even}(X_0, \mathbb{R})$. On I one finds an affine $\mathbb{F}_2^4$ geometry[4] [Ni], which we use to label the fixed points. The four–plane $x_0 \subset H^{even}(T_0, \mathbb{R})$ given in (7) remains unchanged.

By Π we denote the so-called *Kummer lattice*

$$\Pi := \mathrm{span}_{\mathbb{Z}} \left\{ E_i, i \in I; \quad \tfrac{1}{2} \sum_{i \in H} E_i, H \subset I \text{ a hyperplane } \right\}.$$

Its projection $\widehat{\Pi}$ onto $H^2(X_0, \mathbb{Z})$ is the minimal primitive sublattice which contains all Poincaré duals $\widehat{E}_i, i \in I$, of exceptional divisors.

Let $P_{j,k} := \mathrm{span}_{\mathbb{F}_2}(f_j, f_k) \subset \mathbb{F}_2^4$ with $f_j \in \mathbb{F}_2^4$ the j^{th} standard basis vector and $Q_{j,k} := P_{l,m}$ such that $\{j, k, l, m\} = \{1, 2, 3, 4\}$. Then[5] [Ni]

$$M := \left\{ \tfrac{1}{\sqrt{2}} \mu_j \wedge \mu_k - \tfrac{1}{2} \sum_{i \in Q_{j,k}} E_{i+l}, l \in I \right\} \text{ and } \Pi$$

generate a lattice isomorphic to $H^2(X_0, \mathbb{Z})$.

---

[3] Given a lattice Γ, by $\Gamma(n)$ we denote the same $\mathbb{Z}$ module as Γ with quadratic form scaled by a factor of n.

[4] As usual, $\mathbb{F}_p$, p prime, denotes the unique finite field with p elements.

[5] We remark that in [N-W] we missed to exchange $P_{j,k}$ with $Q_{j,k}$, which amounts to translation from homology to cohomology by Poincaré duality.

The lattice $L(T_0)(2) = H^{even}(T_0,\mathbb{Z})(2)$ and M belong to $L(X_0)$, but the E_i do not, since $\pi_* L(T_0) \perp \Pi$ cannot be embedded as sublattice into $L(X_0)$. Instead, $L(X_0) = \mathrm{span}_{\mathbb{Z}}\{\widehat{M} \cup \widehat{\Pi}\}$, where

$$\begin{aligned} &\widehat{M} := M \cup \left\{ \widehat{\upsilon}^0 := \tfrac{1}{\sqrt{2}}\upsilon^0 + \tfrac{1}{4}\sum_{i\in I} E_i + \sqrt{2}\upsilon; \quad \widehat{E}_i := E_i + \tfrac{1}{\sqrt{2}}\upsilon, i \in I \right\} \\ \text{and} \quad &\widehat{\Pi} := \left\{ \pi \in \Pi \mid \forall\, m \in \widehat{M} : \langle \pi, m\rangle \in \mathbb{Z} \right\} \end{aligned} \tag{10}$$

[N-W]. It is important to note that $\widehat{E}_i$ is the two-form contribution to E_i, and vice versa E_i is the orthogonal projection of the lattice vector $\widehat{E}_i$ onto $(\pi_* H^{even}(T_0,\mathbb{R}))^\perp$. The observation that this gives the unique consistent embedding of $\pi_* L(T_0)$ into $L(X_0)$ implies that the B-field in a $\mathbb{Z}_2$ orbifold CFT on $K3$ has value $1/2$ in direction of each exceptional divisor of the blow up [As, N-W]. This observation generalizes to all orbifold CFTs on $K3$ [We].

With the above description of $H^{even}(X_0,\mathbb{Z})$ one checks that the Poincaré duals of generic fiber and generic holomorphic section in our elliptic fibration $p : X \longrightarrow \mathbb{P}^1$ are given by

$$\begin{aligned} &\sqrt{2}\mu_3 \wedge \mu_4, \\ &\tfrac{1}{\sqrt{2}}\mu_1 \wedge \mu_2 - \tfrac{1}{2}\left(E_{(0,0,0,0)} + E_{(0,0,1,0)} + E_{(0,0,0,1)} + E_{(0,0,1,1)} \right), \end{aligned} \tag{11}$$

respectively.

We will now determine the induced mirror map $\gamma_{MS}(X_0) \in \Gamma = Aut(L(X_0)) \cong O^+(4,20;\mathbb{Z})$. The geometric description of mirror symmetry implies that the action on $L(T_0)(2)$ is induced by the action of $\gamma_{MS}(T_0)$ on $L(T_0)$. To extend it to all of $L(X_0)$ we have to find images of E_i, $i \in I$, in $\Pi \otimes \mathbb{Q}$, such that the induced linear map is an automorphism of Γ. We claim that with arbitrary $t_0 \in I$ the following map will do:

$$\begin{aligned} &\forall\, (I,J,K,M) \in \mathbb{F}_2^4 : \\ &\qquad \gamma_{MS}(X_0)(E_{(I,J,K,M)}) := \tfrac{1}{2} \sum_{i,j\in\mathbb{F}_2} (-1)^{iI+jJ} E_{(i,j,K,M)+t_0}. \end{aligned} \tag{12}$$

First, it is easy to see that this map preserves scalar products and acts as involution on $\widehat{\Pi}$. Since $\widehat{\Pi}$ is generated by $E_i \pm E_j$, $i,j \in I$, and $\frac{1}{2}\sum_{i\in H} E_i$, $H \subset I$ a hyperplane, one finds that (12) maps $\widehat{\Pi}$ into itself. Next, we check that $M \subset \widehat{M}$ is mapped into $H^{even}(X_0,\mathbb{Z})$. Namely, there are $\pi_{a,b} \in \widehat{\Pi}$ such that for all $I,J,K,L \in \mathbb{F}_2$

$$\begin{aligned} \gamma_{MS}(X_0) &\left(\tfrac{1}{\sqrt{2}}\mu_1 \wedge \mu_3 - \tfrac{1}{2} \sum_{i\in Q_{1,3}} E_{i+(I,J,K,M)} \right) \\ &= \tfrac{1}{\sqrt{2}}\mu_2 \wedge \mu_3 - \tfrac{1}{2} \sum_{i,m\in\mathbb{F}_2} (-1)^{iI} E_{(i,0,K,m)+t_0} \\ &= \tfrac{1}{\sqrt{2}}\mu_2 \wedge \mu_3 - \tfrac{1}{2} \sum_{i\in Q_{2,3}} E_{i+(0,0,K,0)+t_0} + I\pi_{1,3}, \\ \gamma_{MS}(X_0) &\left(\tfrac{1}{\sqrt{2}}\mu_4 \wedge \mu_2 - \tfrac{1}{2} \sum_{i\in Q_{2,4}} E_{i+(I,J,K,M)} \right) \end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{\sqrt{2}}\mu_1 \wedge \mu_4 - \tfrac{1}{2} \sum_{j,k\in\mathbb{F}_2} (-1)^{jJ} E_{(0,j,k,M)+t_0} \\
&= \tfrac{1}{\sqrt{2}}\mu_1 \wedge \mu_4 - \tfrac{1}{2} \sum_{i\in Q_{1,4}} E_{i+(0,0,0,M)+t_0} + J\pi_{2,4},
\end{aligned}
$$

and $\pi_{(I,J,K,M)} \in \widehat{\Pi}$ with

$$
\begin{aligned}
&\gamma_{MS}(X_0)\left(\tfrac{1}{\sqrt{2}}\mu_3 \wedge \mu_4 - \tfrac{1}{2} \sum_{i\in Q_{3,4}} E_{i+(I,J,K,M)}\right) \\
&\quad = -\tfrac{1}{\sqrt{2}}\upsilon - E_{(0,0,K,M)+t_0} \stackrel{(10)}{=} -\widehat{E}_{(0,0,K,M)+t_0}, \\
&\gamma_{MS}(X_0)\left(\tfrac{1}{\sqrt{2}}\mu_1 \wedge \mu_2 - \tfrac{1}{2} \sum_{i\in Q_{1,2}} E_{i+(I,J,K,M)}\right) \\
&\quad = -\tfrac{1}{\sqrt{2}}\upsilon^0 - \tfrac{1}{4} \sum_{k,m\in\mathbb{F}_2} \big(E_{(0,0,k,m)+t_0} + (-1)^I E_{(1,0,k,m)+t_0} \\
&\qquad\qquad + (-1)^J E_{(0,1,k,m)+t_0} + (-1)^{I+J} E_{(1,1,k,m)+t_0}\big) \\
&\quad = -\tfrac{1}{\sqrt{2}}\upsilon^0 - \tfrac{1}{4}\sum_{i\in I} E_i + \pi_{(I,J,K,M)}.
\end{aligned}
$$

Since $\gamma_{MS}(X_0)_{|\widehat{\Pi}}$ is an involution, and $\gamma_{MS}(X_0) \circ \gamma_{MS}(X_0)_{|\widehat{\Pi}^\perp} = -\mathbb{1}$, this suffices to prove consistency. From the above it also follows that up to automorphisms of $(\pi_* H^{even}(T_0,\mathbb{Z}))^\perp \cap H^{even}(X_0,\mathbb{Z})$, (12) gives the only consistent maps $\gamma_{MS}(X_0)$. We will be more precise about this point at the end of Sects. 4 and 6.

Let us consider the actual geometric action of $\gamma_{MS}(X_0)$. From the above we can easily write out the map on $H^{even}(X_0,\mathbb{Z}) = \mathrm{span}_\mathbb{Z}\{\widehat{M}, \widehat{\Pi}\}$. Hence we have in particular found an explicit continuation of mirror symmetry as induced by fiberwise T–duality to the four singular fibers of $p : X \longrightarrow \mathbb{P}^1$ over $x_3 \in \{0, r_3/2\}, x_4 \in \{0, r_4/2\}$, i.e. with labels $(K,M) \in \mathbb{F}_2^2$. Each of these singular fibers is of type I_0^* in Kodaira's classification [Ko1, Th.6.2] with components dual to

$$
\widehat{E}_{(i,j,K,M)}, i,j \in \mathbb{F}_2, \text{ and } C_{K,M} := \tfrac{1}{\sqrt{2}}\mu_3 \wedge \mu_4 - \tfrac{1}{2} \sum_{i\in Q_{3,4}} \widehat{E}_{i+(0,0,K,M)}.
$$

The latter form $C_{K,M}$ corresponds to the center node of the $\widehat{D}_4$ type Dynkin diagram describing I_0^*. For simplicity, let us set $t_0 = (0,0,0,0)$. Since for suitable $\pi'_{(I,J,K,M)} \in \widehat{\Pi}$ and with the generator $\widehat{\upsilon} = \sqrt{2}\upsilon$ of $H^4(X_0,\mathbb{Z})$ (c.f. [N-W]),

$$
\begin{aligned}
\gamma_{MS}(X_0)(\widehat{E}_{(I,J,K,M)}) &\stackrel{(10)}{=} \gamma_{MS}(X_0)\left(\tfrac{1}{\sqrt{2}}\upsilon + E_{(I,J,K,M)}\right) \\
&= \tfrac{1}{\sqrt{2}}\mu_3 \wedge \mu_4 + \tfrac{1}{2}\big(E_{(0,0,K,M)} + (-1)^I E_{(1,0,K,M)} \\
&\quad +(-1)^J E_{(0,1,K,M)} + (-1)^{I+J} E_{(1,1,K,M)}\big) \\
&= C_{K,M} + \pi'_{(I,J,K,M)} - \delta_{I,0}\delta_{J,0}\widehat{\upsilon}, \\
\gamma_{MS}(X_0)(C_{K,M}) &= -\widehat{E}_{(0,0,K,M)} - \sqrt{2}\mu_3 \wedge \mu_4,
\end{aligned}
$$

we see that up to signs and possible corrections in $\widehat{\Pi}$ and $\pi_* H^{even}(T_0, \mathbb{Z})$, $\gamma_{MS}(X_0)$ exchanges the center node of $I_0^* \cong \widehat{D}_4$ with each of its four legs.

Note also that for $t_0 = (0,0,0,0)$ the Poincaré duals (11) of generic fiber and generic section of our elliptic fibration are simply mapped onto $-\widehat{\upsilon}$, $-\widehat{\upsilon}^0 + \widehat{\upsilon}$ under $\gamma_{MS}(X_0)$, where $\widehat{\upsilon}, \widehat{\upsilon}^0$ are the generators of $H^4(X_0, \mathbb{Z}), H^0(X_0, \mathbb{Z})$, respectively (see (10) and [N-W]).

Next, let us investigate the monodromy $[m], m \in SL(2,\mathbb{Z})$, of the regular two-tori in our fibration $p : X \longrightarrow \mathbb{P}^1$ around a singular fiber, where $[m]$ denotes the conjugacy class of m in $SL(2,\mathbb{Z})$. Since $\gamma_{MS}(X_0)$ acts by $S \in SL(2,\mathbb{Z})$ on the modular parameter of the fiber,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the monodromy should transform as

$$m \quad \longmapsto \quad \left(m^T\right)^{-1} \quad = \quad SmS^{-1}, \tag{13}$$

i.e. $[m]$ should remain invariant under mirror symmetry. In the present case, all singular fibers are of type I_0^*, hence by [Ko1, Th.9.1] their monodromy is

$$m = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{14}$$

which is even invariant under (13). This is in accord with our construction, since our geometric interpretation of a $K3$ theory obtained as $\mathbb{Z}_2$ orbifold on a Kummer surface with $B = 0$ for the underlying toroidal theory is indeed mapped into another such geometric interpretation. The mirror $K3$ is hence expected to be a singular Kummer surface again, with singular fibers of type I_0^*, and with monodromy (14) around each of them.

4. The Mirror Map for Kummer Type Surfaces

In this section, we give the action of mirror symmetry on Kummer type $K3$ surfaces obtained as orbifold limits $\widetilde{T/\mathbb{Z}_N}, N \in \{3,4,6\}$ of $K3$. Since the proofs are analogous to the one for $N = 2$ that has been discussed at length in the previous section, we restrict ourselves to a presentation of the results. For explicit proofs, one needs the description of $L(X_0)$ in terms of $\pi_* L(T_0)^{\mathbb{Z}_N} \cong L(T_0)^{\mathbb{Z}_N}(N)$ and the exceptional divisors obtained by blowing up all singularities, as given in [N-W, We].

Recall the $\mathbb{Z}_N$ orbifold construction of $K3$, $N \in \{3,4,6\}$. Let ζ_N denote a generator of $\mathbb{Z}_N$ and z_1, z_2 complex coordinates on T, such that $T = \widetilde{T}^2 \times T^2$ with elliptic curves $\widetilde{T}^2, T^2$. If both curves are $\mathbb{Z}_N$ symmetric (note that the metric in general need not be diagonal with respect to z_1, z_2), there is an algebraic $\mathbb{Z}_N$ action on T given by

$$\mathbb{Z}_N \ni \zeta_N^l : \quad \zeta_N^l.(z_1, z_2) = (\zeta_N^l z_1, \zeta_N^{-l} z_2).$$

By blowing up the singularities of $T/\mathbb{Z}_N$ and assigning volume zero to all exceptional divisors we obtain a $\mathbb{Z}_N$ orbifold limit X of $K3$. The fixed point set of $\mathbb{Z}_N$ on T will be denoted I in general, and $n(t)$ is the order of the stabilizer group of $t \in I$. For $N=3$ we have $I \cong \mathbb{F}_3^2$, and since the $\mathbb{Z}_4, \mathbb{Z}_6$ orbifold limits can be obtained from the $\mathbb{Z}_2, \mathbb{Z}_3$ orbifold limits by modding out an algebraic $\mathbb{Z}_2$ action, for $N=4,6$ we use $I = \mathbb{F}_2^4/\sim$ and $I = \mathbb{F}_2^4 \cup \mathbb{F}_3^2/\sim$, respectively, where $\sim$ denotes the necessary identifications. To assure compatibility of the $\mathbb{Z}_N$ action with our fibration $p: T \longrightarrow T^2$, we choose $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$ and obtain an induced fibration $p: X \longrightarrow \mathbb{P}^1$ as in the $\mathbb{Z}_2$ case. We can also restrict considerations to appropriate standard tori T_0, T_0' for $N=4, N \in \{3,6\}$, respectively, $X_0^{(\prime)} = T_0^{(\prime)}/\mathbb{Z}_N$. Here, $T_0 = \mathbb{R}^4/\operatorname{span}_\mathbb{Z}\{e_1, \ldots, e_4\}$ with an orthonormal basis $e_1, \ldots, e_4$ as introduced in Sect. 2, and

$$T_0' = \mathbb{R}^4/\operatorname{span}_\mathbb{Z}\{\lambda_1, \ldots, \lambda_4\} \quad \text{with} \quad \begin{cases} \mu_1 = e_1, & \mu_2 = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2, \\ \mu_3 = e_3, & \mu_4 = \frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_4 \end{cases}$$

for the dual basis $\mu_1, \ldots, \mu_4$. The $\mathbb{Z}_3$ action then is given by

$$\zeta_3 : \begin{cases} \mu_1 \longmapsto \mu_2 - \mu_1, & \mu_2 \longmapsto -\mu_1, \\ \mu_3 \longmapsto -\mu_4, & \mu_4 \longmapsto \mu_3 - \mu_4. \end{cases}$$

The exceptional divisor in the blow up of a $\mathbb{Z}_{n(t)}$ type fixed point $t \in I$ has $n(t)-1$ irreducible components with intersection matrix the negative of the Cartan matrix of $A_{n(t)-1}$. The projections to $(\pi_* L(T_0)^{\mathbb{Z}_N})^\perp$ of the Poincaré duals of these (-2) curves are denoted $E_t^{(l)}, l \in \{1, \ldots, n(t)-1\}$, such that

$$\left\langle E_t^{(l)}, E_t^{(m)} \right\rangle = \begin{cases} -2 & \text{for } l = m, \\ 1 & \text{for } |l-m| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$E_t := \sum_{l=1}^{n(t)-1} l E_t^{(l)}, \quad E_t^{(0)} := -\sum_{l=1}^{n(t)-1} E_t^{(l)},$$

as well as a $\mathbb{Z}_{n(t)}$ action on $\mathbb{E}_t := \operatorname{span}_\mathbb{Z}\{E_t^{(l)}, l \in \{1, \ldots, n(t)-1\}\}$ generated by

$$\vartheta(E_t^{(l)}) := \begin{cases} E_t^{(l+1)} & \text{if } l < n(t)-1, \\ E_t^{(0)} & \text{if } l = n(t)-1, \end{cases} \quad \text{such that } (\vartheta - 1)E_t = n(t)E_t^{(0)}. \qquad (15)$$

As in Sect. 3 it suffices to determine the image of each $E_t^{(l)}$ under the extension of the mirror map of the underlying toroidal theory. For the $\mathbb{Z}_4$ orbifold, the latter is $\gamma_{MS}(T_0)$ as in (8), and for the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ orbifolds we have to use $\gamma_{MS}(T_0')$ as defined in (9).

We list the singular fibers of $p: X \longrightarrow \mathbb{P}^1$ in terms of Kodaira's classification [Ko1, Th.6.2], all of which are non-stable. Recall also the labels of the corresponding extended Coxeter Dynkin diagrams:

$$\mathbb{Z}_3: \ IV^* + IV^* + IV^*, \quad \mathbb{Z}_4: \ I_0^* + III^* + III^*, \quad \mathbb{Z}_6: \ I_0^* + II^* + IV^*;$$
$$IV^* \cong \widehat{E}_6, \quad I_0^* \cong \widehat{D}_4, \quad III^* \cong \widehat{E}_7, \quad II^* \cong \widehat{E}_8.$$

By construction, our map $\gamma_{MS}(X_0^{(\prime)})$ acts fiberwise (c.f. (12)), hence it suffices to specify the map on each type of singular fibers. Accordingly, fixed points are only labeled by the fiber coordinates x_1, x_2 in the following.

Fibers of type I_0^* have been discussed in Sect. 3. Type IV^* occurs in both $\mathbb{Z}_3$ and $\mathbb{Z}_6$ orbifolds and contains three A_2 type exceptional divisors with components Poincaré dual to $E_t^{(l)}, l \in \{1,2\}, t \in \mathbb{F}_3$. We find

$$t \in \mathbb{F}_3, l \in \{1,2\}: \quad \gamma_{MS}(X_0')(E_t^{(l)}) = -\tfrac{1}{3} \sum_{k \in \mathbb{F}_3} \vartheta^{l+kt}(E_k), \tag{16}$$

where we have chosen an origin $0 \in \mathbb{F}_3$ and the standard scalar product on $\mathbb{F}_3$.

For III^*, which occurs in the $\mathbb{Z}_4$ orbifold, we have two A_3 type exceptional divisors giving $E_i^{(l)}, l \in \{1,2,3\}, i \in \{(0,0),(1,1)\}$, and one A_1 type exceptional divisor corresponding to $E_{(1,0)}$. Then

$$\begin{aligned} &i \in \mathbb{F}_2, l \in \{1,2,3\}: \\ &\quad \gamma_{MS}(X_0)(E_{(i,i)}^{(l)}) = -\tfrac{1}{2}(-1)^{l+i}(E_{(1,0)}) - \tfrac{1}{4} \sum_{k \in \mathbb{F}_2} \vartheta^{l+2ik}(E_{(k,k)}), \qquad (17) \\ &\quad \gamma_{MS}(X_0)(E_{(1,0)}) = -\tfrac{1}{4}(\vartheta + \vartheta^3) \sum_{k \in \mathbb{F}_2} \vartheta^k(E_{(k,k)}). \end{aligned}$$

Finally, for II^* we have one A_5, A_2, A_1 type exceptional divisor each, corresponding to $E_0^{(l)}, l \in \{1,\dots,5\}$, $E_1^{(l)}, l \in \{1,2\}$, $E_{(1,0)}$. Here,

$$\begin{aligned} l \in \{1,\dots,5\}: \quad & \gamma_{MS}(X_0')(E_0^{(l)}) = -\tfrac{1}{2}(-1)^l(E_{(1,0)}) - \tfrac{1}{3}\vartheta^l(E_1) - \tfrac{1}{6}\vartheta^l(E_0), \\ l \in \{1,2\}: \quad & \gamma_{MS}(X_0')(E_1^{(l)}) = \tfrac{1}{3}\vartheta^l(E_1) - \tfrac{1}{6}(\vartheta^0 + \vartheta^3)\vartheta^l(E_0), \\ & \gamma_{MS}(X_0')(E_{(1,0)}) = -\tfrac{1}{2}E_{(1,0)} - \tfrac{1}{6}(\vartheta + \vartheta^3 + \vartheta^5)(E_0). \qquad (18) \end{aligned}$$

To prove the above, one has to check that scalar products are preserved and that the generator $\widehat{\upsilon}^0$ of $H^0(X_0^{(\prime)}, \mathbb{Z})$ (see (22)) is mapped onto a lattice vector as well as

$$\forall t \in I, l \in \{1,\dots,n(t)-1\}: \quad \widehat{E}_t^{(l)} = E_t^{(l)} + \tfrac{1}{n(t)}\widehat{\upsilon}, \quad \widehat{\upsilon} = \sqrt{N}\upsilon \tag{19}$$

(recall $E_t^{(l)} \notin L(X_0^{(\prime)})$, in general, and that $\widehat{\upsilon} = \sqrt{N}\upsilon$ generates $H^4(X_0^{(\prime)}, \mathbb{Z})$; see [N-W, We] and compare to (10)). This is a straightforward calculation using [We].

For later convenience, (12), (16)-(18) can be summarized in the following formula:

$$\begin{aligned} &t \in I \text{ a fixed point of type } \mathbb{Z}_{n(t)}: \qquad (20) \\ &\quad \gamma_{MS}(X_0^{(\prime)})(E_t^{(L)}) = -\tfrac{1}{N} \sum_{\substack{k \in \text{fiber:} \\ \gcd(n(k),n(t)) \neq 1}} \; \sum_{\substack{0 \le l < N, \\ l \equiv L \bmod n(t)}} \vartheta^{l+kt}(E_k), \end{aligned}$$

where for $n(t) = 2$ we set $E_t^{(1)} := E_t$, and in the sum over $k \in$ fiber each fixed point on the torus is counted separately. As noted in Sect. 2, $\gamma_{MS}(T_0^{(\prime)})$ is a lattice automorphism of order 4, in fact a hyperkähler rotation. For our extension (20) of $\gamma_{MS}(T_0^{(\prime)})$ to the Kummer type lattice Π_N generalizing Π we find

$$\gamma_{MS}(X_0^{(\prime)}) \circ \gamma_{MS}(X_0^{(\prime)})_{|\Pi_N} = -\vartheta^{-1}\iota, \qquad (21)$$
$$\text{where for } t \in I, l \in \{1,\dots,n(t)-1\}: \quad \iota:\ E_t^{(l)} \longmapsto E_{-t}^{(l)}.$$

Hence $\gamma_{MS}(X_0^{(\prime)})$ has order $4, 12, 8, 12$ for $N = 2, 3, 4, 6$, respectively.

For all fibers discussed above, the geometric action is analogous to that on I_0^*: Up to signs and possible corrections in $\pi_* H^{even}(T_0^{(\prime)}, \mathbb{Z})^{\mathbb{Z}_N}$ and $\widehat{\Pi}_N$, the generalization of $\widehat{\Pi}$, $\gamma_{MS}(X_0^{(\prime)})$ exchanges the center node of the extended Dynkin diagram that describes the fiber with each component of its long legs.

Of course, $\gamma_{MS}(X_0^{(\prime)})_{|\Pi_N}$ is not uniquely determined by the two conditions $\gamma_{MS}(X_0^{(\prime)}) \in Aut(L(X_0^{(\prime)}))$ and $\gamma_{MS}(X_0^{(\prime)})_{|\Pi_N^\perp} = \gamma_{MS}(T_0^{(\prime)})$. In fact, assume that $\gamma_{MS}(X_0^{(\prime)})g$ is another consistent extension of $\gamma_{MS}(T_0^{(\prime)})$. Then by definition $g_{|\Pi_N^\perp} = \mathbb{1}$, and g acts as lattice automorphism both on $L(X_0^{(\prime)})$ and Π_N. It is clear that g can incorporate arbitrary shifts of the fixed points in direction of the base in our fibration $p: X \longrightarrow \mathbb{P}^1$. In case $N = 2$ this corresponds to the freedom of choice of $t_0 \in I$ in (12). One checks that among the lattice automorphisms that permute fixed points, g must respect our fibration and therefore can only incorporate shifts on I or the map $\widetilde{\iota}: t \mapsto -t, t \in I$. The latter is nontrivial only in the $\mathbb{Z}_3$ case $I \cong \mathbb{F}_3^2$, where it corresponds to the standard $\mathbb{Z}_2$ action on the underlying torus, i.e. the algebraic automorphism that is modded out from $\widetilde{T_0'/\mathbb{Z}_3}$ to obtain $\widetilde{T_0'/\mathbb{Z}_6}$. Having said this and using the form of (20), in the following we may assume that g acts as lattice automorphism on each $\mathbb{E}_t = \mathrm{span}_{\mathbb{Z}}\{E_t^{(l)}, l \in \{1,\dots,n(t)-1\}\}$ separately. Since g is orthogonal, it must permute the roots in $\mathbb{E}_t$. Recall from [We, Th. 3.3] that

$$\widehat{\upsilon}^0 = \tfrac{1}{\sqrt{N}}\upsilon - \tfrac{1}{N}B_N + \widehat{\upsilon}, \qquad (22)$$
$$\text{where } B_N \in \Pi_N:\ \forall\, t \in I, l \in \{1,\dots,n(t)-1\}:\ \left\langle B_N, E_t^{(l)} \right\rangle = 1.$$

Then $g(\widehat{E}_t^{(l)}) \in L(X_0^{(\prime)})$ with (19) together with the fact that g preserves scalar products implies $g_{|\mathbb{E}_t} = \vartheta^{k_t}$ for some $k_t \in \{1,\dots,n(t)\}$. Moreover, g must obey

$$g(\widehat{\upsilon}^0) - \widehat{\upsilon}^0 = \tfrac{1}{N}(1-g)B_N \in \Pi_N,$$

which restricts the possible combinations of k_t that define g. In fact, with the results of [We] about the form of Π_N, all in all we find that $\gamma_{MS}(X_0)$ is given by (20) up to the above mentioned permutations in I and the action of some

$$g^b \in Aut(\Pi_N):\ g^b_{|\mathbb{E}_t} = \vartheta^{k_t}, \quad g^b\left(\tfrac{1}{N}B_N\right) = \tfrac{1}{N}B_N + b, \quad b \in \Pi_N / \bigoplus_{t\in I}\mathbb{E}_t. \qquad (23)$$

For $N \in \{2,3\}$ these degrees of freedom are parametrized by $Aff(I, \mathbb{F}_N)$, since we have a natural identification

$$g^b \quad \overset{1:1}{\longleftrightarrow} \quad k^b \in Aff(I, \mathbb{F}_N):\quad g^b_{|\mathbb{E}_t} = \vartheta^{\,k^b(t)}.$$

We will give an interpretation of this result at the end of Sect. 6. Note that by the above for any $a \in \mathbb{Z}$, $\gamma_{MS}(X_0^{(\prime)})\vartheta^a = \vartheta^a \gamma_{MS}(X_0^{(\prime)})$ are extensions of $\gamma_{MS}(T_0^{(\prime)})$ to $L(X_0^{(\prime)})$ as well. Hence (21) shows that we can define mirror maps of order 4 on the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ orbifold limits of $K3$.

We can also easily check the action of $\gamma_{MS}(X_0^{(\prime)})$ on the monodromy around the singular fibers. Again from [Ko1, Th.9.1] we read off the monodromy matrices and -apart from I_0^*- find invariance under (13) for III^* type fibers only. It follows that the geometric interpretation as $\mathbb{Z}_4$ orbifold limit obtained from a toroidal theory on a rectangular torus with vanishing B-field is mapped to another such geometric interpretation under mirror symmetry, as expected. For the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ cases, on the other hand, the analogous statement is not true. Though the conjugacy class of the monodromy remains unchanged under $\gamma_{MS}(X_0')$ by definition, its representative changes under (13). This might not be surprising because of the modification (9) by a classical symmetry of order 4 that we had to perform on the hyperkähler rotation $\gamma_{MS}(T_0)$ to gain our lattice automorphism $\gamma_{MS}(T_0')$. It also seems to be due to differences in the complex structure. Namely, for rectangular tori one has a purely imaginary complex structure on the fiber, but for $\mathbb{Z}_3$ or $\mathbb{Z}_6$ orbifolds this is impossible. The images under mirror symmetry will not have purely imaginary Kähler parameters, in other words will have a non-vanishing B-field. It might be interesting to explore this point, which suggests a link between monodromy and complexified Kähler structure within classical geometry.

5. The Mirror Map for $\mathbb{Z}_N$ Orbifold Conformal Field Theories

By our discussion in the Introduction, $\gamma_{MS}(X_0^{(\prime)})$ is to be interpreted as automorphism on the universal cover $\widetilde{\mathcal{M}}$ of the moduli space of $N=(4,4)$ superconformal field theories on $K3$ that identifies equivalent theories. As such it induces an action on the fields of any of our orbifold CFTs, up to holonomy or wave function renormalization. For chiral fields or their spectral flow partners, like the twist fields, the latter is trivial. Their holonomy is trivial, too, as long as we deform within an orbifold component of $\widetilde{\mathcal{M}}$. Since our version of mirror symmetry is induced by T-duality on a $c=3$ toroidal subtheory of the underlying $c=6$ toroidal superconformal field theory, we can determine the action of $\gamma_{MS}(X_0^{(\prime)})$ independently from the geometric results of Sects. 3 and 4. This is the aim of the present section. We remark that our technique is similar to that used in [B-E-R].

Orbifold CFTs have a generic W-algebra given by the invariant part of the current generated algebra on the underlying torus. The Hilbert space of such a theory decomposes into the $\mathbb{Z}_N$ invariant W-algebra representations of the underlying toroidal theory and the so–called twisted sector. The latter consists of W-algebra representations with infinite quantum dimensions, the ground states of which are given by the twist fields. In the following, these twist fields are denoted T_t^l, $l \in \{1,\dots,n(t)-1\}$, where we have chosen T_t of order $n(t)$ for each $t \in I$. We normalize them such that

$$\left\langle T_t^l, T_{t'}^{l'} \right\rangle = \lim_{x\to 0} |x| \left\langle T_t^l(x) T_{t'}^{-l'}(0) \right\rangle = \delta_{t,t'}\delta_{l,l'} \left(2 - \zeta_{n(t)}^l - \zeta_{n(t)}^{-l}\right). \tag{24}$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on the Hilbert space which on $(1,1)$ fields induces the Zamolodchikov metric. In Sect. 6, we will see that the normalization (24) is the natural one.

Twist fields and all the fields of the torus theory can be included together in n-point functions, if one admits ramifications of the world sheet at the twist field positions [H-V, D-F-M-S]. Denote by $V(p,z)$ the vertex operator of momentum $p=(p_l,p_r)$ and conformal dimensions $(h,\bar h)=(p_l^2/2,p_r^2/2)$, where in the notations of Sect. 2 we have $p=(p_l,p_r)=\frac{1}{\sqrt2}(Q_l,Q_r)$, i.e. $\frac{1}{\sqrt2}(Q_l+Q_r,Q_l-Q_r)\in v(\Lambda,B)\mathbb{Z}^{4,4}$. The OPE of $V(p,z)$ with the twist fields $T_t(x), t\in I$, has a contribution given by the twist fields themselves. We write it in the form

$$V(p,z)T_t(x)=(z-x)^{-h}(\bar z-\bar x)^{-\bar h}\sum_{t'\in I}W_{tt'}(p)T_{t'}(x)+\text{ other terms }. \tag{25}$$

We will read off the commutation relations of the matrices $W(p)$ from the four-point function

$$\begin{aligned}&\langle V(p,z)V(p',z')T_t(x)T_{t'}(y)\rangle\\&\qquad=(z-x)^{-h}(\bar z-\bar x)^{-\bar h}(z'-x)^{-h}(\bar z'-\bar x)^{-\bar h}O(z,z',x,y)\langle T_t(x)T_{t'}(y)\rangle.\end{aligned}$$

The function O can be written as the product of two functions which are analytic resp. anti-analytic in their corresponding domains.

Let us assume that $T_t, T_{t'}$ both correspond to $\mathbb{Z}_n$ twists θ, such that by going around $T_t(x)$ or around $T_{t'}(y)$ in the opposite sense, $V(p,z)$ becomes $V(\theta p,z)$. Hence the $V(\theta p,z)$ are well defined on the n-fold cover of the Riemann sphere, which is another Riemann sphere with coordinate $\xi=((z-x)/(z-y))^{1/n}$, and analogously $\eta=((z'-x)/(z'-y))^{1/n}$.

For fixed x,y and the appropriate domains for z,z', the function $O(z,z',x,y)$ is analytic on the cover, with poles given by the known OPE of the vertex operators. This yields

$$O(z,z',x,y)=o(x-y)\prod_{k=1}^{n}(\xi-\zeta_n^k\eta)^{p_l\theta^kp_l'}(\bar\xi-\zeta_n^{-k}\bar\eta)^{p_r\theta^kp_r'},$$

where $\zeta_n:=\exp(2\pi i/n)$ and $o(x-y)$ is easy to calculate but irrelevant for our purpose. Taking the limits $z\to x$ and $z'\to x$ in different orders one finds

$$\begin{aligned}W(p')W(p)\;&=\;\prod_{k=1}^{n}\zeta_n^{kp\theta^kp'}W(p)W(p')=\zeta_n^{\phi_n(p,p')}W(p)W(p'),\\ \text{where}\qquad&\\ \phi_n(p,p')\;:=&\sum_{k=1}^{n}kp\theta^kp'\quad\text{and}\quad qq'\overset{(2)}{=}q_lq_l'-q_rq_r'=Q_lQ_r'+Q_rQ_l'\end{aligned}\tag{26}$$

(see [Le, N-S-V]). Hence the $W(p)$ form a Weyl algebra which is represented on the vector space spanned by the twist fields.

For a given geometric interpretation, where we assume $B=0$ on the underlying toroidal theory, the momentum state vertex operators are characterized by $p_l=p_r$. They therefore form an Abelian subalgebra of the Weyl algebra

(26). With respect to this subalgebra, the representation decomposes into one-dimensional subrepresentations with ground states T_t^l, each of which corresponds to a $\mathbb{Z}_{n(t)/\gcd(l,n(t))}$ twist on a $\mathbb{Z}_{n(t)}$ type fixed point.

To see this explicitly note that for a geometric interpretation with $B=0$ we have

$$\begin{aligned} p = (p_l, p_r) &= \tfrac{1}{\sqrt{2}}(\mu+\lambda, \mu-\lambda) =: p(\mu,\lambda) \\ &\text{with} \quad (\mu,\lambda) \in \mathrm{span}_{\mathbb{Z}}\{\mu_1,\dots,\mu_4\} \oplus \mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\}. \end{aligned}$$

Now (26) takes the form

$$\phi_n(p,p') \equiv \sum_{k=1}^{n} k\left(\mu\theta^k\lambda' - \mu'\theta^k\lambda\right) \bmod n\mathbb{Z}. \tag{27}$$

In fact, the fixed point set I can be interpreted as part of the subgroup of elements of order n with $n\,|\,N$ in the Jacobian torus $H_1(T_0,\mathbb{R})/H_1(T_0,\mathbb{Z})$, where $H_1(T_0,\mathbb{Z}) \cong \mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\}$, and we can use

$$I \hookrightarrow H_1(T_0,\mathbb{Q})/H_1(T_0,\mathbb{Z}), \qquad \text{where } N\lambda \in \mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\} \text{ for } [\lambda] \in I.$$

Then (25) reads, after suitable normalizations of the vertex operators,

$$\begin{aligned} V(p(\mu,\lambda),z)T_t^l(x) &= (z-x)^{-h}(\bar z-\bar x)^{-\bar h}\,\zeta_N^{\mu(Nt)l}T_{t'}^l(x) + \text{ other terms }, \\ t' &:= t + \left[(1-\theta)^{-1}\lambda\right] = t - \left[\tfrac{1}{n(t)}\sum_{k=1}^{n(t)-1} k\theta^k\lambda\right], \end{aligned} \tag{28}$$

$$\text{and} \qquad W(p')W(p)T_t^l = \zeta_{n(t)}^{l\phi_{n(t)}(p,p')}W(p)W(p')T_t^l.$$

Now recall that $\gamma_{MS}(T_0^{(\prime)})$ as discussed in Sect. 2 is given by T-duality on the x_1, x_2 directions of the torus T, which agrees with the Fourier-Mukai transform on the corresponding two-torus [Na1, Na2, Sc, B-vB], see also [Th1]. The standard Fourier transform on $\mathbb{R}^d$ extends to all measurable Abelian groups. On $\mathbb{Z}_N$ it takes the form

$$\widetilde{F}^{(N)}: \mathbb{C}^N \longrightarrow \mathbb{C}^N, \quad \widetilde{F}^{(N)}\left((f_k)_{k=1}^N\right) = \left(\tfrac{1}{\sqrt{N}}\sum_{j=1}^N f_j\,\zeta_N^{jk}\right)_{k=1}^N.$$

In the present context, however, it is natural to restrict $\widetilde{F}^{(N)}$ to $\mathbb{Z}_N \subset U(1) \subset \mathbb{C}^*$ on each component. Let $t_k \in \mathbb{Z}_N$ denote a generator in the k^{th} component, then one has

$$F^{(N)}(t_k^l) = \tfrac{1}{\sqrt{N}}\sum_{j=1}^N t_j^l\,\zeta_N^{jkl},$$

which for simplicity we keep on calling discrete Fourier transform in the following.

Under T-duality, momentum states and winding states are interchanged, and the latter are characterized by $p_l = -p_r$. The ground states of the one-dimensional subrepresentations of the subalgebra of winding states are therefore obtained from the generators $\{T_t^l, t \in I, l \in \{1,\dots,n(t)-1\}\}$ of the twisted sector by performing a $\mathbb{Z}_N$ type discrete Fourier transform. By the above, the resulting map on twist fields has the general form

$$\begin{aligned} l \in \{1, \ldots, n(t)-1\}: & \\ & F(T_t^l) = \sum_{\substack{j \in \text{ fiber:} \\ n(t) | l n(j)}} a^l_{n(j),n(t)} T_j^{ln(j)/n(t)} \zeta_{n(t)}^{ltj}, \qquad (29) \\ \text{with } \forall\, \lambda \in \mathbb{C}: \quad & F(\lambda T_t^l) = \lambda F(T_t^l) \end{aligned}$$

and the coefficients $a^l_{n(j),n(t)}$ to be determined. Here, we use the labeling of fixed points that was introduced in Sect. 4 and which is restricted to the fiber coordinates x_1, x_2. As in (20), each fixed point on the torus contributes separately to the sum $j \in$ fiber.

When N factorizes, the orbifolding can be done stepwise. Therefore,

$$a^{kl}_{n,kn'} = k^{-1/2} a^l_{n,n'}, \qquad \text{and} \qquad a^l_{kn,n'} = k^{1/2} a^l_{n,n'}.$$

This leaves only $a^1_{n,n}$ to be determined. Since F must conserve the normalization, counting fixed points gives

$$|a^1_{n,n}|^{-2} = 2 - \zeta_n - \zeta_n^{-1}.$$

Up to a finite ambiguity, the phase can be determined from the order of F. Instead of a cumbersome direct determination, we fix it by consistency requirements with the results of Sects. 3,4 (see Sect. 6). This yields:

$$a^l_{n,n'} = -\tfrac{1}{\sqrt{nn'}} \sum_{m=1}^{n-1} m \zeta_{n'}^{lm} = \sqrt{\tfrac{n}{n'}} (1 - \zeta^l_{n'})^{-1}.$$

Note in particular that

$$F\left(\overline{T_t^l}\right) = F\left(T_t^{n(t)-l}\right) = \overline{F(T_t^l)}.$$

Explicitly we have in the $\mathbb{Z}_2$ case:

$$t \in \mathbb{F}_2^2: \quad F(T_t) = \tfrac{1}{2} \sum_{j \in \mathbb{F}_2^2} (-1)^{tj} T_j,$$

in the $\mathbb{Z}_3$ case:

$$t \in \mathbb{F}_3: \quad F(T_t) = -\tfrac{i\omega}{\sqrt{3}} \sum_{j \in \mathbb{F}_3} \omega^{tj} T_j, \quad \omega = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2},$$

in the $\mathbb{Z}_4$ case:

$$\begin{aligned} t \in \mathbb{F}_2: \quad F(T_{(t,t)}) &= \tfrac{1+i}{2} \left(T_{(0,0)} + (-1)^t T_{(1,1)}\right), \\ F(T^2_{(t,t)}) &= \tfrac{1}{2} \left(\sqrt{2}(-1)^t T_{(0,1)} + T^2_{(0,0)} + T^2_{(1,1)}\right), \\ F(T_{(0,1)}) &= \tfrac{1}{\sqrt{2}} \left(T^2_{(0,0)} - T^2_{(1,1)}\right), \end{aligned}$$

in the $\mathbb{Z}_6$ case:

$$\begin{aligned} F(T_0) &= \widetilde{\omega}\; T_0, & \widetilde{\omega} &= \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}, \\ F(T_0^2) &= -\tfrac{i\omega}{\sqrt{3}}\left(T_0^2 + \sqrt{2}T_1\right), & \omega &= -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}, \\ F(T_0^3) &= \tfrac{1}{2}\left(T_0^3 + \sqrt{3}T_{(1,0)}\right), \\ F(T_1) &= \tfrac{i\omega}{\sqrt{3}}\left(T_1 - \sqrt{2}T_0^2\right), \\ F(T_{(1,0)}) &= -\tfrac{1}{2}\left(T_{(1,0)} - \sqrt{3}T_0^3\right). \end{aligned}$$

6. Mirror Symmetry, Discrete Fourier Transform, and the McKay Correspondence

In Sects. 4 and 5 we have independently derived the action of mirror symmetry $\gamma_{MS}(X_0^{(\prime)})$ both in terms of geometric data (20) and CFT data (29). Now we shall relate the two approaches and give an interpretation in terms of the classical McKay correspondence.

In general, we assume that the $\mathbb{Z}_N$ orbifold CFT as discussed in Sect. 5 possesses a geometric interpretation on our $\mathbb{Z}_N$ orbifold limit of $K3$ as discussed in Sect. 4. In view of our results on mirror symmetry, we may expect a linear map that intertwines between the pictures. More precisely, on the CFT side the twist fields T_t^l generate deformations of the theory, which act linearly on the cycles Poincaré dual to the $E_t^{(l)}$. Thus they should naturally correspond to linear combinations of these cocycles. We denote the induced linear map from cocycles to twist fields by C. This map should act isometrically with respect to the standard metrics (24) and (4) up to a global sign, and it should obey

$$\forall\, t \in I, l \in \{1, \ldots, n(t)-1\}: \quad F\, C(E_t^{(l)}) \;=\; C\, \gamma_{MS}(X_0^{(\prime)})(E_t^{(l)}). \tag{30}$$

On the CFT level, the sectors of a $\mathbb{Z}_N$ orbifold theory carry distinct representations of the dual of $\mathbb{Z}_N$, with the untwisted sector corresponding to the trivial representation. In particular, one obtains nontrivial representations on all twist fields. More precisely, we have an action of the dual of $\mathbb{Z}_{n(t)}$ on the twist fields T_t^l at a given fixed point $t \in I$. In Sect. 5 we have already labeled the twist fields such that the action is generated by multiplication with $\zeta_{n(t)}^l$ on T_t^l. By abuse of notation we denote the generator of this $\mathbb{Z}_{n(t)}$ action ϑ. It gives the "quantum symmetry" of the orbifold conformal field theory that replaces the "geometric symmetry" on the underlying toroidal theory. It is well known that by modding out the orbifold CFT by the total "quantum" $\mathbb{Z}_N$ symmetry we can reproduce the original toroidal theory.

Geometrically, each fixed point $t \in I$ is left invariant by a subgroup $\mathbb{Z}_{n(t)}$ of $\mathbb{Z}_N$. This induces an action of the dual of $\mathbb{Z}_{n(t)}$ on $\mathbb{E}_t = \operatorname{span}_{\mathbb{Z}}\{E_t^{(l)}, l \in \{0, \ldots, n(t)-1\}\}$. Indeed, its generator ϑ has been introduced in (15) above. The map C can therefore be obtained by decomposing the integral $\mathbb{Z}_{n(t)}$ action on $\mathbb{E}_t$ into one-dimensional representations. We claim that this yields

$$t \in I, l \in \{1, \ldots, n(t)-1\}: \quad C(E_t^{(l)}) := \frac{1}{\sqrt{n(t)}} \sum_{k=1}^{n(t)-1} \zeta_{n(t)}^{lk} T_t^k. \tag{31}$$

Together with (20), (29) one first checks (30). Moreover, the reality condition

$$\forall\, t \in I, l \in \{1,\dots,n(t)-1\}: \quad \overline{C(E_t^{(l)})} \;=\; C(E_t^{(l)}),$$

translates into $\overline{T_t^l} \;=\; T_t^{n(t)-l}$, as expected.

Finally, up to a prefactor, the Hermitean form on $\mathbb{E}_t$ is given by the intersection product (4) on the cocycles, thus by the Cartan matrix of the extended Dynkin diagram. Since the latter can be written as $2-\vartheta-\vartheta^{-1}$, it commutes with ϑ and becomes diagonal in the basis T_t^l of eigenvectors of ϑ. With C a unitary matrix with respect to this basis, the squared norms of the T_t^l yield the corresponding eigenvalues $2-\zeta_{n(t)}^l-\zeta_{n(t)}^{-l}$ of the Cartan matrix, in agreement with (24). All in all, (31) is checked to give an isometry, up to a global sign, on the entire twisted sector.

Now (31) shows that (28) in fact induces an action of $\mathrm{span}_{\mathbb{Z}}\{\mu_1,\dots,\mu_4\} \oplus \mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\}$ on Π_N by

$$\begin{aligned}
(\mu,\lambda) \in \mathrm{span}_{\mathbb{Z}}&\{\mu_1,\dots,\mu_4\} \,\oplus\, \mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\}, t\in I, l\in\{1,\dots,n(t)-1\}: \\
& W(\mu,\lambda)E_t^{(l)} := E_{t'}^{(\mu(nt)+l)} \overset{(15)}{=} \vartheta^{\mu(nt)} E_{t'}^{(l)}, \quad n=n(t), \\
& \qquad t' = t + \left[(1-\theta)^{-1}\lambda\right] = t - \left[\tfrac{1}{n}\sum_{k=1}^{n-1} k\theta^k\lambda\right],
\end{aligned} \tag{32}$$

along with the corresponding Weyl algebra representation given in Sect. 5. In order to work with geometric objects, recall from our discussion in Sect. 4 that rather than the lattice Π_N we have to consider its projection $\widehat{\Pi}_N$ onto $H^2(X_0^{(\prime)},\mathbb{Z})$. By $\widehat{\upsilon}^0, \widehat{\upsilon}$ we denote the generators of $H^0(X_0^{(\prime)},\mathbb{Z}), H^4(X_0^{(\prime)},\mathbb{Z})$, respectively, and recall from (19) that

$$\forall\, t \in I, l \in \{1,\dots,n(t)-1\}: \quad E_t^{(l)} = \widehat{E}_t^{(l)} - \tfrac{1}{n(t)}\widehat{\upsilon}.$$

Hence the $\mathbb{Z}_{n(t)}$ symmetry (15) on the irreducible components of the exceptional divisor over a given fixed point $t \in I$ actually is

$$\vartheta\left(\widehat{E}_t^{(l)}\right) = \begin{cases} \widehat{E}_t^{(l+1)} & \text{if } l < n-1, \\ \widehat{\upsilon} - \sum\limits_{j=1}^{n-1} \widehat{E}_t^{(j)} & \text{if } l = n-1, \end{cases}$$

i.e. $\vartheta(\widehat{\upsilon}) = \widehat{\upsilon}$. By the above, the "quantum" symmetry ϑ has a straightforward geometric meaning.

We also set $\vartheta(\widehat{\upsilon}^0) = \widehat{\upsilon}^0$ and consider the induced action on the basis dual to $\{\widehat{E}_t^{(l)}, l \in \{1,\dots,n(t)-1\}, \widehat{\upsilon} - \sum\limits_{j=1}^{n-1}\widehat{E}_t^{(j)}\}$:

$$\begin{aligned}
\varepsilon_t^{(l)} &:= \widehat{\upsilon}^0 + (\widehat{E}_t^{(l)})^* \text{ where } (\widehat{E}_t^{(0)})^* := 0, \\
&\qquad \langle (\widehat{E}_t^{(l)})^*, \widehat{E}_t^{(l')} \rangle \;=\; \delta_{l,l'} \text{ for } l, l' \in \{1,\dots,n(t)-1\}.
\end{aligned} \tag{33}$$

By (32), $\{\varepsilon^{(0)},\dots,\varepsilon^{(n(t)-1)}\}$ carries the representation

$$
\begin{aligned}
&(\mu,\lambda)\in \mathrm{span}_{\mathbb{Z}}\{\mu_1,\dots,\mu_4\}\oplus\mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\}, t\in I, l\in\{1,\dots,n(t)-1\}:\\
&\qquad W(\mu,\lambda)\varepsilon_t^{(l)} = \varepsilon_{t'}^{(\mu(nt)+l)},\quad t'=t-\Big[\tfrac{1}{n}\sum_{k=1}^{n-1}k\theta^k\lambda\Big],\quad n=n(t), \qquad (34)\\
&W(\mu,\lambda)W(\mu',\lambda') = \vartheta^{\phi_n(p,p')}W(\mu',\lambda')W(\mu,\lambda),\quad p=p(\mu,\lambda), p'=p(\mu',\lambda')
\end{aligned}
$$

with $\phi_n(p,p')$ as in (27).

This is readily interpreted in terms of $\mathbb{Z}_N$ equivariant topological K-theory on the underlying torus T. Namely, consider a $\mathbb{Z}_N$ equivariant flat line bundle $\mathcal{L}$ on T, then by [B-K-R] there is a corresponding line bundle $\pi_*\mathcal{L}$ on our $K3$ surface $X=T/\mathbb{Z}_N$. More precisely, $\mathcal{L}$ is determined by the representation χ_t of $\mathbb{Z}_N$ in each fiber over a fixed point $t\in I$. Then

$$
c_1(\pi_*\mathcal{L})=\sum_{t\in I}c_1(\chi_t),
$$

where $c_1(\chi_t)$ is given by the classical McKay correspondence [Mc1, Mc2, G-S-V, Kn]:

$$
c_1(\chi_t^{(l)})=(\widehat{E}_t^{(l)})^*,\qquad \text{if } \chi_t^{(l)}:\mathbb{C}\longrightarrow\mathbb{C},\quad \chi_t^{(l)}(z)=\zeta_{n(t)}^l\, z.
$$

Hence $\varepsilon_t^{(l)}, t\in I, l\in\{0,\dots,n(t)-1\}$ as in (33) are the $H^0(X,\mathbb{Z})\oplus H^2(X,\mathbb{Q})$ components of the Mukai vectors

$$
ch(\mathcal{E})\sqrt{\widehat{A}(X)}=[\mathrm{rk}\mathcal{E}]\,\widehat{v}^0+c_1(\mathcal{E})+\Big[(c_2-\tfrac{1}{2}c_1^2)(\mathcal{E})[X]+\tfrac{\mathrm{rk}\mathcal{E}}{48}p_1(X)\Big]\,\widehat{v}
$$

for all possible contributions $\mathcal{E}$ to $\pi_*\mathcal{L}$ over $t\in I$. The coefficient of $\widehat{v}$ does not enter into the relation between line bundles and representations.

We therefore find that the action (34) of $\mathrm{span}_{\mathbb{Z}}\{\lambda_1,\dots,\lambda_4\}\cong H_1(T_0^{(\prime)},\mathbb{Z})$ agrees with that of the subgroup $I\subset H_1(T_0^{(\prime)},\mathbb{R})/H_1(T_0^{(\prime)},\mathbb{Z})$ of the Jacobian torus on $\mathbb{Z}_N$ equivariant flat line bundles on T. On the other hand, $\mu\in\mathrm{span}_{\mathbb{Z}}\{\mu_1,\dots,\mu_4\}$ acts by tensoring a line bundle on T with the bundle $\mathcal{L}_\mu$ associated to χ, where

$$
z\in(\mathcal{L}_\mu)_t, t\in I:\quad \chi(z)=\zeta_N^{\mu(n(t)t)}z.
$$

Summarizing, (31) gives an isometry between the $\mathbb{C}$-vector space spanned by the twist fields of an orbifold CFT and a subspace of $H^*(X_0^{(\prime)},\mathbb{C})$, which allows us to identify the natural action of the Weyl algebra (26) on twist fields with the natural action of the same Weyl algebra on the $\mathbb{Z}_N$ equivariant flat line bundles on T. It has been known before, of course, that such identifications should be possible [Wi, Ga, D-G2, B-G-K, dB-D-H$^+$].

Recall from (23) that our formula (20) was only determined up to certain permutations of the fixed points and an action of g^b that is characterized by inducing an integral B–field shift by some $b\in\Pi_N/\bigoplus_{t\in I}\mathbb{E}_t$. Since $\Pi_N/\bigoplus_{t\in I}\mathbb{E}_t$, and for $N\in\{2,3\}$ equivalently $Aff(I,\mathbb{F}_N)$ parametrizes $\mathbb{Z}_N$ equivariant flat line bundles on T, this freedom of choice translates nicely into possible choices of the origin in that parameter space. We have seen, though, that the choice of g^b may influence the order of the resulting $\gamma_{MS}(X_0')$. We find that it then corresponds to a twisting of our mirror map, given by some equivariant flat line bundle on T.

Finally, let us briefly explain the connection of our results to Chen and Ruan's orbifold cohomology [C-R] on $[T/G]$ as discussed in [Ru1, F-G]. In the present case of cyclic $G = \mathbb{Z}_N$ it is isomorphic to

$$H^*_{CR}([T/\mathbb{Z}_N], \mathbb{C}) \cong H^*(T, \mathbb{C})^{\mathbb{Z}_N} \oplus \bigoplus_{t \in I} \bigoplus_{l=1}^{n(t)-1} \mathbb{C}\mathcal{T}(t)_{\zeta^l_{n(t)}},$$

where the generators $\mathcal{T}(t)_{\zeta^l_{n(t)}}$ of the twisted sector corresponding to the fixed point $t \in I$ have scalar product

$$\left(\mathcal{T}(t)_{\zeta^l_{n(t)}}, \mathcal{T}(t')_{\zeta^{l'}_{n(t')}}\right) := \delta_{t,t'}\delta_{l,-l'}. \tag{35}$$

In fact, if instead of (35) one uses the associated sesquilinear form[6] and slightly different normalizations to identify $\mathcal{T}(t)_{\zeta^l_{n(t)}}$ with T_t^l, it agrees with our metric in the twisted sector of the orbifold CFT. More importantly, this means that (31) can be used to prove that for all orbifold limits discussed in this note

$$H^*_{CR}([T_0^{(\prime)}/\mathbb{Z}_N], \mathbb{C}) \cong H^*(X_0^{(\prime)}, \mathbb{C})$$

as metric spaces, confirming part of Ruan's conjecture [Ru1, Conj.6.3] for these cases.

7. Conclusions

In this note, we have analyzed a version of mirror symmetry on elliptically fibred $K3$ surfaces that is induced by fiberwise T-duality on nonsingular fibers. It is straightforward to determine this map for four-tori, which enables us to give the explicit action on $\mathbb{Z}_N$ orbifold limits of $K3$, $N \in \{2, 3, 4, 6\}$, by the techniques of [N-W, We]. In the present case of $N = (4, 4)$ superconformal field theories with central charge $c = 6$ the mirror map can be realized as automorphism on the lattice of integral cohomology on the underlying complex surface (torus or $K3$). While the order of mirror symmetry on the torus is 4, it can take values 4, 8, or 12 on our orbifold limits of $K3$. On the CFT side, the mirror map is induced by a $\mathbb{Z}_N$ type discrete Fourier transform in the fiber acting on the twist fields of the orbifold conformal field theory. Since we are able to derive the mirror map in both the geometric and conformal field theoretic description independently, we can deduce the exact geometric counterparts of orbifold CFT twist fields. Moreover, the correspondence between the twist fields and $\mathbb{Z}_N$ equivariant flat line bundles can be deduced directly. The natural "quantum" $\mathbb{Z}_N$ symmetry in the twisted sector of the orbifold CFT, which can be modded out to retain the original toroidal theory, gains geometric meaning. In fact, by the classical

[6] As remarked in footnote 1, this was explained to us by Yongbin Ruan [Ru2], and goes back to earlier observations by Edward Witten.

McKay correspondence it can be traced back to properties of singularities already investigated in [Mu, Hi].

Our version of mirror symmetry agrees with the one proven more generally in [V-W], which was generalized to the celebrated Strominger/Yau/Zaslow conjecture [S-Y-Z]. We have avoided M-theory language, though, and restricted considerations to the underlying geometry. Together with the rich structure of the $K3$ moduli space this enables us to carry out the construction away from large volume or large complex structure limits in the moduli space and even without touching the issue of its compactification. Note that all our T^2 fibrations $p : X \longrightarrow \mathbb{P}^1$ have non-stable singular fibers. A large complex structure limit in the sense of [G-W2] has therefore not even been defined for those cases we are interested in.

Our results on the explicit prescription for the identification of conformal field theoretic and geometric data might be of interest in their own right. We have pointed out how they resolve the objection in [F-G] to Ruan's conjecture [Ru1, Conj.6.3] (see footnotes 1 and 6; [Ru2]). Note also that we are working within the full CFT, without having to perform a topological twist or introduce boundary states to probe the geometry of our orbifold limits of $K3$.

References

[A-C-R+1] Andreas, B., Curio, G., Ruiperez, D.H., Yau, S.-T.: Fourier-Mukai transform and mirror symmetry for D-branes on elliptic Calabi-Yau. `math.ag/0012196`

[A-C-R+2] Andreas, B., Curio, G., Ruiperez, D.H., Yau, S.-T.: Fibrewise T-duality for D-branes on elliptic Calabi-Yau. JHEP **0103:20** (2001); `hep-th/0101129`

[A-M] Aspinwall, P.S., Morrison, D.R.: String theory on K3 surfaces. In: Greene, B., Yau, S.-T. (eds.), *Mirror symmetry*, vol. **II**, 1994, pp. 703–716; `hep-th/9404151`

[As] Aspinwall, P.S.: Enhanced gauge symmetries and $K3$ surfaces. Phys. Lett. **B357**, 329–334 (1995); `hep-th/9507012`

[Ba] Batyrev, V.: Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. J. Algebr. Geom. **3**, 493–535 (1994); `alg-geom/9310003`

[B-B-R-P] Bartocci, C., Bruzzo, U., Ruiperez, D.H., Munoz Porras, J.M.: Mirror symmetry on elliptic K3 surfaces via Fourier-Mukai transform. Commun. Math. Phys. **195**, 79–93 (1998); `alg-geom/9704023`

[B-E-R] Brunner, I., Entin, R., Römelsberger, Ch.: D-branes on $T^4/\mathbb{Z}_2$ and T-Duality. JHEP **9906:016** (1999); `hep-th/9905078`

[B-G-K] Bergman, O., Gimon, E., Kol, B.: Strings on orbifold lines. JHEP **0105:019** (2001); `hep-th/0102095`

[B-K-R] Bridgeland, T., King, A., Reid, M.: Mukai implies McKay: the McKay correspondence as an equivalence of derived categories. `math.AG/9908027`

[B-S] Bruzzo, U., Sanguinetti, G.: Mirror symmetry on K3 surfaces as a hyper-Kaehler rotation. Lett. Math. Phys. **45**, 295–301 (1998); `physics/9802044`

[B-vB] Braam, P.J., van Baal, P.: Nahm's transformation for instantons. Commun. Math. Phys. **122**, 267–280 (1989)

[C] Cecotti, S.: $N = 2$ supergravity, type IIB superstrings and algebraic geometry. Commun. Math. Phys. **131**, 517–536 (1990)

[C-K] Cox, D.A., Katz, S.: *Mirror symmetry and algebraic geometry.* Providence, RI: Amer. Math. Soc., 1999

[C-O-G-P] Candelas, P., De La Ossa, X.C., Green, P.S., Parkes, L.: A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. Nucl. Phys. **B359**, 21–74 (1991)

[C-R] Chen, W., Ruan, Y.: Orbifold quantum cohomology. `math.AG/0005198`

[dB-D-H+] de Boer, J., Dijkgraaf, R., Hori, K., Keurentjes, A., Morgan, J., Morrison, D.R., Sethi, S.: Triples, fluxes, and strings. `hep-th/0103170`

[D-F-M-S] Dixon, L.J., Friedan, D., Martinec, E., Shenker, S.: The conformal field theory of orbifolds. Nucl. Phys. **B282**, 13–73 (1987)

[D-G1] Distler, J., Greene, B.: Some exact results on the superpotential from Calabi-Yau compactifications. Nucl. Phys. **B309**, 295–316 (1988)

[D-G2] Diaconescu, D.-E., Gomis, J.: Fractional branes and boundary states in orbifold theories. JHEP **0110:001** (2000); `hep-th/9906242`

[Di1] Dijkgraaf, R.: Instanton strings and hyperkaehler geometry. Nucl. Phys. **B543**, 545–571 (1999); `hep-th/9810210`

[Di2] Dixon, L.J.: Some world-sheet properties of superstring compactifications, on orbifolds and otherwise. *Lectures given at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology (Trieste, June 29 - August 7)*

[E-T] Eguchi, T., Taormina, A.: Extended superconformal algebras and string compactifications. Trieste School 1988: Superstrings, pp. 167–188

[F-G] Fantechi, B., Göttsche, L.: Orbifold cohomology for global quotients. `math.AG/0104207`

[Ga] Garcia-Compean, H.: D-branes in orbifold singularities and equivariant K- theory. Nucl. Phys. **B557**, 480–504 (1999); `hep-th/9812226`

[G-H-L] Gomez, C., Hernandez, R., Lopez, E.: S-Duality and the Calabi-Yau interpretation of the $N=4$ to $N=2$ flow. Phys. Lett. **B386**, 115–122 (1996); `hep-th/9512017`

[G-S-V] Gonzalez-Sprinberg, G., Verdier, J.-L.: Construction géométrique de la correspondance de McKay. Ann. Sci. École Norm. Sup. **16**, no. 3, 409–449 (1983)

[G-P] Greene, B.R., Plesser, M.R.: Duality in Calabi-Yau moduli space. Nucl. Phys. **B338**, 15–37 (1990)

[G-W1] Gross, M., Wilson, P.M.H.: Mirror symmetry via 3-tori for a class of Calabi-Yau threefolds. Math. Ann. **309**, no. 3, 505–531 (1997); `alg-geom/9608004`

[G-W2] Gross, M., Wilson, P.M.H.: Large Complex Structure Limits of K3 Surfaces. `math.DG/0008018`

[Hi] Hirzebruch, F.: The topology of normal singularities of an algebraic surface (after D. Mumford). In: *Séminaire Bourbaki*, vol. **8**. Paris, France: Soc. Math. 1995, Exp. No. **250**, pp. 129–137

[H-L] Harvey, R., Lawson Jr., H.B.: Calibrated geometries. Acta Math. **148**, 47–157 (1982)

[H-V] Hamidi, S., Vafa, C.: Interactions on orbifolds. Nucl. Phys. **B279**, 465–513 (1987)

[K-M-P] Katz, S., Morrison, D.R., Plesser, M.R.: Enhanced Gauge Symmetry in Type II String Theory. Nucl. Phys. **B477**, 105–140 (1996); `hep-th/9601108`

[Kn] Knörrer, H.: Group representations and the resolution of rational double points. In: *Finite groups—coming of age (Montreal, Que., 1982)*. Providence, R.I.: Amer. Math. Soc., 1985, pp. 175–222

[Ko1] Kodaira, K.: On compact analytic surfaces: II. Ann. Math. **77**, 563–626 (1963)

[Ko2] Kontsevich, M.: Homological algebra of mirror symmetry. In: *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, vol. **1**,**2**. Basel: Birkhäuser, 1995, pp. 120–139; `alg-geom/9411018`

[K-O-P] Kiritsis, E., Obers, N., Pioline, B.: Heterotic/type II triality and instantons on $K(3)$. JHEP **0001:019** (2000); `hep-th/0001083`

[K-S] Kontsevich, M., Soibelman, Y.: Homological mirror symmetry and torus fibrations. `math.SG/0011041`

[Le] Lepowsky, J.: Calculus of twisted vertex operators. Proc. Nat. Acad. Sci. U.S.A. **82**, no. 24, 8295–8299 (1985)

[L-V-W] Lerche, W., Vafa, C., Warner, N.P.: Chiral rings in $N=2$ superconformal theories. Nucl. Phys. **B324**, 427–474 (1989)

[Mc1] McKay, J.: Graphs, singularities, and finite groups. In: *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*. Providence, R.I.: Amer. Math. Soc., 1980, pp. 183–186

[Mc2] McKay, J.: Cartan matrices, finite groups of quaternions, and Kleinian singularities. Proc. Amer. Math. Soc. **81**, no. 1, 153–154 (1981)

[Mo1] Morrison, D.R.: Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians. J. Amer. Math. Soc. **6**, no. 1, 223–247 (1993); `alg-geom/9202004`

[Mo2] Morrison, D.R.: The Geometry Underlying Mirror Symmetry. In: *New trends in algebraic geometry (Warwick, 1996)*. Cambridge, UK: Cambridge Univ. Press, 1999, pp. 283–310; `alg-geom/9608006`

[Mu] Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Inst. Hautes Études Sci. Publ. Math. **9**, 5–22 (1961)

[Na1] Nahm, W.: The construction of all self–dual monopoles by the ADHM method. In: Craigie, N.S., Goddard, P., Nahm, W. (eds.), *Monopoles in quantum field theory (Trieste, 1981)*. Singapore: World Scientific, 1982, pp. 87–94

[Na2] Nahm, W.: Self–dual monopoles and calorons. In: Denardo, G., et al. (eds.), *Lecture Notes in Physics*, vol. **201**. Berlin-Heidelberg-New York: Springer, 1984, pp. 189–200

[Na3] Nahm, W.: Strings on K3. In: *Proceedings of the ICMP 2000*. London, 2000.

[Na4] Narain, K.S.: New heterotic string theories in uncompactified dimensions < 10. Phys. Lett. **169B**, 41–46 (1986)

[N-S-V] Narain, K.S., Sarmadi, M.H., Vafa, C.: Asymmetric orbifolds. Nucl. Phys. **B288**, 551–557 (1987)

[Ni] Nikulin, V.V.: On Kummer Surfaces. Math. USSR Isv. **9**, 261–275 (1975)

[N-W] Nahm, W., Wendland, K.: A hiker's guide to $K3$ – Aspects of $N = (4,4)$ superconformal field theory with central charge $c = 6$. Commun. Math. Phys. **216**, 85–138 (2001); `hep-th/9912067`

[O-P] Obers, N.A., Pioline, B.: Exact thresholds and instanton effects in string theory. Fortsch. Phys. **49**, 359–375 (2001); `hep-th/0101122`

[Ru1] Ruan, Y.: Stringy geometry and topology of orbifolds. `math.AG/0011149`

[Ru2] Ruan, Y.: Cohomolgy ring of crepant resolutions. `math.AG/0108195`

[Sc] Schenk, H.: On a generalized Fourier transform of instantons over flat tori. Commun. Math. Phys. **116**, 177–183 (1988)

[Se] Seiberg, N.: Observations on the moduli space of superconformal field theories. Nucl. Phys. **B303**, 286–304 (1988)

[S-Y-Z] Strominger, A., Yau, S.-T., Zaslow, E.: Mirror symmetry is T-duality. Nucl. Phys. **B479**, 243–259 (1996); `hep-th/9606040`

[Th1] Thomas, R.P.: Mirror symmetry and actions of braid groups on derived categories. In: *Proceedings of the Harvard Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, January 1999*. International Press, 2001, pp. 187–201

[Th2] Thurston, W.P.: *Three-dimensional geometry and topology*, vol. **1**. Levy, S. (ed.). Princeton, NJ: Princeton University Press, 1997

[vEn] van Enckevort, Ch.: Mirror Symmetry and T–Duality. Ph.D. thesis, Universiteit Utrecht, 2000

[V-W] Vafa, C., Witten, E.: On orbifolds with discrete torsion. J. Geom. Phys. **15**, 189–214 (1995); `hep-th/9409188`

[We] Wendland, K.: Consistency of orbifold conformal field theories on $K3$. `hep-th/0010281`

[Wi] Witten, E.: D-branes and K-theory. JHEP **9812:019** (1998); `hep-th/9810188`

[Za] Zamolodchikov, A.B.: 'Irreversibility' of the flux of the renormalization group in a 2-D field theory. JETP Lett. **43**, 730–732 (1986)