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The associative algebras of conformal field theory

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Abstract

Modulo the ideal generated by the derivative fields, the normal ordered product of holomorphic fields in two-dimensional conformal field theory yields a commutative and associative algebra. The zero mode algebra can be regarded as a deformation of the latter. Alternatively, it can be described as an associative quotient of the algebra given by a modified normal ordered product. We clarify the relation of these structures to Zhu's product and Zhu's algebra of the mathematical literature.

1 Introduction

Apart from free field theories, conformally invariant theories in two dimensions were the first quantum field theories for which an elegant and efficient mathematical theory was developped ([Bor86], [FLM88], [God89], for a recent proposal on axiomatic foundation of conformal field theory see [GG98], where Zhu's algebra is also discussed). Deficiencies of communication between mathematicians and physicists, however, led to parallel developments in both communities, which are inconvenient and often confusing.

A glaring example is the classification of representations of the W-algebra of holomorphic fields. Since the early days of conformal field theory, physicists have used the zero mode algebra for that purpose (e.g. [Zam85], [BG88], [KZ84]).

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Mathematicians talk about Zhu's algebra instead, and define the latter with the help of Zhu's product ([Zhu90], [DLM95]). Its seems that it took them a while to discover that the two algebras are naturally isomorphic, but the fact has been stated in their work ([FZ92], remark after proposition 1.4.2). This fact has been overlooked by physicists who started to use Zhu's algebra under the impression that it is a new structure (e.g. [EG97]). We also failed to notice the remark until we rediscovered the isomorphism by ourselves.

2 Operator product expansion and normal ordered products

The main reason for the unnecessary duplication of concepts seems to be the lack of familiarity of mathematicians with the operator product expansion (OPE).

Consider holomorphic conformal fields ϕ and χ of conformal dimensions h and h'. Their Fourier expansion is of the form

$$\phi(z) = \sum_n z^{n-h}\phi_n$$

and their OPE can be written as

$$\phi(z)\chi(w) = \sum_r (z-w)^r\psi^r(w),$$

where the holomorphic fields ψ^r have dimensions $h+h'+r$. In particular, the sum can be restricted to integers $r \geq -h-h'$. The field

$$\psi^0(w) = \oint_{C(w)} \frac{dz}{2\pi i(z-w)}\phi(z)\chi(w),$$

$C(w)$ a circle around w, is the standard normal ordered product of ϕ and χ. One common notation is $\psi^0 = N(\phi,\chi)$. Writing $C(w)$ as the difference of cycles around zero, one obtains the Fourier coefficients

$$\psi^0_n = \sum_{m\geq h}\phi_m\chi_{n-m} + \sum_{m<h}\chi_{n-m}\phi_m.$$

We grade by the energy, such that creation operators have positive Fourier index.

Instead of splitting the sum at $m=h$ one can use any other fixed value of m. The resulting field differs from $N(\phi,\chi)$ by a linear combination of some fields of lower dimension, namely the ψ^r with $r<0$. Splitting at $m=0$ yields

$$N_0(\phi,\chi)_n = \sum_{m\geq 0}\phi_m\chi_{n-m} + \sum_{m<0}\chi_{n-m}\phi_m.$$

The corresponding field can be obtained from its value at 1 by translation. Since

$$N_0(\phi,\chi)(1) = \sum_n N_0(\phi,\chi)_n = \oint_{C(1)} \frac{dz}{2\pi i(z-1)} z^h \phi(z)\chi(1),$$

one obtains

$$N_0(\phi,\chi)(0) = \oint_{C(0)} \frac{dz}{2\pi i z}(1+z)^h \phi(z)\chi(0).$$

The isomorphism $\psi(0)|vac\rangle = |\psi\rangle$ between fields and states yields

$$|N_0(\phi,\chi)\rangle = \oint_{C(0)} \frac{dz}{2\pi i z}(1+z)^h \phi(z)|\chi\rangle,$$

which coincides with Zhu's definition of the product $\phi * \chi$.

One can modify the N_0-product by scaling its dimension $h+h'-r$ components by λ^r:

$$\oint_{C(0)} \frac{dz}{2\pi i z}(1+\lambda z)^h \phi(z)|\chi\rangle.$$

For $\lambda = 0$ one recovers $N(\phi,\chi)$. Thus the N_0-product can be regarded as a deformation of the N-product.

The derivative of the normal ordered product N is given by

$$\partial N(\phi,\chi) = N(\partial\phi,\chi) + N(\phi,\partial\chi),$$

whereas in the case of the N_0-product ∂ has to be replaced by $L_1 + L_0$ on either side of the equation.

3 Zhu's commutative algebra

The commutator $N(\phi,\chi) - N(\chi,\phi)$ has Fourier coefficients which are given by the commutators $[\phi_m,\chi_n]$. The latter are spanned by Fourier components Ψ_{m+n} of fields of dimension $h(\Psi) < h+h'$. Since the commutator field has dimension $h+h'$, it is a linear combination of derivative fields $\partial^s\Psi$, with $s = h+h'-h(\Psi)$. Similarly, the N-associator

$$A(\phi,\chi,\psi) = N(\phi,N(\chi,\psi)) - N(N(\phi,\chi),\psi)$$

can be calculated in terms of commutators of the Fourier coefficients of ϕ, χ, and ψ. Thus it is a linear combination of fields of the form $N(\Phi,\partial^s\Psi)$, with $s>0$ and $h(\Phi)+h(\Psi) = h(\phi)+h(\chi)+h(\psi)-s$. In other words, modulo the ideal generated by the derivative fields, the N product yields a commutative, associative algebra. Zhu defined this algebra in section 4.4 of [Zhu90], but didn't give it a name. Physicists seem to have overlooked its significance. Provisionally, we shall call it Zhu's commutative algebra. The zero mode algebra is a deformation of it, as we shall see.

4 Zhu's algebra and the algebra of zero modes

We have seen that Zhu's product is just the normal ordered product N_0. Its properties can be derived easily from this fact. Of particular interest is the associator

$$A_0(\phi,\chi,\psi) = N_0(\phi, N_0(\chi,\psi)) - N_0(N_0(\phi,\chi),\psi).$$

One has

$$\begin{aligned} A_0(\phi,\chi,\psi)_n &= \sum_{k<0}\sum_{l=0}^{-k-1}[\phi_{n-k-l},\chi_l]\psi_k - \sum_{k\geq 0}\psi_k\sum_{l=-k}^{-1}[\phi_{n-k-l},\chi_l] \\ &\quad + \sum_{k<0}\sum_{l=k}^{-1}[\phi_{n-l},\psi_{l-k}]\chi_k - \sum_{k\geq 0}\chi_k\sum_{l=0}^{k-1}[\phi_{n-l},\psi_{l-k}]. \end{aligned}$$

Since $\sum_{l=0}^{-k-1}[\phi_{m-l},\chi_l]$ and its continuation $-\sum_{l=-k}^{-1}[\phi_{m-l},\chi_l]$ is a polynomial in k which vanishes for $k=0$, the first two terms on the right hand side yield normal ordered products involving fields with Fourier components $k^s\psi_k$, $s>0$, i.e. linear combinations of normal ordered products of the fields $\partial^s\psi + (h(\psi)+s-1)\partial^{s-1}\psi$. The other remaining terms on the r.h.s. yield normal ordered products of $\partial^s\chi + (h(\chi)+s-1)\partial^{s-1}\chi$. These fields can be written in the form $\partial\Phi + h(\Phi)\Phi$. Modulo normal ordered products of such fields, the product N_0 is associative. This has been discovered by Zhu, though his calculations were not very transparent. The resulting quotient algebra has been called Zhu's algebra by mathematicians, but we see no reason to introduce this term into physics.

Actually, Zhu's formalism was a bit different, but the equivalence is easy to check. He calculated modulo fields of the form

$$\oint_{C(0)} \frac{dz}{2\pi i z^2}\Phi(z)(1+z)^h|\Psi\rangle.$$

When we write $(1+z)^h/z^2 = (1+z)^{h+1}/z^2 - (1+z)^h/z$ and get rid of z^{-2} by a partial integration, this field is reduced to $(\partial\Phi + h(\Phi)\Phi) * \Psi$.

On the ground state vectors v of a representation of the OPE one has

$$N_0(\phi,\chi)_0 v = \phi_0\chi_0 v.$$

This yields a homomorphism of Zhu's algebra to the zero mode algebra. Since the latter is associative, the N_0 associators have to be mapped to zero. Indeed, $(\partial\Phi)_0 + h(\Phi)\Phi_0 = 0$.

Every representation of the zero mode algebra induces a lowest energy representation of the W-algebra of holomorphic fields. If one admits infinite ground state degeneracies, one such representation is the adjoint one, which is faithful. This means that Zhu's algebra is naturally isomorphic to the zero mode algebra, such that we see no need for a terminological distinction. It is not clear, however, if one could make a similar argument by using finite dimensional representations only, apart from the obvious case when the zero mode algebra itself is finite dimensional. In the latter case the theory is rational.

As a vector space, the zero mode algebra is filtered by the conformal dimension. It is not graded, since sums of type $\partial\Phi + h(\Phi)\Phi$ are not homogeneous. The corresponding graded vector space is just the space of all holomorphic fields modulo the derivative fields. The corresponding algebra is Zhu's commutative and associative algebra discussed above.

5 Conclusion

We hope that the present paper will stop the duplication of work in this corner of conformal field theory. Apparently, all of us should work a bit harder to get through the communication barrier. In particular, mathematicians should strive to be less clumsy, and we, perhaps, a bit less sloppy.

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