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Landau–Ginzburg Vacua of String, M- and F-Theory at $c = 12$

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Abstract

Theories in more than ten dimensions play an important role in understanding nonperturbative aspects of string theory. Consistent compactifications of such theories can be constructed via Calabi–Yau fourfolds. These models can be analyzed particularly efficiently in the Landau–Ginzburg phase of the linear σ-model, when available. In the present paper we focus on those σ-models which have both a Landau–Ginzburg phase and a geometric phase described by hypersurfaces in weighted projective five-space. We describe some of the pertinent properties of these models, such as the cohomology, the connectivity of the resulting moduli space, and mirror symmetry among the 1,100,055 configurations which we have constructed.

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1 Introduction

Over the past years much effort has gone into the exploration of Calabi–Yau fourfolds. Such manifolds provide ground states of 4D IIB string theory compactified on manifolds with positive first Chern class and a nontrivial dilaton which was constructed in [1, 2]. The behavior of the dilaton is constrained by the geometry of the curvature of the threefold and can be summarized succinctly by considering a fibered fourfold with a section, providing, in a certain sense, a four-dimensional compactification of a twelve-dimensional theory, called F-theory. Such manifolds furnish three-dimensional compactifications of M-theory and therefore play an important role in the understanding of certain dualities, such as the N=1 4D duality [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]

$$\mathrm{F}_{12}(\mathrm{CY}_4) \quad \longleftrightarrow \quad \mathrm{Het}(V \rightarrow \mathrm{CY}_3), \tag{1}$$

where $V \rightarrow \mathrm{CY}_3$ is a stable vector bundle over a Calabi–Yau threefold determined by the F-theoretic fourfold CY_4.

In this context it is of interest to investigate the space of CY fourfolds in some detail. In the present paper we extend the analysis of Landau–Ginzburg theories at $c = 9$ describing Calabi–Yau threefolds to fourfolds described by models at $c = 12$. We then describe a number of pertinent features of the resulting moduli space of this class of theories. The conjectured F-theory/Heterotic duality in 4D leads to a number of expected properties of the moduli space of CY fourfolds. One natural question raised by the relation (1) is the issue of mirror symmetry. In the context of (2,2) vacua of the heterotic string on Calabi–Yau manifolds mirror symmetry is known to provide a powerful tool for the analysis of such ground states. Much less is know about the general framework of (0,2) vacua, geometrically described by stable vector bundles. As a first step in this direction the existence of (0,2) mirror symmetry between such vector bundles was established in [16] for a large class of (0,2) vacua by generalizing the known (2,2) mirror constructions [17, 18, 19]. This analysis shows that different stable bundles indeed correspond to the same underlying (0,2) superconformal field theory. More recently conjectures have been put forward which aim at a geometrical interpretation of (0,2) mirror symmetry [20, 21]. Given mirror symmetry among vector bundles the duality (1) then leads to the expectation that the moduli space of fourfolds is mirror symmetric, at least in the region described by the subclass of elliptically fibered fourfolds. As will become apparent however, mirror symmetry in fact holds more generally, independent of any fibration considerations. Similar to the case of threefolds we can strengthen the Hodge theoretic mirror symmetry suggested by the comparison of the cohomology group by a direct construction of hypersurface mirror pairs via fractional transformations [18, 22]. For a general D-fold mirror symmetry entails $h^{p,q}(M) = h^{D-p,q}(M^*)$ for a mirror pair (M, M^*). The Hodge diamond of a

Calabi–Yau fourfold contains only four varying Hodge numbers

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \end{array} \qquad (2)$$

three of which are independent. A plot of three independent Hodge numbers turns out to be somewhat unilluminating. As in the case of threefolds however, mirror symmetry for fourfolds distinguishes the combinations $(h^{1,1} - h^{3,1})$ and $(h^{1,1} + h^{3,1})$ and therefore we can summarize mirror symmetry among Calabi–Yau fourfolds in a diagram similar to the mirror plot of [23, 24, 25]. The result for the class of hypersurfaces is shown in Figure 1.

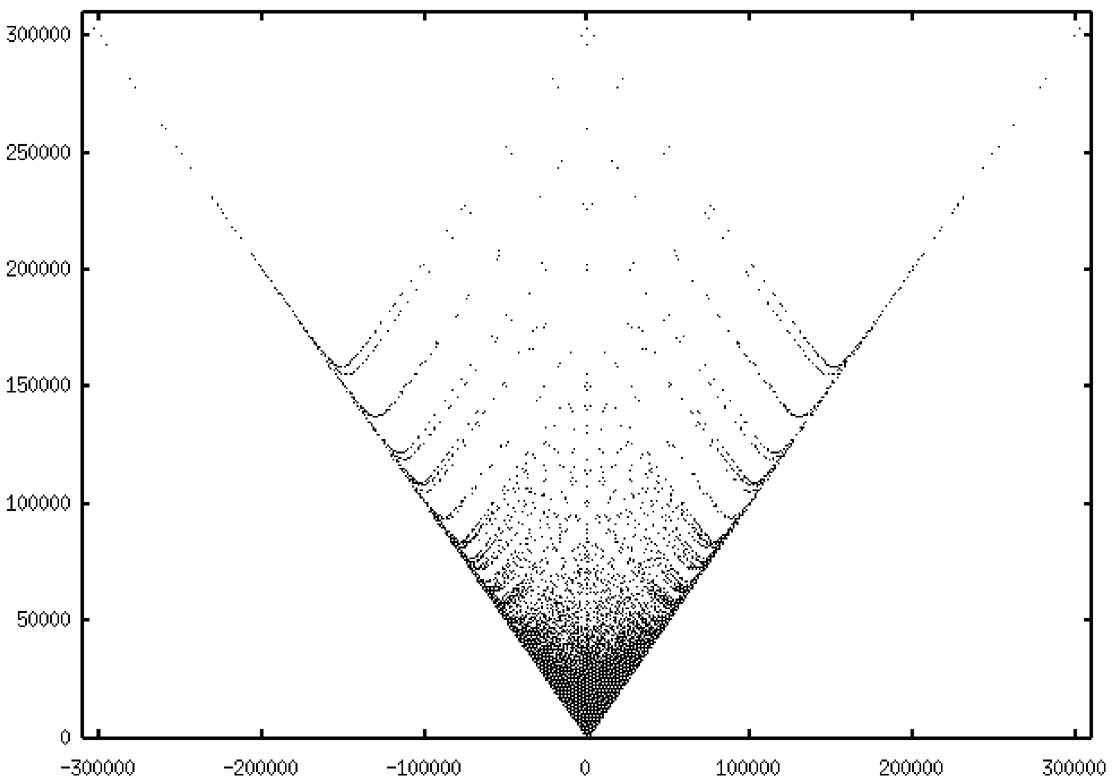

Figure 1: *Plot of* $(h^{(3,1)} + h^{(1,1)})$ *vs.* $(h^{(3,1)} - h^{(1,1)})$ *for the class of Calabi–Yau fourfold hypersurfaces in weighted* $\mathbb{P}_5$.

A further important property of this space is that it is possible to connect manifolds of this space via certain types of phase transitions [7], involving singular configurations whose lower dimensional counterpart are the conifold transitions introduced in [26].

A number of general aspects, such as the problem of finiteness of the number of Landau–Ginzburg configurations and transversality of the allowed potentials are independent of the dimension of the manifolds. We review our earlier discussion of these issues for threefolds [24] in order to make this paper self-contained. The article is organized as follows. After describing in Section 2 and 3 the class of theories we will focus on, we review in Section 3 the computation of the spectrum of such theories. In Section 4 we will turn to a discussion of mirror symmetry and in Section 5 we discuss aspects of the connectedness of the moduli space of the resulting vacua. We then describe the construction of the models in more detail in Sections 6 and 7 and summarize our results in Section 8. Several subclasses of models which are of interest in the context of F-theory and M-theory are described in Section 9.

2 σ-Models

The physical theories for which the class of Calabi–Yau fourfolds provides consistent ground states can be succinctly described via linear σ-models. The starting point of the analysis of [27] is a U(1) gauge theory in N=2 superspace, extending the standard Landau–Ginzburg action for the chiral N=2 superfields to

$$\mathcal{A} = \mathcal{A}_{kin} + \mathcal{A}_{D,\theta} + \mathcal{A}_{kin,\Phi_i} + \mathcal{A}_{W,\Phi_i}. \tag{3}$$

Consider, in the notation of [27], the gauge invariant field strength

$$\mathcal{F} = \frac{1}{2\sqrt{2}}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\}, \tag{4}$$

the kinetic term of which is given by

$$\mathcal{A}_{kin} = -\frac{1}{4e^2}\int d^2z d^4\theta \ \bar{\mathcal{F}}\mathcal{F}. \tag{5}$$

There are two possible interactions, the θ angle term and the Fayet–Illiopoulos D-term. These can be written as

$$\mathcal{A}_{\mathrm{D},\theta} = \frac{it}{2\sqrt{2}}\int d^2z d\theta^+ d\bar{\theta}^- \ \mathcal{F} + h.c. \tag{6}$$

where

$$t = ir + \frac{\theta}{2\pi} \tag{7}$$

and r is the coefficient of the D-term.

To this are added N chiral superfields with U(1)-charge $k_i \in \mathbb{N}$. The kinetic energy of these fields is chosen to be

$$\mathcal{A}_{kin,\Phi_i} = \int d^2z d^4\theta \ \sum_{i=1}^{N} \bar{\Phi}_i \Phi_i \tag{8}$$

and the superpotential is assumed to be of gauge invariant form

$$\mathcal{A}_{W,\Phi} = -\int d^2z d^2\theta \ W(\Phi_i) - h.c. \tag{9}$$

which is supersymmetric because the Φ_i are chiral and W is a holomorphic quasihomogeneous polynomial of the chiral fields.

The constant part of the lowest components of the superfields Φ_i can be thought of as parametrizing the n-dimensional complex space $\mathbb{C}_n$, assuming, as in [27], that the Kähler metric in the kinetic term of the Φ_i should be flat.

The bosonic equations of motion for the auxiliary field D in the superfield $\mathcal{F}$ and the auxiliary fields F_i of the chiral superfields Φ_i become

$$D = -e^2 \left(\sum_i k_i |\phi_i|^2 - r \right) \tag{10}$$

and

$$F_i = \frac{\partial W}{\partial \phi_i}. \tag{11}$$

The bosonic potential that one obtains in terms of the matter fields ϕ_i and the auxiliary fields D and F_i is

$$U(\phi_i, \sigma) = \frac{1}{2e^2} D^2 + \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + 2|\sigma|^2 \sum_i k_i^2 |\phi_i|^2. \tag{12}$$

Assuming now that the superpotential takes the form

$$W(\Phi_i) = \Phi_0 \widetilde{W}(\Phi_1, \ldots, \Phi_N), \tag{13}$$

where $\widetilde{W}$ is a quasihomogeneous polynomial in the variables $(\Phi_0, \ldots, \Phi_N)$ with weights $(k_0, \ldots, k_N)$ which is assumed to be transverse, i.e. the equations

$$\frac{\partial \widetilde{W}}{\partial \Phi_i} = 0 \tag{14}$$

can be solved only at the origin.

With this potential the bosonic potential becomes

$$U(\phi_i, \sigma) = \frac{1}{2e^2} D^2 + \left| \widetilde{W} \right|^2 + |\phi_0|^2 \sum_{i=1}^N \left| \frac{\partial \widetilde{W}}{\partial \phi_i} \right|^2 + 2|\sigma|^2 \left(\sum_{i=1}^N k_i^2 |\phi_i|^2 + k_0^2 |\phi_0|^2 \right) \tag{15}$$

with

$$D = -e^2 \left(\sum_{i=1}^N k_i |\phi_i|^2 - k_0 \phi_0 \bar{\phi}_0 - r \right). \tag{16}$$

All terms in (15) are ≥ 0. Thus in order to minimize the potential U one has to minimize D^2, which leads to different results, depending on what the behavior of the variable r.

The $r \gg 0$ phase: In this case not all ϕ_i can be zero. Since the polynomial $\widetilde{W}$ is transverse everywhere except at the origin $\partial_i \widetilde{W}$ is nonzero for some i. Hence ϕ_0 and σ must be zero. Thus $D = 0$ leads to

$$\sum_{i=1}^{N} k_i \bar{\phi}_i \phi_i = r, \qquad \phi_0 = 0 = \sigma. \tag{17}$$

For $k_i = 1$ this simply defines a sphere $S^{2N-1} \subset \mathbb{C}_N$. Recalling that one has to mod out the U(1) gauge group and also that the sphere S^{2N-1} can be Hopf-fibered $S^1 \longrightarrow S^{2N-1} \longrightarrow \mathbb{P}_{N-1}$ leads to the condition that the constant bosonic components of Φ_i parametrize a projective space $\mathbb{P}_{N-1} = S^{2N-1}/U(1)$. For $k_i \neq 1$ one arrives at a weighted projective space instead. Furthermore the vanishing of $\widetilde{W}$ leads to a geometry for the space of ground states described by a hypersurface embedded in $\mathbb{P}_{N-1}$.

The $r \ll 0$ phase: In this case the vanishing of D leads to $\phi_0 \neq 0$ and hence the term $|\phi_0|^2 \sum_i |\partial_i \widetilde{W}|^2$ enforces that $\phi_i = 0$ since this is the only place where the partial derivatives of the superpotential are allowed to vanish, because of transversality. This fixes the modulus of ϕ_0 to be

$$k_0 |\phi_0| = -r. \tag{18}$$

Because of the gauge invariance the classical vacuum is in fact unique, modulo gauge transformations. Expanding around this vacuum leads to massless fields ϕ_i (for $N \geq 3$). To find the potential for these massless fields one has to integrate out the massive field ϕ_0. Integrating out ϕ_0 means setting ϕ_0 to its expectation value. Thus the effective superpotential of the low energy theory is

$$\widetilde{W} = \sqrt{-k_0 r}\, W(\phi_i). \tag{19}$$

The factor $\sqrt{-k_0 r}$ is inessential since it can be absorbed by rescaling the ϕ_i. Since the origin is a multicritical point, this describes a Landau–Ginzburg orbifold. It is the Landau–Ginzburg formulation of the theory which lends itself for a further analysis of a number of aspects of these vacua.

3 Landau–Ginzburg Theories

The perhaps simplest way to compute the spectrum of the class of models we consider is via their Landau–Ginzburg phase.

3.1 Chiral ring structure

Using a superspace formulation in terms of the coordinates $(z, \bar{z}, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-)$ we can view the Landau–Ginzburg phase of the linear σ-model to be described by the action

$$\mathcal{A} = \int d^2z d^4\theta\ K(\Phi_i, \bar{\Phi}_i) + \int d^2z d^2\theta^-\ \widetilde{W}(\Phi_i) + \int d^2z d^2\theta^+\ \widetilde{W}(\bar{\Phi}_i) \tag{20}$$

where K is the Kähler potential and the superpotential $\widetilde{W}$ is a holomorphic function of the chiral superfields Φ_i. The ground states of the bosonic potential are the critical points of the superpotential of the LG theory and therefore we assume that W has such critical points. We also require that these critical points are isolated since we wish to relate the finite dimensional ring of monomials associated to such a singularity to the chiral ring of physical states in the Landau–Ginzburg theory, in order to construct the spectrum of the corresponding string vacuum. The fact that the fermions in the theory should be massless furthermore leads to the constraint that the critical points are completely degenerate. Finally, we assume that the Landau–Ginzburg potential is quasihomogeneous, i.e. we can assign to each field Φ_i a weight q_i such that for any non-zero complex number $\lambda \in \mathbb{C}^\star$

$$\widetilde{W}(\lambda^{q_1}\Phi_1, \ldots, \lambda^{q_n}\Phi_n) = \lambda \widetilde{W}(\Phi_1, \ldots, \Phi_n). \tag{21}$$

The class of potentials we will focus on thus is comprised of quasihomogeneous polynomials that have an isolated, completely degenerate singularity (which we can always shift to the origin).

Mathematically then a so-called catastrophe is associated to each of the superpotentials $\widetilde{W}(\Phi_i)$, obtained by first truncating the superfield Φ_i to its lowest bosonic component $\phi_i(z,\bar{z})$, and then going to the field theoretic limit of the string by assuming ϕ_i to be constant $\phi_i = z_i$. Writing the weights as $q_i = k_i/d$, we will denote by $\mathbb{C}_{(k_1,k_2,\ldots,k_n)}[d]$ the set of all catastrophes described by the zero locus of polynomials of degree d in variables z_i of weight k_i.

The affine varieties described by these polynomials are not compact and hence it is necessary to implement a projection in order to compactify these spaces. In Landau–Ginzburg language, this amounts to an orbifolding of the theory with respect to a discrete group $\mathbb{Z}_d$ the order of which is the degree of the LG potential [28]. The spectrum of the orbifold theory will contain twisted states which, together with the monomial ring of the potential, describe the complete spectrum of the corresponding Calabi–Yau manifold. We will denote the orbifold of a Landau–Ginzburg theory by

$$\mathbb{C}^\star_{(k_1,k_2,\ldots,k_n)}[d] \tag{22}$$

and call it a configuration.

In the manifold context we are now interested in complex four-dimensional Kähler manifolds, with vanishing first Chern class. For a general Landau–Ginzburg theory no unambiguous universal prescription for doing so has been found, and none can exist [24]. One way to compactify amounts to simply imposing projective equivalence

$$(z_1, \ldots, z_n) \equiv (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n) \tag{23}$$

which embeds the hypersurface described by the zero locus of the polynomial into a weighted projective space $\mathbb{P}_{(k_1,k_2,\ldots,k_n)}$ with weights k_i. The set of hypersurfaces of degree d embedded in weighted projective space will be denoted by

$$\mathbb{P}_{(k_1,k_2,\ldots,k_n)}[d]. \tag{24}$$

For a potential with six scaling variables this construction is completely sufficient in order to pass from the Landau–Ginzburg theory to a string vacuum, provided $d = \sum_{i=1}^{6} k_i$, which is the condition that these hypersurfaces have vanishing first Chern class. For more than six variables, however, this type of compactification does not lead to a string vacuum and the geometric phase is in fact described by higher codimension manifolds embedded in products of weighted projective space. A simple example is furnished by the LG potential in six variables

$$W = \Phi_1\Psi_1^2 + \Phi_2\Psi_2^2 + \sum_{i=1}^{4}\Phi_i^{12} + \Phi_5^4 \tag{25}$$

which corresponds to the exactly solvable model described by the tensor product of $N = 2$ minimal superconformal theories at the levels

$$(22^2 \otimes 10^2 \otimes 2)_{D^2\otimes A^3}, \tag{26}$$

where the subscripts indicate the affine invariants chosen for the individual factors[1]. This theory belongs to the LG configuration

$$\mathbb{C}^{\star}_{(2,11,2,11,2,2,6)}[24] \tag{27}$$

whose geometrical phase is described by the weighted complete intersection Calabi–Yau (CICY) manifold in the configuration

$$\begin{matrix} \mathbb{P}_{(1,1,1,1,3,6)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} 1 & 12 \\ 2 & 0 \end{bmatrix} \tag{28}$$

described by the intersection of the zero locus of the two potentials

$$\begin{aligned} p_1 &= x_1^2 y_1 + x_2^2 y_2 \\ p_2 &= y_1^{12} + y_2^{12} + y_3^{12} + y_4^{12} + y_5^4 + y_6^2. \end{aligned} \tag{29}$$

Here we have added a trivial factor Φ_6^2 to the potential and again taken the field theory limit via $\phi_i(z,\bar{z}) = y_i$ and $\psi_j(z,\bar{z}) = x_j$, where ϕ_i and ψ_j are the lowest components of the chiral superfield Φ_i and Ψ_j. The first column in the degree matrix (28) indicates that the first polynomial is of bidegree (2,1) in the coordinates (x_i, y_j) of the product of the projective line $\mathbb{P}_1$ and the weighted projective space $\mathbb{P}_{(1,1,1,3,6)}$ respectively, whereas the second column shows that the second polynomial is independent of the projective line and of degree 12 in the coordinates of the weighted $\mathbb{P}_4$.

Even though the assumptions just described [30, 31] may seem rather reasonable it is clear that it is not the most general class of F-theory, M-theory, or string vacua. Although it provides a rather large set of different models there are vacua which cannot be described in this framework. An interesting project for the future would be the complete construction of hypersurfaces in toric varieties. First steps in this direction have been taken in [10].

[1] Further explanations and references can be found in [29].

3.2 Landau–Ginzburg cohomology computation

The cohomology of Calabi–Yau fourfolds is most efficiently computed via the Landau–Ginzburg model [4] along the lines described in [28]. The simplest part of the computations pertains to the Euler number which can be obtained via

$$\chi = \frac{1}{d}\sum_{l,r=0}^{d-1} \prod_{lq_i, rq_i \in \mathbf{Z}} \frac{d-k_i}{k_i}. \tag{30}$$

The Landau–Ginzburg construction does better and allows to compute all the Hodge numbers independently. To do so one constructs a Poincaré-type polynomial for the $l^{\rm th}$ twisted sector

$$P_l(t,\bar{t}) = \prod_{lq_i\in \mathbf{Z}} \left(\frac{1-(t\bar{t})^{d-k_i}}{1-(t\bar{t})^{k_i}}\right) \tag{31}$$

which leads to the trace

$$Tr_l\left((t\bar{t})^{dJ_0}\right) = t^{d\left(k_l+\frac{1}{6}c_T\right)}\bar{t}^{d\left(-Q_l+\frac{1}{6}c_T\right)} \prod_{lq_i\in \mathbf{Z}} \left(\frac{1-(t\bar{t})^{d-k_i}}{1-(t\bar{t})^{k_i}}\right) \tag{32}$$

with charges

$$Q_l = \sum_{lq_i\notin \mathbf{Z}} \left(lq_i - [lq_i] - \frac{1}{2}\right) \tag{33}$$

and the central charge of those fields which transform nontrivially under the twist l

$$\frac{1}{6}c_T = \sum_{lq_i\ \notin \mathbf{Z}} \left(\frac{1}{2} - q_i\right). \tag{34}$$

Here t and $\bar{t}$ are formal variables, d is the degree of the Landau–Ginzburg potential, the $q_i = k_i/d$ are the normalized weights of the fields and $[lq_i]$ is the integer part of lq_i. Expanding this polynomial in powers in t and $\bar{t}$ it is possible to read off the contributions to the various cohomology groups from the different sectors of the twisted LG-theory. The (2,1)-forms for example are given by the number of fields of charge (1,1), i.e. the coefficient of $(t\bar{t})^d$. In general, the number of (p,q)-forms are given by the coefficient of $t^{(3-p)d}\bar{t}^{qd}$ in the Poincare polynomials summed over all sectors $l = 0,\ldots,d-1$. In more detail, the basic observation is that in the l^{th} twisted sector the charges of the states are of the form

$$(Q_l, -Q_l) + (r,r). \tag{35}$$

The charges Q_l, given by (33), are the contributions to the charge coming from the twisted fields and r is any of the charges generated by the subring of those fields that are invariant under the l-twist. These charges are generated by the Poincaré polynomial (31) of the invariant fields. Given fields of integral charge one can generate the cohomology classes: fields with charges $(p,q)\in \mathbb{Z}\times\mathbb{Z}$ will generate the cohomology groups $\mathrm{H}^{(r,s)}$ according to the relation

$$(p,q) \longleftrightarrow \mathrm{H}^{(D-p,q)} \tag{36}$$

where D is, in general, the complex dimension of the manifold.

It was first pointed out in [32] that in higher dimensions the dimensions of the cohomology groups are not necessarily independent, in contradistinction to surfaces and threefolds. For Calabi–Yau fourfolds this fact leads to the relation [3, 8]

$$44 + 4h^{(1,1)} + 4h^{(3,1)} - 2h^{(2,1)} - h^{(2,2)} = 0 \tag{37}$$

and therefore we only have three independent cohomology dimensions in the present context.

To illustrate this method we consider the fourfold $\mathbb{P}_{(1,1,2,4,4,4)}[16]$. The Landau–Ginzburg computation leads to the following contributions to the cohomology:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 3 & & 0 & & \\ & 0 & & 3 & & 3 & & 0 & \\ 1 & & 440 & & 1810 & & 440 & & 1. \end{array} \tag{38}$$

The Euler number agrees with the previous computations. The contributions of the various twisted sectors to the individual Hodge numbers are listed in Table 1.

Cohomology Group	Contributing Twisted Sectors $h_l^{(p,q)} = h_{d-l}^{(D-p,D-q)}$	Dimension
$\mathrm{H}^{(1,1)}$	$h_7^{(1,1)} + h_{11}^{(1,1)} + h_{14}^{(1,1)}$	1+1+1=3
$\mathrm{H}^{(2,2)}$	$h_0^{(2,2)} + h_3^{(2,2)} + h_6^{(2,2)} + h_8^{(2,2)} + h_{10}^{(2,2)} + h_{13}^{(2,2)}$	1787+1+1+19+1+1=1810
$\mathrm{H}^{(1,3)}$	$h_0^{(1,3)} + h_8^{(1,3)}$	439+1=440
$\mathrm{H}^{(2,3)}$	$h_4^{(2,3)}$	3

Table 1: *Sector contributions to the cohomology* $\mathrm{H}^{p,q}(\mathbb{P}_{(1,1,2,4,4,4)}[16])$.

3.3 Geometry of fourfolds

Similar to the case of threefolds weighted projective hypersurface fourfolds are in general singular and must be resolved. These resolutions correspond to the twisted contributions in the Landau–Ginzburg formulation of the previous subsections. It turns out however that the details of the geometric resolution are quite different from the resolution of threefolds.

As an example which illustrates this consider again the the fourfold $\mathbb{P}_{(1,1,2,4,4,4)}[16]$ as one of the simplest spaces with a singular curve (this curve itself is smooth). With $c_4(\mathbb{P}_{(1,1,2,4,4,4)}[16]) = 21,288\, h^4$ and the fourfold singularities described by the $\mathbb{Z}_2$-singular surface

$$\begin{aligned} \mathbb{Z}_2 \quad : \quad & S = \mathbb{P}_{(1,2,2,2)}[8] \\ & c_2(S) = 26h^2, \quad \chi_{\rm sing}(S) = 26 \end{aligned} \tag{39}$$

and the $\mathbb{Z}_4$-singular curve

$$\begin{aligned} \mathbb{Z}_4 \quad : \quad & C = \mathbb{P}_2[4], \\ & \chi_C = -4, \quad \text{genus} = 3 \end{aligned} \tag{40}$$

one finds the Euler number

$$\begin{aligned} \chi_4 &= \frac{21,288 \cdot 16}{2 \cdot 4 \cdot 4 \cdot 4 \cdot 4} - \frac{1}{2}\left(26 - \frac{1}{2}(-4)\right) + 2\left(26 - \frac{1}{2}(-4)\right) - \frac{1}{4}(-4) + 4(-4) \\ &= 2688. \end{aligned} \tag{41}$$

This fourfold is a fibration over the base $\mathbb{P}_1$ whose generic quasismooth fiber threefold configuration is given by $F = \mathbb{P}_{(1,1,2,2,2)}[8]$. This generic fiber degenerates over $N = 16$ points on the base with singular fibers $F^\sharp = \mathbb{P}_{(1,2,2,2)}[8]$. We can use this fibration structure to compute the Euler number independently via the fibration formula

$$\chi_{4,\text{fib}} = \chi(\mathbb{P}_1 - N)\chi(F) + N\chi(F^\sharp). \tag{42}$$

Hence the fibration formula gives

$$\chi_{4,\text{fib}} = (2 - 16) \cdot (-168) + 16(20 + 1) = 2688, \tag{43}$$

in agreement with the resolution formula.

We now see that the resolution of fourfolds leads to different ingredients compared to those of threefolds. The resolution of the singular curve $C = \mathbb{P}_2[4]$ introduces in the present case 3 (2,1)-forms and 3 (1,2)-forms even though this is a $\mathbb{Z}_4$-curve of genus $g = 3$. On a threefold its resolution would have a divisor D which is a fiber bundle over the curve C with fiber $\mathbb{P}_1 \vee \mathbb{P}_1 \vee \mathbb{P}_1$. Instead, on the fourfold the resolution of this genus 3 curve generates only three (2,1)-forms. Similar differences appear for the remaining nontrivial cohomology groups.

4 Connectedness of Moduli Space

For a number of reasons we would expect the moduli space of Calabi–Yau fourfolds to be connected [7, 33]. First, we can consider fibered fourfolds and consider degenerations in the fibers. This type of transition is particularly obvious if the fibers are threefolds because we can then consider conifold transitions, for instance [26]. It is also possible however to consider lower-dimensional fibers, such as K3 surfaces or elliptic surfaces. Even though the cohomology of the fibers cannot change in such transitions the resulting transitions of the total space are generically nontrivial. This generalizes the observations of [22] to fourfolds.

4.1 Transitions between hypersurfaces in weighted projective spaces

Simple transitions between hypersurfaces can be constructed via discriminantal splits of fourfolds [33]. An example is provided by the sextic hypersurface which is connected via such a split to a hypersurface of degree twelve

$$\mathbb{P}_{(1,1,2,2,2,4)}[12]_{2592} \quad \longleftrightarrow \quad \mathbb{P}_5[6]_{2610}. \tag{44}$$

This can be seen by rewriting the lhs hypersurface as a codimension two complete intersection manifold

$$\mathbb{P}_{(1,1,2,2,2,4)}[12] \quad \sim \quad \begin{array}{c} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(1,1,1,1,1,2)} \end{array} \begin{bmatrix} 2 & 0 \\ 1 & 6 \end{bmatrix}. \tag{45}$$

This can be achieved by first going into the Landau–Ginzburg phase via the associated linear σ-model. We denote this LG configuration by $\mathbb{C}^*_{(1,1,2,2,2,4)}[12]$, in which we can consider the Fermat section of the moduli space for concreteness. Once we are in the LG phase we can add two mass terms y_i^2, $i = 1, 2$, without changing the renormalization group fixed point. This leads to an equivalent description of this model as

$$\mathbb{C}_{(1,1,6,6,2,2,2,4)}[12] \ \ni \ \left\{ \sum_{i=1}^{2} \left(x_i^{12} + y_i^2 \right) + \sum_{j=3}^{5} x_j^6 + x_6^3 = 0 \right\}. \tag{46}$$

This theory however is isomorphic to the orbifold

$$\mathbb{C}_{(1,6,1,6,2,2,2,4)}[12] \Big/ \mathbb{Z}_2^2 : \ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 \end{bmatrix} \ \sim \ \mathbb{C}_{(2,5,2,5,2,2,2,4)}[12] \tag{47}$$

as can be seen with fractional transformations [18] and therefore, by reversing the above steps we see that the geometrical phase of this LG theory is described by the rhs of equation (45). By applying the discriminantal transition to this codimension two manifold we finally arrive at the rhs of equation (44).

4.2 Connectedness to the space of CICYs

It has been shown in [7] that more generally the space of all Calabi–Yau fourfolds embedded in products of ordinary projective space

$$\begin{array}{c} \mathbb{P}_{n_1} \\ \mathbb{P}_{n_2} \\ \vdots \\ \mathbb{P}_{n_F} \end{array} \begin{bmatrix} d_1^1 & d_2^1 & \dots & d_N^1 \\ d_1^2 & d_2^2 & \dots & d_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_1^F & d_2^F & \dots & d_N^F \end{bmatrix} \tag{48}$$

is connected via splitting type transitions. By repeatedly applying $\mathbb{P}_1$-splits of the type

$$X \ = \ Y[(u+v) \quad M] \quad \longleftrightarrow \quad \begin{array}{c} \mathbb{P}_1 \\ Y \end{array} \begin{bmatrix} 1 & 1 & 0 \\ u & v & M \end{bmatrix} \ = \ X_{\text{split}} \tag{49}$$

to any of the projective factors with $n_i > 1$ until all corresponding $d_a^i = 1$ and contracting the $\mathbb{P}_{n_i}$ via

$$X = Y \left[\sum_{a=1}^{n+1} u_a \quad M \right] \quad \longleftrightarrow \quad \begin{array}{c} \mathbb{P}_n \\ Y \end{array} \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ u_1 & u_2 & \cdots & u_{n+1} & M \end{bmatrix} \ = \ X_{\text{split}} \tag{50}$$

it follows that all these manifolds are connected to the simple configuration

$$\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \end{array} \left[\begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{array}\right]_{1440} \tag{51}$$

with Euler number $\chi = 1440$ which can be determined via Cherning. From Lefshetz' hyperplane theorem we know that $h^{(1,1)} = 5$ and $h^{(2,1)} = 0$. The dimension of $\mathrm{H}^{(3,1)}$ for this manifold can be determined by counting complex deformations with the result $h^{(3,1)} = 227$. From the Euler number we can then determine that final remaining Hodge number to obtain the complete Hodge diamond

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 5 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 227 & & 972 & & 227 & & 1. \end{array} \tag{52}$$

After having shown that the space of CICYs is connected it is natural to ask whether it is non-simply connected. This a more difficult question to answer. There are however many loop type transitions which can be constructed within the splitting construction of [7]. An example of such a loop is provided by the spaces of Figure 2.

$$\begin{array}{ccccc}
\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_{(1,1,2,2,2,4)} \end{array} \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 2 & 6 \end{array}\right] & \longleftrightarrow & \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_{(1,1,2,2,2,4)} \end{array} \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 2 & 2 & 6 \end{array}\right] & \longleftrightarrow & \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_{(1,1,2,2,2,4)} \end{array} \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 8 \end{array}\right] \\
\updownarrow & & & & \updownarrow \\
\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_{(1,1,2,2,2,4)} \end{array} \left[\begin{array}{cc} 1 & 1 \\ 4 & 8 \end{array}\right] & \longleftrightarrow & \mathbb{P}_{(1,1,2,2,2,4)}[12] & \longleftrightarrow & \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_{(1,1,2,2,2,4)} \end{array} \left[\begin{array}{cc} 1 & 1 \\ 2 & 10 \end{array}\right] \\
 & & \updownarrow & & \\
\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_5 \end{array} \left[\begin{array}{cc} 1 & 1 \\ 1 & 5 \end{array}\right] & \longleftrightarrow & \mathbb{P}_4[6] & \longleftrightarrow & \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_5 \end{array} \left[\begin{array}{cc} 1 & 2 \\ 1 & 4 \end{array}\right] \\
\updownarrow & & & & \updownarrow \\
\begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_5 \end{array} \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 4 \end{array}\right] & \longleftrightarrow & \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_5 \end{array} \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right] & \longleftrightarrow & \begin{array}{c} \mathbb{P}_1 \\ \mathbb{P}_1 \\ \mathbb{P}_5 \end{array} \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{array}\right]
\end{array} \tag{53}$$

Figure 2: *An example of two connected loop transitions.*

The existence of these loops indicates that the moduli spaces of threefolds and fourfolds and, more generally n-folds, is not simply connected. If this turns out to be correct then this will have important consequences for approaches to the connectivity of the moduli space via direct cohomological methods because it would notably impact the splitting properties of the long exact sequences which have to be computed[2] [34].

5 Mirror Symmetry

It is clear from Figure 1 and Figure 2 that the space of Calabi–Yau fourfold hypersurfaces exhibits a high degree of symmetry. The total number of 667,954 distinct Hodge diamonds leads to 583,824 distinct pairs of combinations

$$(h^{(3,1)} + h^{(1,1)}, h^{(3,1)} - h^{(1,1)}) \tag{54}$$

which are shown in Figure 1. The degree of mirror symmetry is roughly 70% and 1205 of the 202,492 mirror pairs are Hodge theoretically self-mirror, i.e. $h^{(3,1)} = h^{(1,1)}$.

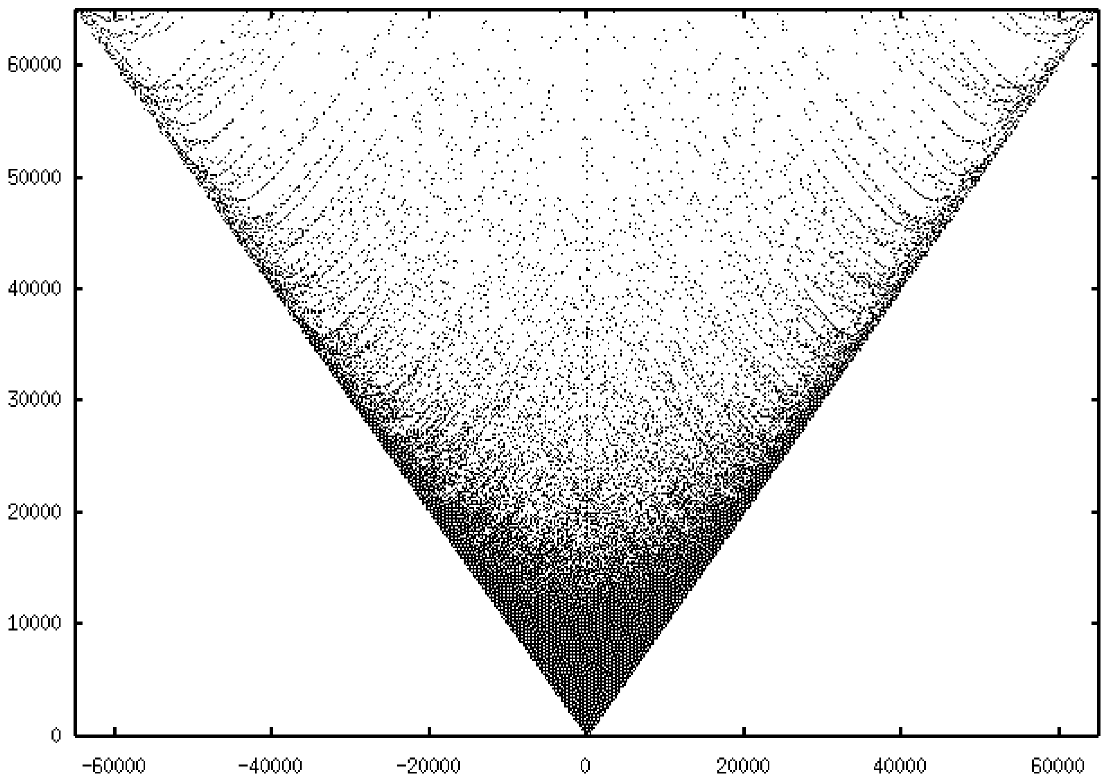

Figure 3: *A zoom of the plot of Figure 1.*

[2]We are grateful to Werner Nahm for discussions on these matters.

Similarly to the case of threefold mirror symmetry we can ask whether potential mirror pairs can be related via fractional transformations. As expected this is indeed the case.

The essential ingredient of the fractional transformation mirror construction [18, 22] is the basic isomorphism

$$\begin{aligned} &\mathbb{C}_{\left(\frac{b}{g_{ab}},\frac{a}{g_{ab}}\right)}\left[\frac{ab}{g_{ab}}\right] \ni \left\{z_1^a + z_2^b = 0\right\} \Big/ \mathbb{Z}_b : [\,(b-1) \quad 1\,] \\ \sim \; &\mathbb{C}_{\left(\frac{b^2}{h_{ab}},\frac{a(b-1)-b}{h_{ab}}\right)}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \left\{y_1^{a(b-1)/b} + y_1 y_2^b = 0\right\} \Big/ \mathbb{Z}_{b-1} : [\,1 \quad (b-2)\,] \end{aligned} \tag{55}$$

induced by the fractional transformations

$$\begin{aligned} z_1 &= y_1^{1-\frac{1}{b}}, & y_1 &= z_1^{\frac{b}{b-1}} \\ z_2 &= y_1^{\frac{1}{b}} y_2, & y_2 &= \frac{z_2}{z_1^{\frac{1}{b-1}}} \end{aligned} \tag{56}$$

in the path integral of the theory. Here g_{ab} is the greatest common divisor of a and b and h_{ab} is the greatest common divisor of b^2 and $(ab-a-b)$. The action of a cyclic group $\mathbb{Z}_b$ of order b denoted by $[m \;\; n]$ indicates that the symmetry acts like $(z_1, z_2) \mapsto (\alpha^m z_1, \alpha^n z_2)$ where α is the b^{th} root of unity[3].

As an example we consider the simplest fourfold hypersurface configuration $\mathbb{P}_5[6]$ described by polynomials of degree six in ordinary projective 5-space. As a first step we consider the action of the cyclic group of order six defined by

$$\mathbb{Z}_6 \ni \alpha : \quad (z_1, z_2, z_3, z_4, z_5, z_6) \mapsto (\alpha^5 z_1, \alpha z_2, z_3, z_4, z_5, z_6), \tag{57}$$

where α is the sixth root of unity. Applying the above isomorphism we see that the weighted hypersurface transform of the orbifold $\mathbb{P}_5[6]\big/\mathbb{Z}_6 : [5\ 1\ 0\ 0\ 0\ 0]$ is given by

$$\mathbb{P}_{(6,4,5,5,5,5)}[30] = \text{Fractional Transform}\left(\mathbb{P}_5[6]\big/\mathbb{Z}_6 : \;[5\ 1\ 0\ 0\ 0\ 0]\right). \tag{58}$$

In principle we have to implement an orbifolding also on the space on the lhs. But since this $\mathbb{Z}_5$ is part of the projective equivalence of the lhs configuration the relation (58) in fact holds.

Applying fractional transformations iteratively to the sextic fourfold $\mathbb{P}_5[6]$ with $(h^{(3,1)}, h^{(2,1)}, h^{(1,1)}, h^{(2,2)}) = (426, 0, 1, 1752)$ leads to the results in Table 2.

[3] A detailed discussion of the rational structure of the twisted states implied by the construction of [18] can be found in [22].

Cover Space: $\mathbb{P}_5[6]$			
Group	Action	FT Image	$(h^{(3,1)}, h^{(2,1)}, h^{(1,1)}, h^{(2,2)})$
$\mathbb{Z}_6$:	$[5\ 1\ 0\ 0\ 0\ 0]$	$\mathbb{P}_{(6,4,5,5,5,5)}[30]$	(164,0,6,724)
$\mathbb{Z}_6^2$:	$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \end{bmatrix}$	$\mathbb{P}_{(30,24,21,25,25,25)}[150]$	(30,101,30,82)
$\mathbb{Z}_6^3$:	$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \end{bmatrix}$	$\mathbb{P}_{(150,120,126,104,125,125)}[750]$	(6,0,164,724)
$\mathbb{Z}_6^3$:	$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \end{bmatrix}$	$\mathbb{P}_{(750,600,630,624,521,625)}[3750]$	(1,0,426,1752)

Table 2: *Fractional transforms of a number of group actions on the sextic fourfold.*

From this we recognize that the last space is indeed the mirror of the sextic hypersurface and that also the first space and the third in the table are mirrors of each other, whereas the second entry is Hodge self-mirror.

6 Transversality of Catastrophes

The most explicit way of constructing a Landau–Ginzburg vacuum is, of course, to exhibit a specific potential that satisfies all the conditions imposed by the requirement that it ought to describe a consistent ground state of the string. Knowledge of the explicit form of the potential of a LG theory is very useful information when it comes to the detailed analysis of such a model. It is however not necessary if only limited knowledge, such as the computation of the spectrum of the theory, is required. In fact the only ingredients necessary for the computation of the spectrum of a LG vacuum [28] are the anomalous dimensions of the scaling fields as well as the fact that in a configuration of weights there exists a polynomial of appropriate degree with an isolated singularity. However, it is much easier to check whether there exists such a polynomial in a configuration than to actually construct such a potential. The reason is a theorem by Bertini which asserts that if a polynomial does have an isolated singularity on the base locus then, even though this potential may have worse singularities away from the base locus, there exists a deformation of the original polynomial that only admits an isolated singularity anywhere. Hence we only have to find criteria that guarantee at worst an isolated singularity on the base locus. It is precisely this problem that was addressed in the mathematics literature [35] at the same time as the explicit construction of LG vacua was started in ref. [23]. For the sake of completeness we briefly review the main point of the argument of ref. [35].

Suppose we wish to check whether a polynomial in n variables z_i with weights k_i has an isolated singularity, i.e. whether the condition

$$dp = \sum_i \frac{\partial p}{\partial z_i} dz_i = 0 \tag{59}$$

can be solved at the origin $z_1 = \cdots = z_n = 0$. According to Bertini's theorem, the singularities of a general element in $\mathbb{C}_{(k_1,\ldots,k_n)}[d]$ will lie on the base locus, i.e., the intersection of the hypersurface and all the components of the base locus, described by coordinate planes of dimension $k = 1, \ldots, n$. Let $\mathcal{P}_k$ be such a k-plane, which we may assume to be described by setting the coordinates $z_{k+1} = \cdots = z_n$ to zero. Expand the polynomials in terms of the non-vanishing coordinates $z_1, \ldots, z_k$

$$p(z_1, \ldots, z_n) = q_0(z_1, \ldots, z_k) \ + \sum_{j=k+1}^{n} q_j(z_1, \ldots,_k) z_j + h.o.t. \tag{60}$$

Clearly, if $q_0 \neq 0$ then $\mathcal{P}_k$ is not part of the base locus and hence the hypersurface is transverse. If on the other hand $q_j = 0$, then $\mathcal{P}_k$ is part of the base locus and singular points can occur on the intersection of the hypersurfaces defined by $\mathcal{H}_j = \{q_j = 0\}$. If, however, we can arrange this intersection to be empty, then the potential is smooth on the base locus.

Thus we have found that the conditions for transversality in any number of variables is the existence for any index set $\mathcal{I}_k = \{1, \ldots, k\}$ of

1. either a monomial $z_1^{a_1} \cdots z_k^{a_k}$ of degree d
2. or of at least k monomials $z_1^{a_1} \cdots z_k^{a_k} z_{e_i}$ with distinct e_i.

Assume on the other hand that neither of these conditions holds for all index sets and let $\mathcal{I}_k$ be the subset for which they fail. Then the potential has the form

$$p(z_1, \ldots, z_n) = \sum_{j=k+1}^{n} q_j(z_1, \ldots, z_k) z_j \ + \ \cdots \tag{61}$$

with at most $k-1$ non-vanishing q_j. In this case the intersection of the hypersurfaces $\mathcal{H}_j$ will be positive and hence the polynomial p will not be transverse.

As an example for the considerable ease with which one can check whether a given configuration allows for the existence of a potential with an isolated singularity, consider the polynomial of Orlik and Randall

$$p = z_1^3 + z_1 z_2^3 + z_1 z_3^5 + z_4^{45} + z_2^2 z_3^4 z_4. \tag{62}$$

Condition (59) is equivalent to the system of equations

$$\begin{array}{rclrcl} 0 & = & 3z_1^2 + z_2^3 + z_3^5, & 0 & = & 3z_1 z_2^2 + 2z_2 z_3^4 z_4 \\ 0 & = & 5z_1 z_3^4 + 4z_2^2 z_3^3 z_4, & 0 & = & z_2^2 z_3^4 + 45 z_4^{44}. \end{array} \tag{63}$$

which, on the base locus, collapses to the trivial pair of equations $z_2z_3 = 0 = z_2^3 + z_5^5$. Hence this configuration allows for such a polynomial. To check the system away from the base locus clearly is much more complicated.

By adding two variables of combined weight 13 it is possible to define a Calabi–Yau deformation class $\mathbb{P}_{(1,5,6,8,10,13,15)}$[45].

7 Finiteness Considerations

As in the case of threefolds [24, 25] the problem of finiteness has two aspects: first one has to find a constraint on the number of scaling fields that can appear in the LG theory and then one has to determine limits on the exponents with which the variables occur in the superpotential. Both of these constraints follow from the fact that the central charge of a Landau–Ginzburg theory with fields of charge q_i

$$c = 3\sum_{i=1}^{r}(1-2q_i) =: \sum_{i=1}^{r} c_i \tag{64}$$

has to be $c = 12$ in order to describe a string, F-theory, or M-theory vacuum of the type relevant for the structure of the 4D dualities of interest. The following considerations follow closely the discussion in [24], adapted suitably to fourfolds.

It follows from the above considerations that we have to assume, in order to avoid redundant reconstructions of LG theories, that the central charge c_i of all scaling fields of the potential should be positive. In order to relate the potentials to manifolds, we may then add one or several trivial factors or more complicated theories with zero central charge.

Using the above results we will derive more detailed finiteness conditions on the number of fields and the size of the exponents in the superpotential.

To get a lower bound on the number of scaling fields, observe that from (64) written as

$$\sum_{i=1}^{r} q_i = \frac{r}{2} - \frac{c}{6} := \hat{c}^{(1)} \tag{65}$$

we obtain $r > \frac{c}{3}$ using the positivity of the charges.

Now let p be a polynomial of degree d in r variables. For the one-element index set $\mathcal{I}_1$ the conditions (1.) and (2.) for transversality imply the existence of integers $n_i \geq 2$, $1 \leq i \leq r$ and of a map $\sigma : \mathcal{I}_r \rightarrow \mathcal{I}_r$ such that for all i one has

$$n_i q_i + q_{\sigma(i)} = 1, \tag{66}$$

where $\sigma(i) = i$ if condition (1.) and $\sigma(i) \neq i$ if condition (2.) holds, respectively.

Let us now see how many nontrivial fields can occur at most. Fields which have charge $q_i \leq \frac{1}{3}$ contribute $c_i \geq 1$ to the conformal anomaly. Next, consider fields with larger charge. Since we assume $c_i > 0$, their charges are in the range $\frac{1}{3} < q_i < \frac{1}{2}$. It seems that these fields may cause a problem because $c_i \to 0$ as $q_i \to \frac{1}{2}$ which would a priori allow infinitely many fields. However, among these fields the transversality condition (1.) cannot hold, because two of them are not enough and three of them are too many fields in order to form a monomial of charge one. Transversality condition (2.) implies that a field z_i among them has to occur together with a partner field $z_{\sigma(i)}$. These pairs contribute $c_i + c_{\sigma(i)} > 2$ to the conformal anomaly according to (64) and (66), so we can conclude that $r \leq c$.

In order to construct all transverse LG potentials for a given total central charge c, we choose a specific r in the range obtained above and consider all possible maps σ of which there are r^r. Without restriction on the generality, we may assume the n_i to be ordered: $n_1 \leq \cdots \leq n_r$. Starting with (65) we obtain via (66) and the positivity of the charges a bound $n_1 < \frac{r}{\hat{c}^{(1)}}$.

Now we choose n_1 in the allowed range and use (66) in order to eliminate q_1 in favour of the q_i, $i > 1$. This yields an equation of the general form

$$\sum_{i=p}^{r} w_i^{(p)} q_i = \hat{c}^{(p)}. \tag{67}$$

In this step we have $p = 2$, in (65) we had $p = 1$ and $w_i^{(p)} = 1$. If $\hat{c}^{(p)} \neq 0$, (67) allows us to derive a finite bound N_p for n_p:

$$n_p < \frac{1}{\hat{c}^{(p)}} \sum_{i \in \mathcal{I}_\pm} w_i^{(p)} =: N_p, \tag{68}$$

where $\mathcal{I}_\pm$ are the indices of the positive/negative $w_i^{(p)}$; the choice depends on the sign of $\hat{c}^{(p)}$. If $N_p < n_{p-1}$ we increment n_{p-1} as long as it does not hit its bound and so on.

What to do in the case $\hat{c}^{(p)} = 0$? If the $w_i^{(p)}$ are indefinite we get no bound from (67). However, we will see that the existence of monomials for certain index sets, which are required by the transversality conditions, implies a bound for n_p. Let $\mathcal{I}_a$ denote the indices of the already bounded n_i and $\mathcal{I}_b$ the others. How can indefinite w's arise? If there is a chain of indices $a_0 = a, a_1 = \sigma(a), \ldots, a_l = \sigma^{l-1}(a) =: b(a)$ linked by the map σ, the charge of the field z_a with $a \in \mathcal{I}_a$ will depend on the unknown charge of a field $z_{b(a)}$ with $b \in \mathcal{I}_b$. The charge of z_a is then given by

$$q_a = \frac{1}{n_a} - \frac{1}{n_a n_{a_1}} + \cdots - \frac{(-1)^l}{n_a \ldots n_{a_{l-1}}} + \frac{(-1)^l}{n_a \ldots n_{a_{l-1}}}\, q_{b(a)}. \tag{69}$$

Indefiniteness of the $w_i^{(p)}$ can only occur if there are fields z_a, $a \in \mathcal{I}_a$, with odd l, i.e. the coefficient of $q_{b(a)}$ is negative. Call the index set of these fields $\mathcal{I}^*$. Assume first that the transversality condition (1.) holds. This implies the existence of positive integers m_i such that $\sum_{i \in \mathcal{I}^*} m_i q_i = 1$. From (65) and the positivity of the charges it follows that $m_i < 2n_i$. For the unknown q_i, $i \in \mathcal{I}_b$, we get an equation of the form $\sum_{i\in\mathcal{I}_b} v_i q_i = \varepsilon$, which yields a bound for n_k with $k = \min_{a\in\mathcal{I}^*} b(a)$, since all $v_i = v_{b(a)} = \frac{m_a}{n_a \ldots n_{a_{l-1}}}$ are positive. The lowest possible positive value ε_{min} for

$$\varepsilon = -1 + \sum_{a\in\mathcal{I}^*} m_a \left(\frac{1}{n_a} - \ldots - \frac{(-1)^l}{n_a \ldots n_{a_{l-1}}} \right). \tag{70}$$

can be obtained by minimizing ε with respect to the set $\{m_i\}$ leading to the bound

$$n_i < \frac{2}{\varepsilon_{\text{min}}} \sum_{a\in\mathcal{I}^*} \frac{1}{n_{a_1} \ldots n_{a_{l-1}}}. \tag{71}$$

If there is no such ε_{min} we have to increment n_{p-1}.

If transversality condition (2.) applies, we have $|\mathcal{I}^*|$ equations of the form $\sum_{i\in\mathcal{I}^*} m_i^{(j)} q_i = 1 - q_{e_j}$ which can be rewritten as $\sum_{i\in\mathcal{I}_b} v_i^{(j)} q_i = \varepsilon^{(j)}$. Deriving a bound is similar to the case discussed above. Only if all $v_i^{(j)}$ happen to be indefinite and all $\varepsilon^{(j)}$ are zero we get from this condition. In [24] it has been argued that this cannot occur, again due to the positivity of the charges.

Finally, the q_i are obtained from the n_i by solving the upper triangular linear equation system (67). The weights k_i and the degree d are then given by $q_i = \frac{k_i}{d}$ with minimal denominators and numerators.

This procedure of restricting the bound for n_p, given $n_1, \ldots, n_{p-1}$, for each map σ was implemented in a C Code. It allows all configurations to be found without testing unnecessarily many combinations of the n_i.

In the five and six-variable case we have found 360,346 and 739,709 transverse configurations, respectively. By adding a trivial mass term z_6^2 in the five-variable case, the configurations mentioned so far lead to four-dimensional Calabi–Yau manifolds described as hypersurfaces in a five-dimensional weighted projective space by a one-polynomial constraint. The list of all these examples, including the Hodge numbers, can be found on the web [36]. The complete computer run, carried out on a cluster of 100 MHz DEC Alpha machines, required a total amount of CPU time of the order of ten years. In order to avoid integer overflow the computation is performed most easily by using 64-bit arithmetics. The computation of the Hodge numbers on the other hand needed only a few months of CPU time on an ordinary Pentium PC. Only for the models with the highest degrees did we use an Ultra Sparc multi processor machine.

The algorithm can be directly extended to the cases with more variables, i.e. Calabi–Yau manifolds with codimension>1. When dealing with n variables, however, one needs to consider n^n maps σ. This number of maps grows much faster than the size of integer weights of these higher codimension configurations decreases. Therefore such a run would need even more computer power than the five/six variable case and is at present beyond the means of the resources available to us. In order to get some estimate of the number of expected models we did an exploratory run which we stopped after 30 years of CPU time. A brute force run would take the degree d, start with the minimal value and compute all partitions into n numbers of kd (k denotes the codimension). For each partition we check whether the corresponding model is transverse. Then we increment d and so on. Of course, this algorithm will fail when d reaches higher values but the generated number of models is large enough in order to estimate the total number of configurations. Such a brute force run was performed by Kreuzer and Skarke [10] generating approx. 500,000 models with codimension one with $d \leq 4000$. During our run for codimension one we could estimate the total number of configuration with the help of their data with the expected result of more than one million models, very close to our actual result. We therefore expect that the following scaling argument holds: Suppose we find N transverse configurations with a brute force run and M of them are contained in the results of our algorithm which has computed K configurations, but only with a small part of all maps σ. Then there should exist approximately $K\,\frac{N}{M}$ transverse configurations.

From this estimate we expect about 800,000 models of codimension two, about 32,500 models of codimension three and about 40 models of codimension four, respectively. Our run has found 88% of them. To actually perform this computation will require much larger resources than the ones we have at our disposal.

Concerning the Hodge numbers, the models with higher codimensions give rise to 98,402=17% additional spectra. Applying the scaling argument we expect 112,000 additional spectra, i.e. a total of about 700,000.

Finally, we mention that the degree of mirror symmetry does not change significantly when the higher codimension models are included. About 28% of the spectra do not have a mirror partner, so the degree of mirror symmetry is about 72%. A plot of the resulting spectra is very similar to the codimension one case, hence we do not present it in this paper.

8 Results

We have constructed 1,100,055 Landau–Ginzburg theories at $c = 12$, where 360,346 configuration exhibit a trivial quadratic factor and the remaining 739,709 correspond to more general conformal field theories. This class of models leads to 667,954 different spectra, i.e. different Hodge triplets which determine the complete Hodge diamond $(h^{(1,1)}, h^{(2,1)}, h^{(3,1)}, h^{(2,2)})$ of the geometric phase. The configuration with the maximal degree is given by

$$\mathbb{P}_{(1806,151662,931638,2173882,3260733)}[6521466], \tag{72}$$

with Hodge numbers ($h^{(2,2)} = 1,213,644$, $h^{(3,1)} = 252$, $h^{(2,1)} = 0$, $h^{(1,1)} = 303,148$). Only 204 of the more than one million models lead to a negative Euler number. The 24 different negative Euler numbers that occur are collected in Table 3.

−6	−12	−18	−24	−30	−36
−42	−48	−60	−66	−72	−84
−90	−96	−120	−132	−138	−144
−168	−180	−192	−198	−240	−252

Table 3: *The negative Euler numbers of hypersurface fourfolds in weighted* $\mathbb{P}_5$.

More details on the spectrum of the individual models can be obtained from the websites [36] where we list all the configurations and their cohomology groups. As mentioned in the cohomology section above not all four cohomology groups are independent because of the relation (37). Any triplet of cohomology dimensions thus captures the complete information. In Figure 4 we present a threedimensional plot where we suppress the cohomology group $\mathrm{H}^{(2,2)}$.

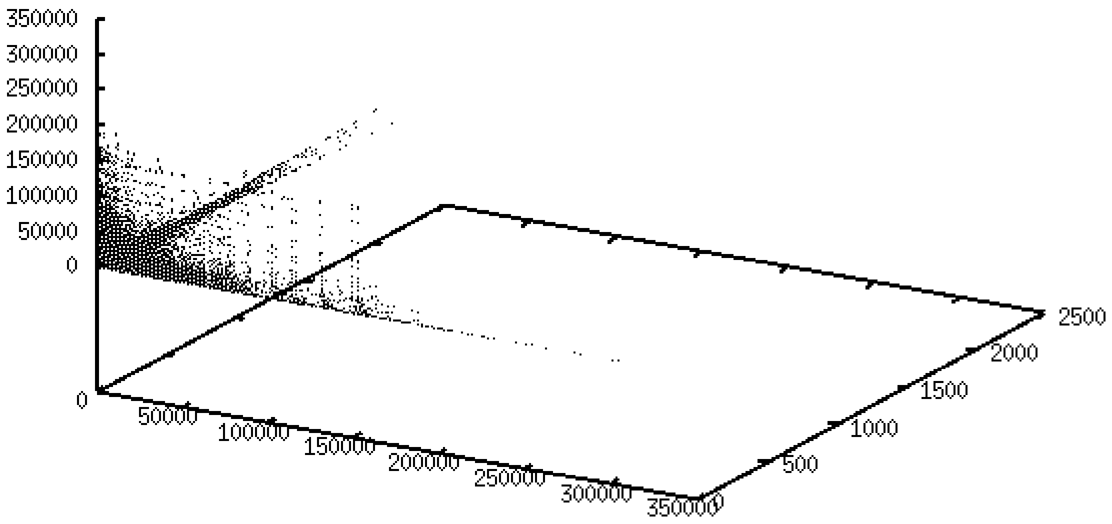

Figure 4: *A plot of three independent cohomology dimensions* $h^{(1,1)}, h^{(2,1)}, h^{(3,1)}$.

The massless spectrum is very rough information about these models and as in the case of threefolds Hodge isomorphic theories will differ in their couplings. As in the case of threefolds there is some redundancy in our list of spaces. Consider e.g. the two models

$$\mathbb{P}_{(2,2,2,2,1,9)}[18] \ni \{z_1^9 + \cdots + z_4^9 + z_5^{18} + z_6^2 = 0\} \tag{73}$$

and

$$\mathbb{P}_{(1,1,1,1,1,4)}[9] \ni \{y_1^9 + \cdots + y_4^9 + y_5^9 + y_5 y_6^2 = 0\}. \tag{74}$$

Using the fractional transform we see that these two theories are in fact identical, mapped into each other via an orbifolding which is part of the projective equivalence.

In the fourfold mirror plot of Figure 1 a remarkable parabolic structure appears in the upper regions of the plot. In the lower region of Figure 1 this feature is obscured by the high density of points. The close-up view of the mirror plot in Figure 3 shows that this feature persists even for manifolds with smaller cohomology groups. In Figure 5 we show a close-up view of such curves which suggests that they do indeed describe parabolas (with a regression coefficient $r > 0.999$).

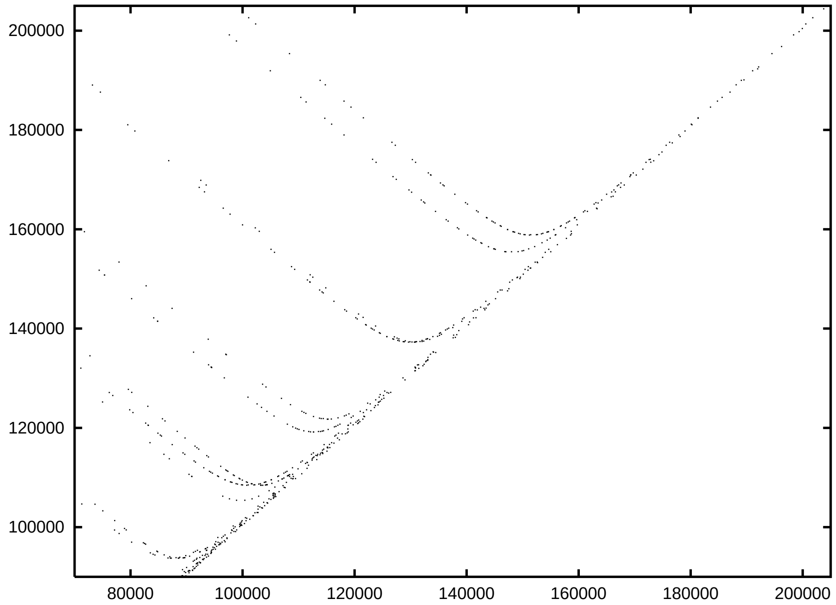

Figure 5: *A close-up view of one of the parabola-like curves.*

A further zoom in Figure 6 shows that the parabolic structure is not perfect.

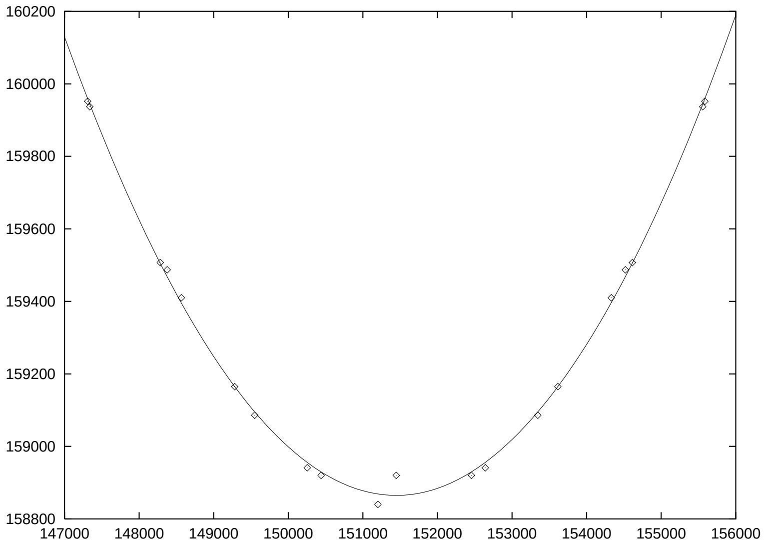

Figure 6: *A close-up view of one of the parabola-like curves.*

Besides the rigorous relation (37) one could ask for further relations among the Hodge numbers which may only hold approximately. Indeed, if we plot $h^{(2,2)}$ vs. $(h^{(3,1)}+h^{(1,1)})$ like displayed in Figure

7, we get the linear relation

$$h^{(2,2)} \approx 4.00037\,(h^{(3,1)} + h^{(1,1)}) - 7.6 \tag{75}$$

with surprisingly good accuracy, i.e. the regression coefficient is $r = 0.999992$.

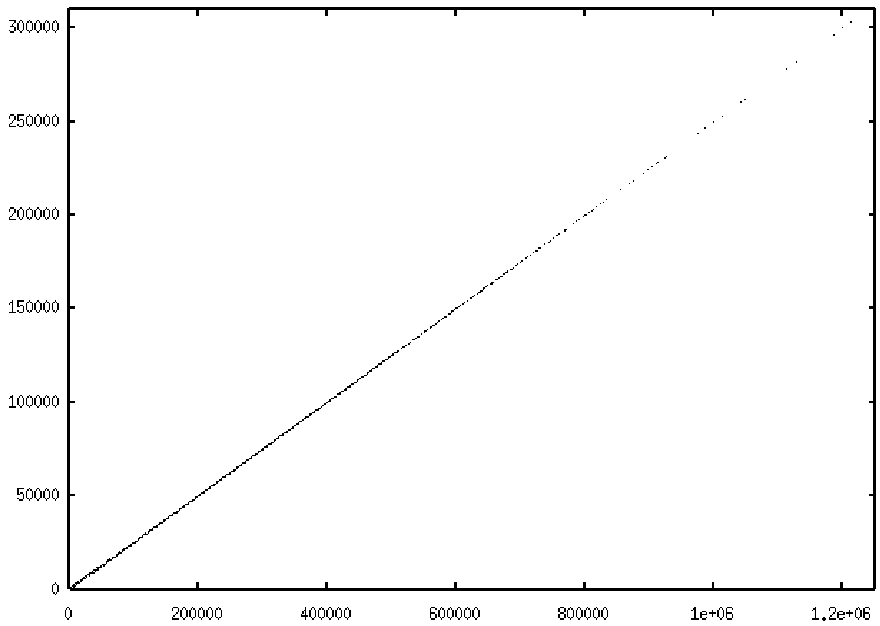

Figure 7: *Plot of $h^{(2,2)}$ vs. $(h^{(3,1)}+h^{(1,1)})$ for the class of Calabi–Yau fourfold hypersurfaces in weighted $\mathbb{P}_5$.*

However, (75) does not hold exactly. To illustrate this, we show a plot of $(4.00037\,(h^{(3,1)}+h^{(1,1)}) - 7.6 - h^{(2,2)})$ vs. $\log(h^{(2,2)})$ in Figure 8.

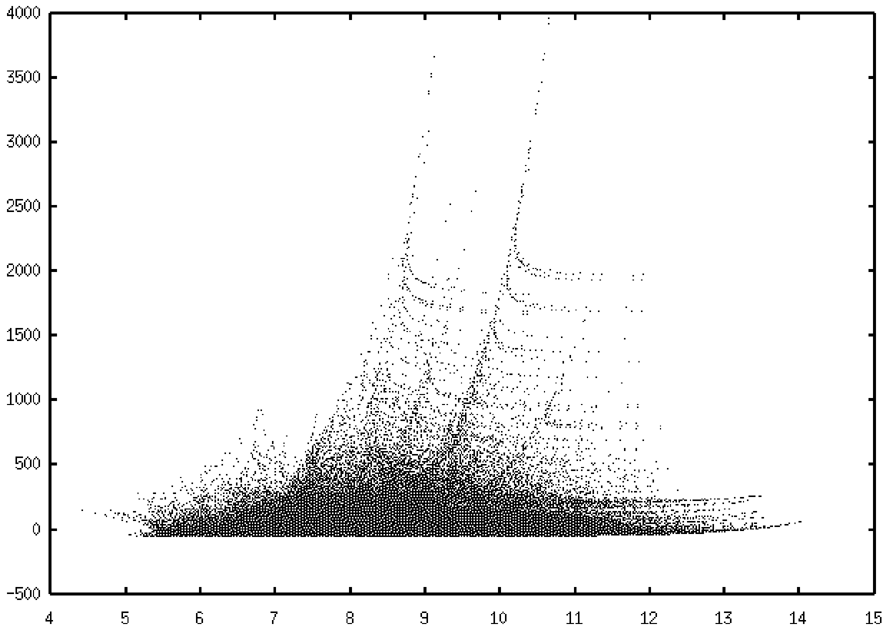

Figure 8: *Plot of* $(4.00037\,(h^{(3,1)} + h^{(1,1)}) - 7.6 - h^{(2,2)})$ *vs. log*$(h^{(2,2)})$ *for the class of Calabi–Yau fourfold hypersurfaces in weighted* $\mathbb{P}_5$.

9 Fibration Structure

In the context of the recent discussions on duality between F-theory, M-theory and heterotic string ground states manifolds that are fibered have become of particular interest [37]. Some simple sufficient criteria for the existence of fibrations have been formulated in [38]. The idea is to check whether the $(n-2)$-dimensional subvariety CY_{n-2} defined by a family of divisors on a Calabi–Yau $(n-1)$-fold $\mathrm{CY}_{n-1} = \mathbb{P}_{(\tilde{k}_1,\ldots,\tilde{k}_{n+1})}[\tilde{d}]$ as

$$\mathcal{D}_i = \{z_i = p_i(z_{j\neq i})\} \cap \mathrm{CY}_n \tag{76}$$

is itself a Calabi–Yau space. In order to do this a number of conditions have to be satisfied. If we focus on families of the type (76) we need to be able to partition the weight k_i, leading to the condition (i) $\tilde{k}_i = \sum_{j\neq i} b_j \tilde{k}_j$.

Furthermore we have to normalize the $(n-1)$-hypersurface appropriately such that after deleting one variable the remaining ones do not have a nontrivial common divisor. To this effect we denote the n remaining weights by

$$(k_1, \ldots, k_n) = (\tilde{k}_1, .., \hat{\tilde{k}}_i, ..., \tilde{k}_{n+1}). \tag{77}$$

The divisor thus defines a hypersurface in $\mathbb{P}_{(k_1,\ldots,k_n)}[d]$ with $d = \tilde{d}$ the original degree. It is this configuration for which it can happen that after deleting a variable the remaining weights *do* have a common denominator, in which case we would not get a useful configuration. The normalized weight vector which maps this into a proper configuration is defined as follows: for each i consider the remaining weights and denote by ρ_i the greatest common divisor of these weights

$$\rho_i := \mathrm{gcd}(k_1, \ldots, \hat{k}_i, \ldots, k_n). \tag{78}$$

Given these ρ_i we can consider the transformation

$$\mathbb{P}_{(k_1,\ldots,k_n)}[d] \longrightarrow \mathbb{P}_{(\hat{k}_1,\ldots,\hat{k}_n)}[\hat{d}] \tag{79}$$

defined by the map

$$(z_1, ..., z_n) \mapsto (x_1, ..., x_n) := (z_1^{\rho_1}, ..., z_n^{\rho_n}). \tag{80}$$

with the weights $\hat{k}_i = \rho_i k_i$. This maps the the hypersurface defined by the polynomial

$$p = \sum_{(a_1,\ldots,a_n), i_j \in \mathbb{N}} \alpha_{a_1,\ldots,a_n} z_1^{a_1} \cdots z_n^{a_n}. \tag{81}$$

where $k_j \cdot a_j = d$ into the hypersurface

$$\bar{p} = \sum_{(\bar{a}_1,\ldots,\bar{a}_n)} \bar{\alpha}_{\bar{a}_1,\ldots,\bar{a}_n} x_1^{\bar{a}_1} \cdots x_n^{\bar{a}_n} \tag{82}$$

with $\bar{k}_i \cdot \bar{a}_i = \bar{d}$. Hence we need the condition (ii) $\rho_i | a_i$.

Now, in general the weights $\hat{k}_i$, $i = 1, ..., n$ will have a common divisor and in order to obtain a well-defined configuration we have to divide all weights by this divisor. To do so consider the set of ρ_i's and define

$$\delta_i := \mathrm{lcm}(\rho_1, \ldots, \hat{\rho}_i, \ldots, \rho_n). \tag{83}$$

Given these, one defines the normalized weight vector of the final configuration $\mathbb{P}_{(\bar{k}_1,\ldots,\bar{k}_n)}[\bar{d}]$ by setting $\bar{k}_i = \frac{k_i}{\delta_i}$ and the normalized degree by $\bar{d} = \frac{d}{\kappa}$ with $\kappa = \rho_i \delta_i$ for any i.

With this one can write the original divisor in the threefold whose normalization defines the fiber as

$$\sum_{\bar{A}} \bar{\alpha}_{\bar{a}_1,\ldots,\bar{a}_n} z_1^{\rho_1 \bar{a}_1} \cdots z_n^{\rho_n \bar{a}_n}. \tag{84}$$

Since this is a hypersurface of degree d we are interested in a vector $\bar{A}$ such that

$$\sum_i \rho_i \bar{a}_i k_i = d. \tag{85}$$

Now, *if* the fiber is a Calabi–Yau configuration then we have a monomial with $\bar{a}_i = 1, \forall i$ and hence we have the condition (iii) $\sum_{i=1}^n k_i \rho_i = d$.

Finally, we can check the transversality of the fiber configuration by comparing it to the known lists of weighted elliptic curves, weighted K3 hypersurfaces [35] and weighted Calabi–Yau threefold hypersurfaces [24, 25].

By implementing these criteria we can search for a number of different fibration types among the fourfolds. These criteria are sufficient but not necessary and hence will not yield an exhaustive list. They do however provide a wealth of examplest. The simplest type examples for which one can analyze problems in F/Heterotic duality are fourfolds which have an iterative fibration structure in which an elliptic fourfold is not only a K3-fibration as well but also is CY_3-fibered. The iterative fibrations of such manifolds show a nested structure which can be summarized in the diagram:

$$\begin{array}{ccccccc} T^2 & \longrightarrow & K3 & \longrightarrow & CY_3 & \longrightarrow & CY_4 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{P}_1 & & \mathbb{P}_1 & & \mathbb{P}_1. \end{array} \tag{86}$$

A search for fibrations for which the generic smooth fiber is a Calabi–Yau threefolds leads to a list 35,540 examples. Restricting further to such fibrations which are also K3 fibrations leads to 8,270 examples. These criteria lead to 4,305 examples of this kind which are also elliptic.

For F-theory the threefold fibration structure is not necessary and we can consider more general fourfolds which are either K3 fibered or elliptic. The above criteria lead to a list of 49,751 elliptic fibrations the Hodges combinations $(h^{(1,1)} \pm h^{(3,1)})$ of which we display in Figure 9.

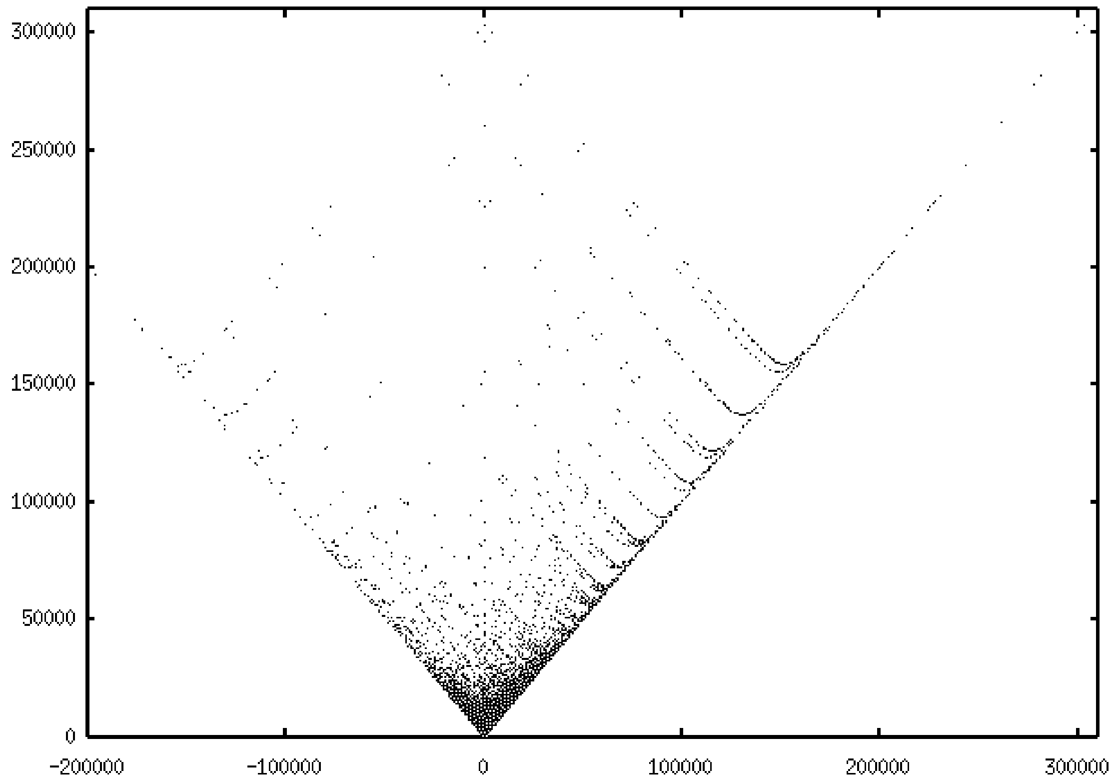

Figure 9: *Plot of* $(h^{(3,1)}+h^{(1,1)})$ *vs.* $(h^{(3,1)}-h^{(1,1)})$ *for 49,751 elliptic Calabi–Yau fourfold hypersurfaces in weighted* $\mathbb{P}_5$.

Of these elliptic fibrations we find that 13,285 pass the test of being also K3 fibrations. Those spaces we display in Figure 10.

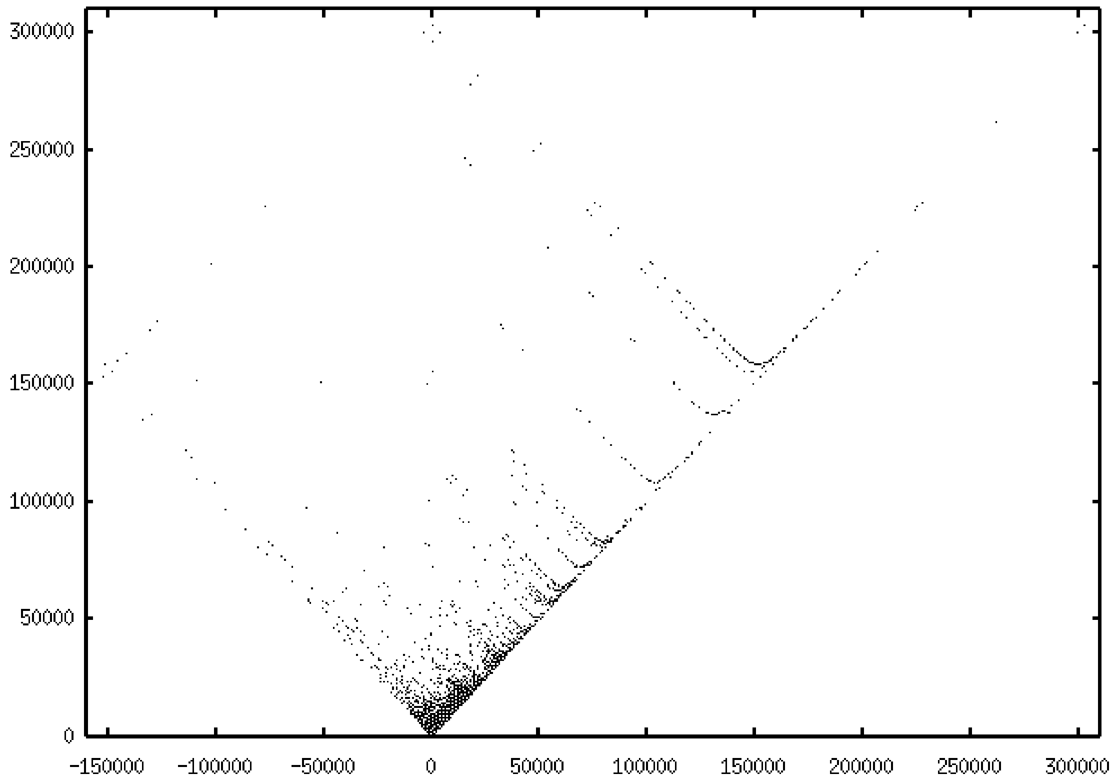

Figure 10: *Plot of* $(h^{(3,1)}+h^{(1,1)})$ *vs.* $(h^{(3,1)}-h^{(1,1)})$ *for 13,285 elliptic K3-fibered Calabi–Yau fourfold hypersurfaces in weighted* $\mathbb{P}_5$.

We finally present a plot of the 27099 fourfolds in our list of Calabi–Yau fourfolds which are K3-fibrations in Figure 11.

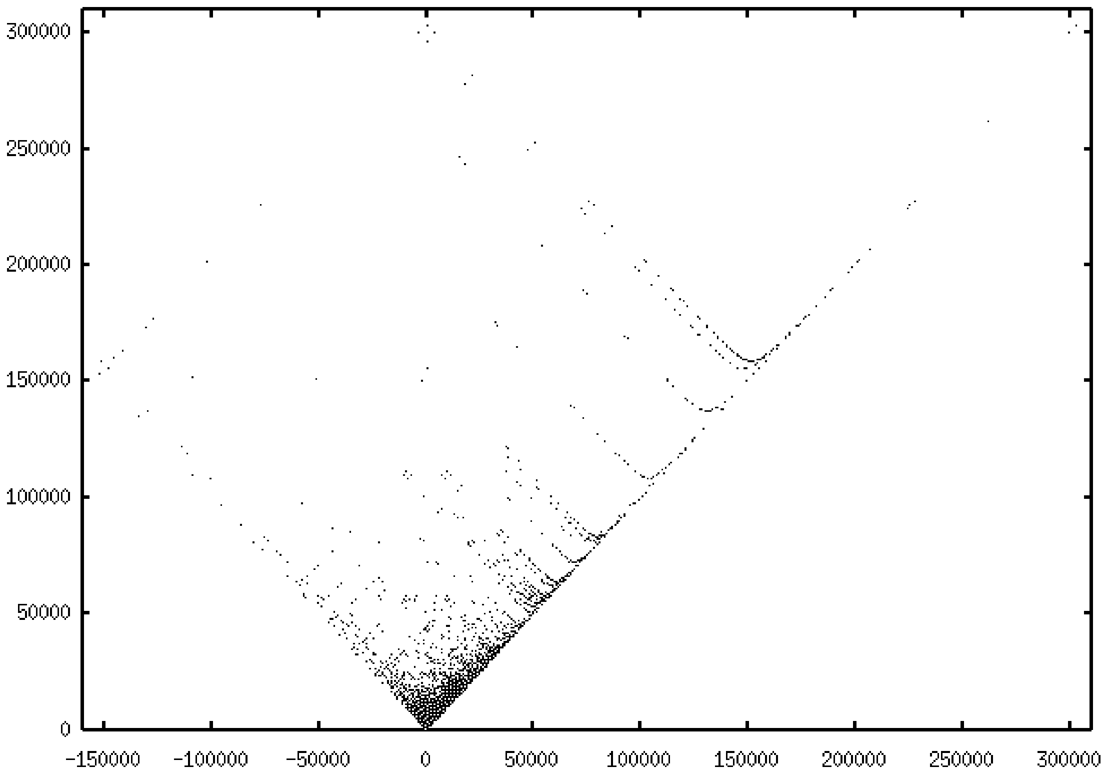

Figure 11: *Plot of* $(h^{(3,1)} + h^{(1,1)})$ *vs.* $(h^{(3,1)} - h^{(1,1)})$ *for 27,099 K3-fibered Calabi–Yau fourfold hypersurfaces in weighted* $\mathbb{P}_5$.

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