A Hiker's Guide to K3.
Aspects of $N = (4,4)$ Superconformal Field Theory with Central Charge $c = 6$

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Abstract: We study the moduli space $\mathcal{M}$ of $N = (4,4)$ superconformal field theories with central charge $c = 6$. After a slight emendation of its global description we find the locations of various known models in the component of $\mathcal{M}$ associated to $K3$ surfaces. Among them are the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold theories obtained from the torus component of $\mathcal{M}$. Here, $SO(4,4)$ triality is found to play a dominant role. We obtain the B-field values in direction of the exceptional divisors which arise from orbifolding. We prove T-duality for the $\mathbb{Z}_2$ orbifolds and use it to derive the form of $\mathcal{M}$ purely within conformal field theory. For the Gepner model $(2)^4$ and some of its orbifolds we find the locations in $\mathcal{M}$ and prove isomorphisms to nonlinear σ models. In particular we prove that the Gepner model $(2)^4$ has a geometric interpretation with Fermat quartic target space.

This paper aims to make a contribution to a better understanding of the $N = (4,4)$ superconformal field theories with left and right central charge $c = 6$. Ultimately, one would like to know their moduli space $\mathcal{M}$ as an algebraic space, their partition functions as functions on $\mathcal{M}$ and modular functions on the upper half plane, and an algorithm for the calculation of all operator product coefficients, depending again on $\mathcal{M}$. This would constitute a good basis for the understanding of quantum supergravity in six dimensions, and presumably for an investigation of the more complicated physics in four dimensions.

The moduli space $\mathcal{M}$ has been identified with a high degree of plausibility, though a number of details remain to be clarified. It has two components, $\mathcal{M}^{tori}$ and $\mathcal{M}^{K3}$, one 16–dimensional associated to the four–torus and one 80–dimensional associated to $K3$. The superconformal field theories in $\mathcal{M}^{tori}$ are well understood. One also understands some varieties of theories which belong to $\mathcal{M}^{K3}$, including about 30 isolated Gepner type models and varieties which contain orbifolds of theories in $\mathcal{M}^{tori}$. In the literature one can find statements concerning intersections of these subvarieties, but not all of them are correct. Indeed, their precise positions in $\mathcal{M}$ had not been studied up to now. One difficulty is due to the fact that the standard description of $\mathcal{M}^{tori}$ is based on the odd cohomology of the torus, which does not survive the orbifolding.

As varieties of superconformal theories $\mathcal{M}^{tori}$ and $\mathcal{M}^{K3}$ cannot intersect for trivial reasons. As ordinary conformal theories without $\mathbb{Z}_2$ grading intersections are possible and will be shown to occur.

The plan of our paper is as follows. In section 1 we will review known results following [A-M, As2]. We correct some of the details and add proofs for well–known conjectural features. In section 1.1 we explain the connection between our description of $\mathcal{M}^{tori}$ in terms of the even cohomology and the one given by Narain much earlier by odd cohomology [C-E-N-T, Na]. Both are eight–dimensional, and they are related by $SO(4,4)$ triality. Section 2 deals with $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold conformal field theories. We arrive at a description for the subvarieties of these theories within $\mathcal{M}^{K3}$. In particular, we present a proof for the well–known conjecture that orbifold conformal field theories tend to give the value $B=\frac{1}{2}$ [As2, §4] to the B-field in direction of the exceptional divisors gained from the orbifold procedure and determine the correct B-field values for $\mathbb{Z}_4$ orbifolds. Our results are in agreement with those of [Do, B-I], that were obtained in a different context. We calculate the conjugate of torus T-duality under the $\mathbb{Z}_2$ orbifolding map to $\mathcal{M}^{K3}$ and find that it is a kind of squareroot of the Fourier–Mukai T-duality on $K3$. This yields a proof of the latter and allows us to determine the form of $\mathcal{M}^{K3}$ purely within conformal field theory, without having recourse to Landau-Ginzburg arguments. We disprove the conjecture that $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold moduli spaces meet in the Gepner model $(2)^4$ [E-O-T-Y]. We show that the $\mathbb{Z}_4$ orbifold of the nonlinear σ model on the torus with lattice $\Lambda=\mathbb{Z}^4$ has a geometric interpretation on the Fermat quartic hypersurface.

Section 3 is devoted to the study of special points with higher discrete symmetry groups in the moduli space, namely Gepner models (actually $(2)^4$ and some of its orbifolds by phase symmetries). We stress that our approach is different from the one advocated in [F-K-S-S, F-K-S] where massless spectra and symmetries of all Gepner models and their orbifolds were matched to those of algebraic manifolds corresponding to these models. The correspondence there was understood in terms of Landau-Ginzburg models, a limit which we do not make use of at all. We instead explicitly prove equivalence of the Gepner models under investigation to nonlinear σ models. This also enables us to give the precise location of the respective models within the moduli space $\mathcal{M}^{K3}$. We prove that the Gepner model $(2)^4$ is isomorphic to the $\mathbb{Z}_4$ orbifold and therefore to the Fermat quartic model studied in the previous section. We moreover find two meeting points of $\mathcal{M}^{K3}$ and $\mathcal{M}^{tori}$ generalizing earlier results for bosonic theories [K-S] to the corresponding $N=(4,4)$ supersymmetric models. We find a meeting point of the moduli spaces of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold conformal field theories different from the one conjectured in [E-O-T-Y]. In section 4 we conclude by gathering the results and joining them to a panoramic view of part of the moduli space (figure 4.1).

In the context of σ models we must fix our α' conventions. For ease of notation we use the rather unusual $\alpha'=1$, so T-duality for a bosonic string compactified on a circle of radius R reads $R\mapsto\frac{1}{R}$. We hoped to save us a lot of factors of $\sqrt{2}$ this way.

Often, the left–right transformed analogue of some statement will not be mentioned explicitly, in order to avoid tedious repetitions. Fourier components of holomorphic fields are labelled by the energy, not by its negative.

1. The moduli space of $N=(4,4)$ superconformal field theories with central charge $c=6$

We consider unitary two dimensional superconformal quantum field theories. They can be described as Minkowskian theories on the circle or equivalently as euclidean theories on tori with parameter τ in the upper complex halfplane. The worldsheet coordinates are called σ_0, σ_1.
The space of states $\mathcal{H}$ of a quantum field theory has a real structure given by CPT. For any $N=(4,4)$ superconformal theory $\mathcal{H}$ contains four–dimensional vector spaces Q_l and Q_r of real left and right supercharges. Since we consider left and right central charge $c=6$, we use the extension of the $N=(2,2)$ superconformal algebra by an $su(2)\oplus su(2)$ current algebra of level 1 [A-B-D+]. The (3+3)-dimensional Lie group generated by the corresponding charges will be denoted by $SU(2)_l^{susy}\times SU(2)_r^{susy}$ and its $\{(\mathbb{1},\mathbb{1}),(-\mathbb{1},-\mathbb{1})\}$ quotient by $SO(4)^{susy}$. The commutant of $SU(2)_l^{susy}$ in $SO(Q_l)$ will be called $SU(2)_l$. Here and in the following we use the notation $SO(W)$ for the special orthogonal group of a real vector space W with given scalar product.
One can identify $SU(2)_l^{susy}$ with $SU(2)_l$ by selecting one vector in Q_l. The subgroup of $SO(Q_l)$ which fixes this vector is an $SO(3)$ group with surjective projections to the two $SU(2)$ groups modulo their centers and allows an identification of the images. Such an identification seems to be implicit in many discussions in the literature, but will not be used in this section.
We will consider canonical subspaces of $\mathcal{H}$ spanned by the states with specified conformal dimensions $(h;\overline{h})$ which belong to some irreducible representation of $SU(2)_l^{susy}\times SU(2)_r^{susy}$. The latter are labelled by the charges $(Q;\overline{Q})$ with respect to a Cartan torus of $SU(2)_l^{susy}\times SU(2)_r^{susy}$. Since any two Cartan tori are related by a conjugation, the spectrum does not depend on the choice of this torus. Charges are normalized to integral values, as has become conventional in the context of extended supersymmetry.
We assume the existence of a quartet of spectral flow fields with $(h,Q;\overline{h},\overline{Q}) = (\frac{1}{4},\varepsilon_1;\frac{1}{4},\varepsilon_2), \varepsilon_i\in\{\pm 1\}$. Operator products with each of them yield a combined left+right spectral flow. Instead of using $N=(4,4)$ supersymmetry it suffices to start with $N=(2,2)$ and this quartet. Indeed, the operator product of a pair of quartet fields yields lefthanded flow operators with $(h,Q;\overline{h},\overline{Q})=(1,\pm 2;0,0)$, and analogously on the righthanded side for another pair. These enhance the $u(1)_l^{susy}\oplus u(1)_r^{susy}$ subalgebra of the $N=(2,2)$ superconformal algebra to an $A_1^{(1)}\times A_1^{(1)}$ Kac-Moody algebra. Thus the $N=(2,2)$ superconformal algebra is enhanced to $N=(4,4)$ [E-O-T-Y].
Our assumptions are natural in the context of superstring compactification. There, unbroken extended spacetime supersymmetry is obtained from $N=(2,2)$ worldsheet supersymmetry with spectral flow operators [Se1, Se2]. Thus our superconformal theories may be used as a background for $N=4$ supergravity in six dimensions. Here, however, we concentrate on the internal conformal field theory. External degrees of freedom are not taken into account.

Let us give a brief summary on what is known about the moduli space $\mathcal{M}$ so far. The spaces of states of the conformal theories form a bundle with local grading by finite dimensional subbundles over $\mathcal{M}$. They can be decomposed into irreducible representations of the left and right $N=4$ supersymmetries. The irreducible representations are determined by their lowest weight values of (h,Q). These representations can be deformed continuously with respect to the value of h, except for the representations of non-zero Witten index, also called *massless representations* [E-T1, E-T2, Ta]. Apart from the vacuum representation with $(h,Q)=(0,0)$, the lowest weight states of massless representations are labelled by $(h,Q)=(\frac{1}{2},\pm 1)$ in the Neveu-Schwarz sector and by $(h,Q)=(\frac{1}{4},\pm 1)$ or $(h,Q)=(\frac{1}{4},0)$ in the Ramond sector. Let us enumerate the representations which are massless with respect to both the left and the right handed side. Apart from the vacuum we already mentioned the spectral flow operators with $(h,Q;\overline{h},\overline{Q})=(\frac{1}{4},\varepsilon_1;\frac{1}{4},\varepsilon_2), \varepsilon_i\in\{\pm 1\}$. They form a vector multiplet under $SO(4)^{susy}$. Since the vacuum is unique, there is exactly one multiplet of such fields. On the other hand, the dimension of the vector space of real $(\frac{1}{4},0;\frac{1}{4},0)$ fields is not fixed a priori. We shall denote it by $4+\delta$. With a slight abuse of notation, the orthogonal group of this vector space will be called $O(4+\delta)$. These are all the possibilities of massless representations in the Ramond sector. The corresponding ground state fields describe the entire cohomology of Landau-Ginzburg or σ model descriptions of our theories [L-V-W].

If in a given model there is a field with $(h,Q;\overline{h},\overline{Q})=(\frac{1}{2},\pm 1;0,0)$, application of $su(2)_l$ and supersymmetry operators yields four lefthanded Majorana fermions and the corresponding abelian currents. As we shall see below, this suffices to show that the model has an interpretation as nonlinear σ model on a torus, with the currents as generators of translation and the fermions as parallel sections of a flat spin bundle. Such models have $\delta=0$ and constitute the component $\mathcal{M}^{tori}$ of $\mathcal{M}$.

The vector space $\mathcal{F}_{1/2}$ spanned by the fields with $(h,Q;\overline{h},\overline{Q})=(\frac{1}{2},\varepsilon_1;\frac{1}{2},\varepsilon_2), \varepsilon_i\in\{\pm 1\}$ is obtained from the $(\frac{1}{4},0;\frac{1}{4},0)$ Ramond fields by spectral flow. Thus it gives an irreducible $4(4+\delta)$–dimensional representation of $su(2)_l^{susy}\oplus su(2)_r^{susy}\oplus o(4+\delta)$. It determines the supersymmetric deformations of the theory, as will be considered below.

The massless representations cannot be deformed, so δ is constant over the generic points of a connected component of $\mathcal{M}$ and can only increase over non-generic ones. Tensor products of a massive lefthanded representation with a righthanded massless representation cannot be deformed either, since $h-\overline{h}$ must remain intergral. The span of such tensor products in the space of states yields a string theoretic generalization $\mathcal{E}$ of the elliptic genus [S-W1, S-W2], which is constant for all theories within a connected component of $\mathcal{M}$. Since for $c=6$ and theories with merely integer charges $\mathcal{E}$ is a theta function of level 2 and characteristic q^{-1}, by its properties under modular transformations one can show that $\mathcal{E}$ is a multiple of the elliptic genus $\mathcal{E}_{K3}$ of a $K3$ surface. According to their charges, the numbers of $(\frac{1}{4},\frac{1}{4})$ fields can be arranged into a Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & n_l & & n_r & \\ 1 & & 4+\delta & & 1 \\ & n_r & & n_l & \\ & & 1 & & \end{array}$$

where by the above $n_l \in \{0,2\}$ also yields the number of left handed Dirac fermions. The uniqueness of the left and right elliptic genera shows $n_l = n_r$ and $\delta = 16 - 8n_l$. Moreover, left and righthanded elliptic genera have the same power series expression. They vanish over $\mathcal{M}^{tori}$. In particular, as was anticipated above, the existence of one field with $(h, Q; \overline{h}, \overline{Q}) = (\frac{1}{2}, \pm 1; 0, 0)$ suffices to show that the theory is toroidal. The elliptic genus on $\mathcal{M}$ is interpreted as index of a supercharge acting on the loop space of $K3$ [Wi1,Wi2]. We call one of our conformal field theories *associated to torus or* $K3$, depending on the elliptic genus. For the theories associated to $K3$ one has $\delta = 16$.
To understand the local structure of the moduli space $\mathcal{M}$ we must determine the tangent space $\mathcal{H}_1$ in a given point of $\mathcal{M}$, i.e. describe the deformation moduli of a given theory. This space consists of real fields of dimensions $h = \overline{h} = 1$ in the space of states $\mathcal{H}$ over the chosen point. The Zamolodchikov metric [Za] on the space of such fields establishes on $\mathcal{M}$ the structure of a Riemannian manifold, with holonomy group contained in $O(\mathcal{H}_1)$. To preserve the supersymmetry algebra, $\mathcal{H}_1$ must consist of $SO(4)^{susy}$ invariant fields in the image of $\mathcal{F}_{1/2}$ under $(Q_l)_{1/2} \otimes (Q_r)_{1/2}$, where the latter subscripts denote Fourier components. Accordingly, $\mathcal{F}_{1/2} \oplus \mathcal{H}_1$ yields a well–known representation of the $osp(2,2)$ superalgebra spanned by $(Q_l)_{\pm 1/2}$, $su(2)_l^{susy}$ and the Virasoro operator L_0. In particular, $\mathcal{H}_1$ should be $4(4+\delta)$–dimensional and form an irreducible representation of $su(2)_l \oplus su(2)_r \oplus o(4+\delta)$. We shall assume that all elements of $\mathcal{H}_1$ really give integrable deformations, as has been shown to all orders in perturbation theory [Di]. Note, however, that there is no complete proof yet.
The holonomy group of $\mathcal{M}$ projects to an $O(4+\delta)$ action on the uncharged massless Ramond representations and to an $SO(4)$ action on $Q_l \otimes Q_r$. Thus its Lie algebra is contained in $su(2)_l \oplus su(2)_r \oplus o(4+\delta)$. The two Lie algebras are equal for $\mathcal{M}^{tori}$ and one expects the same for $\delta = 16$. Below we shall find an isometry from $\mathcal{M}^{tori}$ to a subvariety of $\mathcal{M}^{K3}$, such that the holonomy Lie algebra of the latter space is at least $su(2)_l \oplus su(2)_r \oplus so(4)$. Moreover, this isometry shows that $\mathcal{M}^{K3}$ is not compact. Since one has the inclusion

$$su(2) \oplus su(2) \oplus o(4+\delta) \cong sp(1) \oplus sp(1) \oplus o(4+\delta) \hookrightarrow sp(1) \oplus sp(4+\delta),$$

the moduli space of $N = (4,4)$ superconformal field theories with $c = 6$ associated to torus or $K3$ is a quaternionic Kähler manifold of real dimension $4(4+\delta)$. To determine its local structure, recall that we are looking for a noncompact space. By Berger's classification of quaternionic Kähler manifolds [Be] it can only be reducible or quaternionic symmetric [Si, Th. 9]. Because non–Ricci flat quaternionic Kähler manifolds are (even locally) de Rham irreducible [Wo], this means that it can only be Ricci flat or quaternionic symmetric. The former is excluded because geodesic submanifolds on which all holomorphic sectional curvatures are negative and bounded away from zero have been found [P-S, C-F-G, Ce1]. Hence the moduli space must locally be the Wolf space

$$\begin{aligned} \mathcal{T}^{4,4+\delta} &= O^+(4, 4+\delta; \mathbb{R}) / SO(4) \times O(4+\delta) \\ &\cong SO^+(4, 4+\delta; \mathbb{R}) / SO(4) \times SO(4+\delta), \end{aligned} \tag{1.1}$$

i.e. one component of the Grassmannian of oriented spacelike four–planes $x \subset \mathbb{R}^{4,4+\delta}$ [Ce2], reproducing Narain's and Seiberg's previous results [C-E-N-T, Na, Sei]. Here $SO^+(W)$ denotes the identity component of the special orthogonal

group $SO(W)$ of a vector space W with given scalar product. The space of maximal positive definite subspaces of W has two components, and $O^+(W)$ denotes the subgroup of elements of $O(W)$ which do not interchange these components. Note that for positive definite W we have $SO(W) = O^+(W)$. The Zamolodchikov metric on $\mathcal{T}^{4,4+\delta}$ is the group invariant one.
From the preceeding discussion, x can be interpreted as the $SO(4)^{susy}$ invariant part of the tensor product of $Q_l \otimes Q_r$ with the four-dimensional space of charged Ramond ground states. Note that the action of $so(4) = su(2)_l \oplus su(2)_r$ discussed above generates orthogonal transformations of the four–plane $x \in \mathcal{T}^{4,4+\delta}$ corresponding to the theory under inspection, whereas $o(4+\delta)$ acts on its orthogonal complement.
We repeatedly used the splitting $so(4) = su(2)_l \oplus su(2)_r$. Consider the antisymmetric product $\Lambda^2 x$ of the above four–plane x. We choose the orientation of x such that $su(2)_l$ fixes the anti–selfdual part $(\Lambda^2 x)^-$ of $\Lambda^2 x$ with respect to the group invariant metric on $O^+(4, 4+\delta; \mathbb{R})$. When the theory has a parity operation which interchanges Q_l and Q_r, this induces a change of orientation of x. The choice of an $N = (2,2)$ subalgebra within the $N = (4,4)$ superconformal algebra corresponds to the selection of a Cartan torus $u(1)_l \oplus u(1)_r$ of $su(2)_l \oplus su(2)_r$. This induces the choice of an oriented two–plane in x. The rotations of x in this two–plane are generated by $u(1)_{l+r}$, those perpendicular to the plane by $u(1)_{l-r}$. Thus the moduli space of $N = (2,2)$ superconformal field theories with central charge $c = 6$ is given by a Grassmann bundle over $\mathcal{M}$, with fibre $SO(4)/(SO(2)_{l+r} \times SO(2)_{l-r}) \cong \mathbb{S}^2 \times \mathbb{S}^2$.
Generic examples for our conformal theories are the nonlinear σ models with the oriented four–torus or the $K3$ surface as target space X. In the $K3$ case, the existence of these quantum field theories has not been proven yet, but their conformal dimensions and operator product coefficients have a well defined perturbation theory in terms of inverse powers of the volume. We tacitly make the assumption that a rigorous treatment is possible and warn the reader that many of our statements depend on this assumption.
A nonlinear σ model on X assigns an action to any twocycle on X. This action is the sum of the area of the cycle for a given Ricci flat metric plus the image of the cycle under a cohomology element $B \in H^2(X,\mathbb{R})$. Since integer shifts of the action are irrelevant, the physically relevant B-field is the projection of B to $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$. Thus the parameter space of nonlinear σ models has the form $\{\mathit{Ricci\ flat\ metrics}\} \times \{B - \mathit{fields}\}$. The corresponding Teichmüller space is

$$\mathcal{T}^{3,3+\delta} \times \mathbb{R}^+ \times H^2(X,\mathbb{R}). \tag{1.2}$$

Its elements will be denoted by (Σ, V, B). The first factor of the product is the Teichmüller space of Ricci flat metrics of volume 1 on X, the second parametrizes the volume, and the last one represents the B-field. The Zamolodchikov metric gives a warped product structure to this space. Worldsheet parity transformations $(\sigma_0, \sigma_1) \mapsto (-\sigma_0, \sigma_1)$ change the sign of the cycles, or equivalently the sign of B, which yields an automorphism of the parameter space.
Target space parity for $B = 0$ yields a specific worldsheet parity transformation and thus an identification of $su(2)_l$ with $su(2)_r$. The corresponding diagonal Lie algebra $su(2)_{l+r}$ generates an $SO(3)$ subgroup of $SO(4)$. Under the action of this subgroup x decomposes into a line and its orthogonal three–plane $\Sigma \subset x$. The $\mathbb{S}^2 \times \mathbb{S}^2$ bundle over $\mathcal{M}$ now has a diagonal $\mathbb{S}^2$ subbundle. Each point in the fibre

corresponds to the choice of an $SO(2)$ subgroup of $SO(3)$ or a subalgebra $u(1)_{l+r}$ of $su(2)_{l+r}$. Geometrically this yields a complex structure in the target space. Thus the $\mathbb{S}^2$ bundle over the $B=0$ subspace of $\mathcal{M}$ is the bundle of complex structures over the moduli space of Ricci flat metrics on the target space.
Recall some basic facts about the Teichmüller space $\mathcal{T}^{3,3+\delta}$ of Einstein metrics on an oriented four–torus or $K3$ surface X. We consider the vector space $H^2(X,\mathbb{R})$ together with its intersection product, such that $H^2(X,\mathbb{R}) \cong \mathbb{R}^{3,3+\delta}$. In other words, positive definite subspaces have at most dimension three, negative definite ones at most dimension $3+\delta$. On $K3$ this choice of sign determines a canonical orientation. When one wants to study $\mathcal{M}^{tori}$ by itself, the choice of a torus orientation is superfluous. Our main interest, however, is the study of torus orbifolds. For a canonical blow–up of the resulting singularities one needs an orientation. The effect of a change of orientation on the torus will be considered below.
Metric and orientation on X define a Hodge star operator, which on $H^2(X,\mathbb{R})$ has eigenvalues +1 and -1. The corresponding eigenspaces of dimensions three and $3+\delta$ are positive and negative definite, respectively. Let $\Sigma \subset H^2(X,\mathbb{R})$ be the positive definite three–plane obtained in this way. The orientation on X induces an orientation on Σ. One can show that Ricci flat metrics are locally uniquely specified by Σ, apart from a scale factor given by the volume. Since the Hodge star operator in the middle dimension does not change under a rescaling of the metric, the volume V must be specified separately. It follows that $\mathcal{T}^{3,3+\delta}\times\mathbb{R}^+$ is the Teichmüller space of Einstein metrics on X. Explicitly, we have

$$\mathcal{T}^{3,3+\delta} = O^+(H^2(X,\mathbb{R}))/SO(3)\times O(3+\delta). \tag{1.3}$$

The $SO(3)$ group in the denominator is to be interpreted as $SO(\Sigma_0)$ for some positive definite reference three–plane in $H^2(X,\mathbb{R})$, while $O(3+\delta)$ is the corresponding group for the orthogonal complement of Σ_0. Equivalently, $\mathcal{T}^{3,3+\delta}$ could have been written as $SO^+(H^2(X,\mathbb{R}))/SO(3)\times SO(3+\delta)$. We choose the description (1.3) for later convenience in the construction of the entire moduli space.
For higher dimensional Calabi-Yau spaces the σ model description works only for large volume due to instanton corrections. In our case, however, the metric on the moduli space does not receive corrections [N-S]. Therefore the Teichmüller space (1.2) of σ models on X should be a covering of a component of $\mathcal{M}$, thus isomorphic to the Teichmüller space $\mathcal{T}^{4,4+\delta}$ obtained in (1.1). Indeed, for $\delta=16$ a natural isomorphism

$$\mathcal{T}^{4,4+\delta} \cong \mathcal{T}^{3,3+\delta}\times\mathbb{R}^+\times H^2(X,\mathbb{R}) \tag{1.4}$$

was given in [A-M, As2], with a correction and clarification by [R-W, Di]. The same construction actually works for $\delta=0$, too. It uses the identification

$$\mathcal{T}^{4,4+\delta} = O^+(H^{even}(X,\mathbb{R}))/SO(4)\times O(4+\delta),$$

where $SO(4)$ is to be interpreted as $SO(x_0)$ for some positive definite reference four–plane in $H^{even}(X,\mathbb{R})$, while $O(4+\delta)$ is the corresponding group for the orthogonal complement of x_0. In other words, the elements of $\mathcal{T}^{4,4+\delta}$ are interpreted as positive definite oriented four–planes $x \subset H^{even}(X,\mathbb{R})$ by $H^{even}(X,\mathbb{R}) \cong \mathbb{R}^{4,4+\delta}$. Note that all the cohomology of $K3$ is even, whereas $H^{odd}(X,\mathbb{R}) \cong \mathbb{R}^{4,4}$ when X is a four–torus.

To explicitly realize the isomorphism (1.4) one also needs the positive generators υ of $H^4(X,\mathbb{Z})$ and υ^0 of $H^0(X,\mathbb{Z})$, which are Poincaré dual to points and to the whole oriented cycle X, respectively. They are nullvectors in $H^{even}(X,\mathbb{R})$ and satisfy $\langle \upsilon, \upsilon^0\rangle = 1$. Thus over $\mathbb{Z}$ they span an even, unimodular lattice isomorphic to the standard hyperbolic lattice U with bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now consider a triple (Σ, V, B) in the right hand side of (1.4). Define

$$\begin{aligned} &\xi : \Sigma \to H^{even}(X,\mathbb{R}), \quad \xi(\sigma) := \sigma - \langle B, \sigma\rangle \upsilon, \\ &x := \operatorname{span}_{\mathbb{R}} \left(\xi(\Sigma),\ \xi_4 := \upsilon^0 + B + \left(V - \tfrac{\|B\|^2}{2} \right) \upsilon \right). \end{aligned} \tag{1.5}$$

Then $\widetilde{\Sigma} = \xi(\Sigma)$ is a positive definite oriented three–plane in $H^{even}(X,\mathbb{R})$, and the vector ξ_4 is orthogonal to $\widetilde{\Sigma}$. Since $\|\xi_4\|^2 = 2V$, it has positive square. Together, $\widetilde{\Sigma}$ and ξ_4 span an oriented four–plane $x \subset H^{even}(X,\mathbb{R})$. Obviously, the map $(\Sigma, V, B) \mapsto x$ is invertible, once υ and υ^0 are given.
To describe the projection from Teichmüller space to $\mathcal{M}$ we need to consider the lattices $H^2(X,\mathbb{Z})$ and $H^{even}(X,\mathbb{Z})$. They are even, unimodular, and have signature $(p, p+\delta)$ with $p=3$ and $p=4$, respectively. Such lattices are isometric to $\Gamma^{p,p+\delta} = U^p \oplus (E_8(-1))^{\delta/8}$. Here each summand is a free $\mathbb{Z}$ module, E_8 has as bilinear form the Cartan matrix of E_8, and for any lattice Γ we denote by $\Gamma(n)$ the same $\mathbb{Z}$ module Γ with quadratic form scaled by n.
We now consider the projection from Teichmüller space to $\mathcal{M}$. First we have to identify all points in $\mathcal{T}^{3,3+\delta}$ which yield the same Ricci flat metric. This means that we have to quotient the Teichmüller space (1.3) by the so–called *classical symmetries*. The projection is given by

$$O^+(H^2(X,\mathbb{Z}))\backslash \mathcal{T}^{3,3+\delta} \tag{1.6}$$

[K-T]. Here we use the notation $O^+(\Gamma)$ for the intersection of $O^+(W)$ with the automorphism group of a lattice $\Gamma \subset W$. The interpretation of the quotient space (1.6) as moduli space of Einstein metrics of volume 1 on X is straightforward in the torus case, but for $X = K3$ one has to include orbifold limits (see section 2). The corresponding σ models are not expected to exist for all values of B [Wi3]. To simplify the discussion we include such *conifold points* in $\mathcal{M}$. On $\mathcal{T}^{4,4+\delta}$ the group of classical symmetries lifts by (1.5) to the subgroup of $O^+(H^{even}(X,\mathbb{Z}))$ which fixes both lattice vectors υ and υ^0.
Next we consider the shifts of B by elements $\lambda \in H^2(X,\mathbb{Z})$, which neither change the physical content. One easily calculates that this also yields a left action on $\mathcal{T}^{4,4+\delta}$ by a lattice automorphism in $O^+(H^{even}(X,\mathbb{Z}))$, generated by $w \mapsto w - \langle w, \lambda\rangle \upsilon$ for $\langle w, \upsilon\rangle = 0$ and $\upsilon^0 \mapsto \upsilon^0 + \lambda - \frac{\|\lambda\|^2}{2}\upsilon$. These transformations fix υ and shift υ^0 to arbitrary nullvectors dual to υ. Thus the choice of υ^0 is physically irrelevant.
We shall argue that the projection from Teichmüller space to $\mathcal{M}$ is given by

$$\mathcal{T}^{4,4+\delta} \longrightarrow O^+(H^{even}(X,\mathbb{Z}))\backslash \mathcal{T}^{4,4+\delta}. \tag{1.7}$$

The group $O^+(H^{even}(X,\mathbb{Z}))$ acts transitively on pairs of primitive lattice vectors of equal length [L-P,Ni3]. Thus (1.7) would imply that different choices of υ, υ^0 are equivalent. Anticipating this result in general, we call the choice of an arbitrary primitive nullvector $\upsilon \in H^{even}(X,\mathbb{Z})$ a *geometric interpretation* of a positive oriented four–plane $x \subset H^{even}(X,\mathbb{Z})$. Such a choice yields a family of σ models with physically equivalent data (Σ, V, B). A conformal field theory has various different geometric interpretations, and the choice of υ is comparable to a choice of a chart of $\mathcal{M}$.

Aspinwall and Morrison also identify theories which are related by the worldsheet parity transformation [A-M]. We regard the latter as a symmetry of $\mathcal{M}$. It is given by change of orientation of the four–plane x or equivalently by a conjugation of $O^+(H^{even}(X,\mathbb{R}))$ with an element of $O(H^{even}(X,\mathbb{R}))-O^+(H^{even}(X,\mathbb{R}))$ which transforms the lattice $H^{even}(X,\mathbb{Z})$ and the reference four–plane x_0 into themselves. To stay in the classical context, one may choose an element which fixes υ and υ^0. More canonically, parity corresponds to $(\upsilon,\upsilon^0) \mapsto (-\upsilon,-\upsilon^0)$. The latter induces $\xi_4 \mapsto -\xi_4$ and $(\Sigma, V, B) \mapsto (\Sigma, V, -B)$.

Let us consider the general pattern of identifications. When two points in Teichmüller space are identified the same is true for their tangent spaces. Higher derivatives can be treated by perturbation theory in terms of tensor products of the tangent spaces $\mathcal{H}_1$. Assuming the convergence of the perturbation expansion in conformal field theory, any such isomorphism can be transported to all points of $\mathcal{T}^{4,4+\delta}$. Therefore σ model isomorphisms are given by the action of a group $\mathcal{G}^{(\delta)}$ on this space. In the previous considerations we have found a subgroup of $\mathcal{G}^{(\delta)}$.

Below we shall prove that the interchange of υ and υ^0, which is the Fourier-Mukai transform [R-W], also belongs to $\mathcal{G}^{(\delta)}$. When $B=0$, this yields the map $(\Sigma, V, 0) \mapsto (\Sigma, V^{-1}, 0)$. In the torus case, it is known as T-duality and it seems natural to extend this name to $X = K3$. We will not use the name mirror symmetry for this transformation.

It is obvious that classical symmetries, integral B-field shifts, and T-duality generate all of $O^+(H^{even}(X,\mathbb{Z}))$. Thus $\mathcal{G}^{(\delta)}$ contains all of this group. As argued in [A-M,As2], it cannot be larger, since otherwise the quotient of $\mathcal{T}^{4,4+\delta}$ by $\mathcal{G}^{(\delta)}$ plus the parity automorphism would not be Hausdorff [Al]. For a proof of the Hausdorff property of $\mathcal{M}$ one will need some features of the superconformal field theories, which should be easy to verify once they are somewhat better understood. First, one has to check that all fields are generated by the iterated operator products of a finite dimensional subspace of basic fields. Next one has to show that the operator product coefficients are determined in terms of a finite number of basic coefficients, and that the latter are constrained by algebraic equations only. This would show that $\mathcal{M}$ is an algebraic space. In particular, every point has a neighborhood which contains no isomorphic point. All of these features are true in the known examples of conformal field theories with finite effective central charge, in particular for the unitary theories. They certainly should be true in our case.

In the context of σ models it often is useful to choose a complex structure on X. When such a structure is given, the real and imaginary parts of any generator of $H^{2,0}(X,\mathbb{C})$ span an oriented two-plane $\Omega \subset \Sigma$. Conversely, any such subspace Ω defines a complex structure. This means that the choice of an Einstein metric is nothing but the choice of an $\mathbb{S}^2$ of complex structures on

X, in other words a *hyperkähler structure*. In terms of cohomology, Ω specifies $H^{2,0}(X,\mathbb{C}) \oplus H^{0,2}(X,\mathbb{C})$. The orthogonal complement of Ω in $H^2(X,\mathbb{R})$ yields $H^{1,1}(X,\mathbb{R})$. Any vector $\omega \in H^{1,1}(X,\mathbb{R})$ of positive norm yields a Kähler class compatible with the complex structure and the hyperkähler structure Σ spanned by Ω and ω.

Since $H^2(X,\mathbb{Z})$ is torsionfree for tori and $K3$ surfaces, the Néron-Severi group $NS(X)$ can be identified with $Pic(X) := H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{R})$, the *Picard lattice* of X. By a result of Kodaira's, X is algebraic, if $NS(X)$ contains an element ρ of positive length squared [Ko]. Given a hyperkähler structure Σ we can always find $\Omega \subset \Sigma$ such that X becomes an algebraic surface. It suffices to choose ω as the projection of ρ on Σ and Ω as the corresponding orthogonal complement. The projection is non-vanishing, since the orthogonal complement of Σ in $H^2(X,\mathbb{R})$ is negative definite. Varying ρ one obtains a countable infinity of algebraic structures on X. Thus the occasionally encountered interpretation of moduli of conformal field theories as corresponding to nonalgebraic deformations of $K3$ surfaces does not make sense (this was already pointed out in [Ce2] by different arguments).

The choice of $\Omega \subset \Sigma$ lifts to a corresponding choice of a two-plane $\widetilde{\Omega} \subset x$. As discussed above this selects a $(2,2)$ subalgebra of the $(4,4)$ superalgebra. We will refer to the choice of such a two–plane as fixing a complex structure. More precisely, the two–plane $\widetilde{\Omega}$ specifies a complex structure *in every geometric interpretation* of the conformal field theory.

1.1. Moduli space of theories associated to tori. Originally, Narain determined the moduli space $\mathcal{M}^{tori}$ of superconformal field theories associated to tori by explicit construction of nonlinear σ models [C-E-N-T,Na]. With the above formalism we can reproduce his description as follows.

Let us consider tori of arbitrary dimension d. We change the notation by transposing the group elements, which exchanges left and right group actions. This yields

$$\mathcal{M}^{Narain} = O(d) \times O(d) \backslash O(d,d) / O(\Gamma^{d,d}).$$

This moduli space has a symmetry given by worldsheet parity. We shall see that its action on $O(d,d)$ exchanges the two $O(d)$ factors. For later convenience we are going to use the cover $SO(d) \times SO(d) \backslash SO^+(d,d)/SO^+(\Gamma^{d,d})$ of $\mathcal{M}^{Narain}$. For even d this is a four–fold cover, for odd d a two–fold one. The $\mathbb{R}$–span of $\Gamma^{d,d}$ is naturally isomorphic to $\mathbb{R}^d \oplus (\mathbb{R}^d)^*$, where $\mathbb{R}^d$ is considered as an isotropic subspace and W^* denotes the dual of a vector space W, and analogously for lattices. Thus $O(d,d)$ can be considered as the orthogonal group of a vector space with elements (α,β), $\alpha,\beta \in \mathbb{R}^d$ and scalar product

$$(\alpha,\beta)\cdot(\alpha',\beta') = \alpha\cdot\beta' + \alpha'\cdot\beta.$$

There is a canonical maximal positive definite d-plane given by $\alpha=\beta$ in $\mathbb{R}^d \oplus (\mathbb{R}^d)^* = \mathbb{R}^{d,d}$. The group $SO(d)\times SO(d)$ is supposed to describe rotations in this d-plane and in its orthogonal complement. In this description, the parity transformation consists of interchanging these two orthogonal d-planes, plus a sign change of the bilinear form on $\mathbb{R}^{d,d}$.

Now we use the isometry

$$V : SO(d)\backslash GL^+(d) \times Skew(d\times d,\mathbb{R}) \longrightarrow SO(d)\times SO(d)\backslash SO^+(d,d) \;\cong\; \mathcal{T}^{d,d} \tag{1.8}$$

given by

$$V(\Lambda,B) = \begin{pmatrix} (\Lambda^T)^{-1} & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} \mathbb{1} & -B \\ 0 & \mathbb{1} \end{pmatrix}. \tag{1.9}$$

We identify $\Lambda \in GL^+(d)$ with the image of $\mathbb{Z}^d$ under Λ. Finally we change coordinates by $p_l := (\alpha+\beta)/\sqrt{2},\ p_r := (\alpha-\beta)/\sqrt{2}$, such that the scalar product becomes

$$(p_l;p_r)\cdot(p_l';p_r') := p_l p_l' - p_r p_r'. \tag{1.10}$$

This means that the positive definite d-plane is given by $p_r = 0$ and its orthogonal complement by $p_l = 0$. Altogether, with $\widetilde{B} := (\Lambda^T)^{-1}B\Lambda^{-1}$ a point in $\mathcal{M}^{tori}$ is now described by the lattice

$$\begin{aligned} \Gamma(\Lambda,B) = \Big\{ &(p_l(\lambda,\mu);p_r(\lambda,\mu)) \\ &:= \tfrac{1}{\sqrt{2}}\left(\mu - \widetilde{B}\lambda + \lambda; \mu - \widetilde{B}\lambda - \lambda\right)\Big|\,(\lambda,\mu)\in\Lambda\oplus\Lambda^*\Big\}. \end{aligned} \tag{1.11}$$

The corresponding σ model has the real torus $T = \mathbb{R}^d/\Lambda$ as target space and $B\in H^2(T,\mathbb{R})\cong Skew(d\times d,\mathbb{R})$ as B-field. Introducing d Majorana fermions $\psi_1,\dots,\psi_d$ as superpartners of the abelian currents $j_1,\dots,j_d$ on the torus one constructs an $N=(2,2)$ superconformal field theory with central charge $c = 3d/2$ which will be denoted by $\mathcal{T}(\Lambda,B)$. From equation (1.9) it is clear that integral shifts of B and lattice automorphisms yield isomorphic theories.
The theory is specified by its charge lattice $\Gamma(\Lambda,B)$. Namely, to any pair $(\lambda,\mu)\in\Lambda\oplus\Lambda^*$ there corresponds a vertex operator $V_{\lambda,\mu}$ with charge $(p_l(\lambda,\mu);p_r(\lambda,\mu))$ with respect to $(j_1,\dots,j_d;\overline{\jmath}_1,\dots,\overline{\jmath}_d)$ and with dimensions $(h;\overline{h}) = (\frac{1}{2}p_l^2;\frac{1}{2}p_r^2)$. Thus h and $-\overline{h}$ are the squares of the projections of $(p_l;p_r)$ to the positive definite d-plane and its orthogonal complement, respectively. In this description, the parity operation is represented by the interchange of the latter two planes plus a sign change in the quadratic form on $\mathbb{R}^{d,d}$. The transformations which exchange the sheets of our covering of Narain's moduli space $\mathcal{M}^{Narain}$ are given by target space orientation change and T–duality, as can be read off from equation (1.11).
The partition function of this theory is

$$\begin{aligned} Z(\tau,z) &= Z_{\Lambda,B}(\tau)\cdot\tfrac{1}{2}\sum_{i=1}^{4}\left|\frac{\vartheta_i(\tau,z)}{\eta(\tau)}\right|^d, \\ Z_{\Lambda,B}(\tau) &= \frac{1}{|\eta(\tau)|^{2d}}\sum_{(\lambda,\mu)\in\Lambda\oplus\Lambda^*} q^{\frac{1}{2}(p_l(\lambda,\mu))^2}\,\overline{q}^{\frac{1}{2}(p_r(\lambda,\mu))^2}, \end{aligned} \tag{1.12}$$

where $q=\exp(2\pi i\tau)$ and analogously for $\overline{q}$. The functions $\vartheta_j(\tau,z), j=1,\dots,4$ are the classical theta functions and $\eta(\tau)$ is the Dedekind eta function. For ease of notation we will write $\eta=\eta(\tau), \vartheta_j(z)=\vartheta_j(\tau,z)$, and $\vartheta_j=\vartheta_j(\tau,0)$ in the following.
By considering $\mathcal{H}_1$ one easily checks that all theories in $\mathcal{M}^{tori}$ are described by some even unimodular lattice Γ. We want to show that every such lattice

has a σ model interpretation $\Gamma = \Gamma(\Lambda, B)$ (see also [As2]). Choose a maximal nullplane $Y \subset \mathbb{R}^{d,d} = \mathbb{R}^d \oplus (\mathbb{R}^d)^*$ such that $Y \cap \Gamma \subset \Gamma$ is a primitive sublattice. Apply an $SO(d) \times O(d)$ transformation such that the equation of this plane becomes $\beta = 0$. Put $Y \cap \Gamma = (\Lambda^*, 0)$. Next choose a dual nullplane Y^0 such that $Y \oplus Y^0 = \mathbb{R}^{d,d}$ and $Y^0 \cap \Gamma \subset \Gamma$ is a primitive lattice, too. Existence of Y^0 can be shown by a Gram type algorithm. Then $Y^0 = \{(-B\beta, \beta) \mid \beta \in \mathbb{R}^d\}$ for some skew matrix B, and $\Gamma = \Gamma(\Lambda, B)$. Note that different choices of Y^0 merely correspond to translations of B by integral matrices. So the geometric interpretation is actually fixed by the choice of Y alone as soon as B is viewed as an element of $Skew(d)/Skew(d \times d; \mathbb{Z})$.
In this interpretation, $\mathbb{R}^d$ is identified with the cohomology group $H^1(\mathbb{R}^d/\mathbb{Z}^d, \mathbb{R})$ of the reference torus $T = \mathbb{R}^d/\mathbb{Z}^d$. In addition to its defining representation, the double cover of the group $SO^+(d,d)$ also has half-spinor representations, namely its images in $SO^+(H^{odd}(T,\mathbb{R}))$ and in $SO^+(H^{even}(T,\mathbb{R}))$. For $d = 4$ one has the obvious isomorphism $SO^+(4,4) \cong SO^+(H^{odd}(T,\mathbb{R}))$, which together with $SO^+(4,4) \cong SO^+(H^{even}(T,\mathbb{R}))$ yields the celebrated D_4 triality [L-M, I.8]. It is the latter automorphism which we will need in this paper, since the odd cohomology of X does not survive orbifold maps.
Note that for $Spin(4,4)$ representations on $\mathbb{R}^{4,4}$ there is the same triality relation as for $Spin(8)$ representations on $\mathbb{R}^8$, i.e. an $\mathcal{S}_3$ permuting the vector representation, the chiral and the antichiral Weyl spinor representation. The role of triality is already visible upon comparison of the geometric interpretations, where the analogy between choices of nullplanes Y, Y^0 as described above and nullvectors υ, υ^0 in (1.5) is apparent. Indeed, part of the triality manifests itself in a one to one correspondence between maximal isotropic subspaces $Y \subset \mathbb{R}^{4,4}$ and null Weyl spinors υ such that $Y = \{y \in \mathbb{R}^{d,d} \mid c(y)(\upsilon) = 0\}$ where c denotes Clifford multiplication on the spinor bundle [B-T]. One can regard this as further justification for the interpretation of υ as volume form which generates $H^4(T,\mathbb{Z})$ in our geometric interpretation. Recall also that in both cases different choices of Y^0, υ^0 correspond to B-field shifts by integral forms.
We now explicitly describe the isomorphism (1.8) to show that it is a triality automorphism. First compare (1.8) to (1.4) and notice that $Skew(4) \cong \mathbb{R}^{3,3}$ which will simply be written $Skew(4) \ni B \mapsto b \in \mathbb{R}^{3,3}$ in the following. Moreover, because $|\det \Lambda|$ is the volume of the torus $T = \mathbb{R}^d/\Lambda$, we can decompose $SO(4)\backslash GL^+(4) \cong SO(4)\backslash SL(4) \times \mathbb{R}^+$. Now let $T_{\Lambda_0} = \mathbb{R}^4/\Lambda_0$ where Λ_0 is a lattice of determinant 1 and is viewed as element of $SL(4)$. Consider the induced representation ρ of $SL(4)$ on the exterior product $\Lambda^2(\mathbb{R}^4)$ which defines an isomorphism $\Lambda^2(\Lambda_0) \cong H_2(T_{\Lambda_0}, \mathbb{Z})$ for every $\Lambda_0 \in SL(4)$. Because ρ commutes with the action of the Hodge star operator $*$ and $*^2 = \mathbb{1}$ on twoforms, $SL(4)$ is actually represented by $SO^+(3,3)$. In terms of coordinates as in (1.9) and with $\Lambda = V^{1/4}\Lambda_0 = (\lambda_1, \ldots, \lambda_4), V = |\det \Lambda|$, we can write

$$\begin{aligned}\rho\,(\Lambda_0) = V^{-1/2}\,(\lambda_1 \wedge \lambda_2\,, \lambda_1 \wedge \lambda_3\,, \lambda_1 \wedge \lambda_4\,, \lambda_3 \wedge \lambda_4\,, \lambda_4 \wedge \lambda_2\,, \lambda_2 \wedge \lambda_3) \\ \in SO^+\,(H_2(T,\mathbb{R})) \cong SO^+(3,3).\end{aligned} \tag{1.13}$$

Because $SO^+(3,3) \cong SL(4)/\mathbb{Z}_2$ and $SO(3) \times SO(3)/\mathbb{Z}_2 \cong SO(4)$ we find $SO(4)\backslash SL(4) \cong \mathcal{T}^{3,3}$ and all in all have

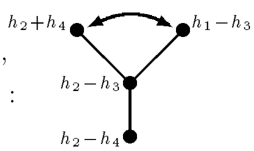
$$\mathcal{T}^{4,4} \overset{(1.8)}{\cong} SO(4)\backslash GL^+(4) \times Skew(4) \xrightarrow{\cong} \mathcal{T}^{3,3} \times \mathbb{R}^+ \times \mathbb{R}^{3,3} \overset{(1.4)}{\cong} \mathcal{T}^{4,4}. \tag{1.14}$$

By (1.14) the geometric interpretation of a superconformal field theory is translated from a description in terms of the lattice of the underlying torus, i.e. in terms of $\Lambda \cong H_1(T_\Lambda, \mathbb{Z})$, to a description in terms of $H_2(T_\Lambda, \mathbb{Z}) \cong \Lambda^2(\Lambda)$. This translation is essential for understanding the relation between the moduli spaces $\mathcal{M}^{tori}$ and $\mathcal{M}^{K3}$. To actually arrive at the description (1.4) in terms of hyperkähler structures, i.e. in terms of $H^2(T, \mathbb{Z})$, we have to apply Poincaré duality or use the dual lattice Λ^* instead of Λ. This distinction will no longer be relevant after theories related by T–duality have been identified.
We insert the coordinate expressions in (1.9) and (1.5) into (1.14), write $\Lambda = V^{1/4}\Lambda_0, V = |\det \Lambda|$ as before and arrive at

$$V(\Lambda, B) \longmapsto S(\Lambda, B) = \left(\begin{array}{c|c|c} V^{1/2} & 0 & 0 \\ \hline 0 & \rho(\Lambda_0) & 0 \\ \hline 0 & 0 & V^{-1/2} \end{array}\right) \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline b & \mathbb{1} & 0 \\ \hline -\frac{\|B\|^2}{2} & -b^T & 1 \end{array}\right). \tag{1.15}$$

Observe that (1.15) is a homomorphism $\mathcal{T}^{4,4} \to \mathcal{T}^{4,4}$ and thus gives a natural explanation for the quadratic dependence on B in (1.5). Moreover, (1.15) reveals the structure of the warped product (1.4) alluded to before. But above all on Lie algebra level one can now easily read off that (1.15) is the triality automorphism exchanging the two half spinor representations V and S. Namely, let $h_1, \ldots, h_4$ denote generators of the Cartan subalgebra of $so(4,4)$. Here h_i generates dilations of the radius R_i of our torus in direction λ_i. Since $\exp(\vartheta h_i)$ scales $V^{\pm 1/2}$ by $e^{\pm\vartheta/2}$ and with (1.13) one then finds that (1.15) indeed is induced by the triality automorphism which acts on the Cartan subalgebra by

$$h_1 \mapsto \tfrac{1}{2}(h_1 + h_2 + h_3 + h_4),\ h_2 \mapsto \tfrac{1}{2}(h_1 + h_2 - h_3 - h_4),$$
$$h_3 \mapsto \tfrac{1}{2}(h_1 - h_2 + h_3 - h_4),\ h_4 \mapsto \tfrac{1}{2}(h_1 - h_2 - h_3 + h_4):$$



Note that triality interchanges the outer automorphisms of $SO^+(4,4)$ related to worldsheet parity and target space orientation.
Triality considerations have a long history in superstring and supergravity theories, see for example [Sha, Cu, G-O]. Concerning recent work, as communicated to us by N. Obers, $SO(4,4)$ is crucial in the conjectured duality between heterotic strings on the fourtorus and type IIA on $K3$ [O-P1, K-O-P]. In connection with the calculation of $G(\mathbb{Z})$ invariant string theory amplitudes one can use triality to write down new identities for Eisenstein series [O-P1, O-P2].
We now come to a concept which is of major importance in the context of Calabi-Yau compactification and nonlinear σ models, namely the idea of *large volume limit*. A precise notion is necessary of how to associate a unique geometric interpretation to a theory described by an even self dual lattice Γ when parameters of volume go to infinity. Intuitively, because of the uniqueness condition, this should describe the limit where all the radii of the torus in this particular geometric interpretation are large. Because in the charge lattice (1.11) $\lambda \in \Lambda$ and $\mu \in \Lambda^*$ are interpreted as winding and momentum modes, the corresponding

nullplane Y should have the property

$$\begin{aligned} Y \cap \Gamma &= \mathrm{span}_{\mathbb{Z}} \left\{ \tfrac{1}{\sqrt{2}}(\mu;\mu) \in \Gamma \,\middle|\, \|\mu\|^2 \ll 1 \right\} \\ &\subset \mathrm{span}_{\mathbb{Z}} \left\{ (p_l;p_r) \in \Gamma \,\middle|\, \|p_l\|^2 \ll 1, \|p_r\|^2 \ll 1 \right\} =: \widetilde{\Gamma}. \end{aligned} \tag{1.16}$$

Because $\|p_l\|^2 - \|p_r\|^2 \in \mathbb{Z}$, for $(p_l;p_r) \in \widetilde{\Gamma}$ we have $\|p_l\|^2 = \|p_r\|^2$. This shows $Y \cap \Gamma = \widetilde{\Gamma}$ because any $(p_l;p_r) \notin Y^\perp = Y$ must have large components. Moreover, if a maximal isotropic plane Y as in (1.16) exists, then it is uniquely defined, thus yielding a sensible notion of large volume limit. Large volume and small volume limits are exchanged by T–duality.
For our embedding of torus orbifold theories into the $K3$ moduli space $\mathcal{M}^{K3}$ we have to keep target space orientation. We also want to keep the left–right distinction in the conformal field theory. Torus T–duality just yields a reparametrization of the theory and should be divided out of the moduli space. Thus for us the relevant moduli space of torus theories is given by

$$\mathcal{M}^{tori} = SO(d) \times O(d) \backslash O^+(d,d)/O^+(\Gamma^{d,d}). \tag{1.17}$$

Notice that this is a double cover of $\mathcal{M}^{Narain}$.

1.2. Moduli space of theories associated to $K3$ surfaces. We now give some more details about the moduli space of conformal field theories associated to $K3$ which we will concentrate on for the rest of the paper, namely

$$\mathcal{M}^{K3} = O^+(H^{even}(X,\mathbb{Z})) \backslash \mathcal{T}^{4,20} \tag{1.18}$$

by (1.7). For other presentations see [A-M, R-W, Di].
In the decomposition (1.4) we determine the product metric such that it becomes an isometry. In particular, it faithfully relates moduli of the conformal field theory to deformations of geometric objects. Recall that the structure of the tangent space $\mathcal{H}_1$ of $\mathcal{M}^{K3}$ in a given superconformal field theory is best understood by examining the $(\frac{1}{2},\frac{1}{2})$-fields in $\mathcal{F}_{1/2}$. In our case we have related it to the $su(2)_l^{susy} \oplus su(2)_r^{susy}$ invariant subspace of the tensor product $Q_l \otimes Q_r \otimes \mathcal{H}^{(4)}_{1/4} \otimes \mathcal{H}^{(0)}_{1/4}$, where $\mathcal{H}^{(4)}_{1/4}$ denotes the charged and $\mathcal{H}^{(0)}_{1/4}$ the uncharged Ramond ground states. The invariant subspace of $Q_l \otimes Q_r \otimes \mathcal{H}^{(4)}_{1/4}$ yields a four–plane with an orthogonal group generated by $su(2)_l \oplus su(2)_r$. When a frame in $Q_l \otimes Q_r$ is chosen, the latter tensor product factor can be omitted. The description of $\mathcal{M}$ implies that $\mathcal{H}^{(4)}_{1/4} \oplus \mathcal{H}^{(0)}_{1/4}$ has a natural non-degenerate indefinite metric and remains invariant under deformations, but it has not been understood how this comes about. In terms of the four–plane $x \in \mathcal{T}^{4,20}$ giving the location of our theory in moduli space, specific vectors in the tangent space $T_x\mathcal{T}^{4,20}$ are described by infinitesimal deformations of one generator $\xi \in x$ in direction $x^\perp$ that leaves $\xi^\perp \cap x$ invariant.
To formulate this in terms of a geometric interpretation (Σ, V, B) specified by (1.5), pick a basis $\eta_1, \dots, \eta_{19}$ of $\Sigma^\perp \subset H^2(X,\mathbb{R}) \cong \mathbb{R}^{3,19}$. Then $x^\perp$ is spanned by $\{\eta_i - \langle \eta_i, B\rangle\, \upsilon; i = 1, \dots, 19\}$ and $\eta_{20} := \upsilon^0 + B - (\frac{\|B\|^2}{2} + V)\upsilon$. In each of the $SO(4)$ fibres of $\mathcal{H}_1$ over $\eta_i - \langle \eta_i, B\rangle\, \upsilon, i = 1, \dots, 19$ we find a threedimensional

subspace deforming generators of Σ by η_i, as well as the deformation of B in direction of η_i. The fibre over η_{20} contains B-field deformations in direction of Σ and the deformation of volume. All in all, a $3 \cdot 19 = 57$ dimensional subspace of $\mathcal{H}_1 = T_x\mathcal{M}^{K3}$ is mapped onto deformations of Σ by $(1,1)$-forms $\eta \in \Sigma^\perp \cap H^2(X,\mathbb{R}) \subset H^{1,1}(X,\mathbb{R})$, no matter what complex structure we pick in Σ. The 23 dimensional complement of this subspace is given by $19+3$ deformations of the B-field by forms $\eta \in H^2(X,\mathbb{R})$ and the volume deformation.

One of the most valuable tools for understanding the structure of the moduli space is the study of symmetries. So the next question to be answered is how to translate symmetries of our superconformal field theory to its geometric interpretations. Those symmetries which commute with the $su(2)_l \oplus su(2)_r$ action leave the four–plane x invariant and are called *algebraic symmetries*. When the $N=(4,4)$ supersymmetric theories are constructed in terms of $(2,2)$ supersymmetric theories one has a natural framing. In this context, algebraic symmetries are those which leave the entire vector space $Q_l \otimes Q_r$ of supercharges invariant. More generally, any abelian symmetry group of our theory projects to a $u(1)_l \oplus u(1)_r$ subgroup of $su(2)_l \oplus su(2)_r$ and fixes the corresponding $N=(2,2)$ subalgebra. When corresponding supercharges are fixed, the abelian symmetry group acts diagonally on the charge generators $J^\pm, \overline{J}^\pm$ of $su(2)_l^{susy} \oplus su(2)_r^{susy}$. The algebraic subgroup of this symmetry group is the one which fixes these charges.

If the primitive nullvector υ specifying our geometric interpretation (Σ, V, B) is invariant upon the induced action of an algebraic symmetry we call the latter a *classical symmetry* of the geometric interpretation (Σ, V, B). Because a classical symmetry α^* fixes x by definition we get an induced automorphism of $H^2(X,\mathbb{R})$ which leaves $\Sigma \subset H^2(X,\mathbb{R})$ and $B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$ invariant. Moreover, because ξ_4 in (1.5) is invariant as well, $\eta_{20} = \upsilon^0 + B - (\frac{\|B\|^2}{2} + V)\upsilon$ is fixed. Thus α^* acts trivially on moduli of volume and B-field deformation in direction of Σ. Because α^* acts as automorphism on $H^{1,1}(X,\mathbb{R}) = \Omega^\perp \cap H^2(X,\mathbb{R})$ for any choice of complex structure $\Omega \subset \Sigma$ on X leaving the onedimensional $H^{1,1}(X,\mathbb{R}) \cap \Sigma$ invariant, all in all, $x \mapsto (\Sigma, V, B)$ maps the action of α^* to an automorphism of $H^2(X,\mathbb{R})$ which on $H^{1,1}(X,\mathbb{R})$ has exactly the same spectrum as α^* on $(\frac{1}{2}, \frac{1}{2})$-fields with charge, say, $Q = \overline{Q} = 1$.

If the integral action of α^* on $H^2(X,\mathbb{C})$ is induced by an automorphism $\alpha \in Aut(X)$ of finite order of the $K3$ surface X, then by definition, because α^* acts trivially on $H^{2,0}(X,\mathbb{C})$, α is an *algebraic automorphism* [Ni2]. This notion of course only makes sense after a choice of complex structure, or in conformal field theory language an $N=(2,2)$ subalgebra of the $N=(4,4)$ superconformal algebra fixing generators $J, J^\pm, \overline{J}, \overline{J}^\pm$ of $su(2)_l \oplus su(2)_r$. Still, because we always assume the metric to be invariant under α^* as well, i.e. $\Sigma \subset H^2(X,\mathbb{R})^{\alpha^*}$, this is no further restriction. On the other hand, given an algebraic automorphism α of X which induces an automorphism of $H^2(X,\mathbb{R})$ that leaves the B-field invariant, α induces a symmetry of our conformal field theory which leaves $J, J^\pm, \overline{J}, \overline{J}^\pm$ invariant. This gives a precise notion of how to continue such an algebraic automorphism to the conformal field theory level.

We are thus naturally led to a discussion of algebraic automorphisms of $K3$ surfaces, which are mathematically well understood thanks to the work of Nikulin [Ni2] for the abelian and Mukai [Mu] for the general case. The first to explicitly take advantage of their special properties in the context of conformal field theory was P.S. Aspinwall [As1]. From [Ni2, Th. 4.3,4.7,4.15] one can deduce the following consequence of the global Torelli theorem:

Theorem 1.1
Let g denote an automorphism of $H^2(X,\mathbb{C})$ of finite order which maps forms corresponding to effective divisors of self intersection number -2 in $Pic(X)$ to forms corresponding to effective divisors. Then g is induced by an algebraic automorphism of X iff $\left(H^2(X,\mathbb{Z})^g\right)^\perp \cap H^2(X,\mathbb{Z}) \subset Pic(X)$ is negative definite with respect to the intersection form and does not contain elements of length squared -2.

If for a geometric interpretation (Σ, V, B) of $x \in O^+(H^{even}(X,\mathbb{Z}))\backslash\mathcal{T}^{4,20}$ we have classical symmetries which act effectively on what we read off as $H^2(X,\mathbb{C})$ but are not induced by an algebraic automorphism of the $K3$ surface X by theorem 1.1, then our interpretation of x as giving a superconformal field theory breaks down. Such points should be conifold points of the moduli space $\mathcal{M}^{K3}$, characterized by too high an amount of symmetry. One can regard Nikulin's theorem 1.1 as harbinger of Witten's result that in points of enhanced symmetry on the moduli space of type IIA string theories compactified on $K3$ the conformal field theory description breaks down [Wi3].
By abuse of notation we will often renounce to distinguish between an algebraic automorphism on $K3$ and its induced action on cohomology.
From Mukai's work [Mu, Th. 1.4] one may learn that the induced action of any algebraic automorphism group G on the total rational cohomology $H^*(X,\mathbb{Q})$ is a *Mathieu representation* of G over $\mathbb{Q}$, i.e. a representation with character

$$\chi(g) = \mu(\mathrm{ord}(g)), \text{ where for } n \in \mathbb{N}: \; \mu(n) := \frac{24}{n \prod\limits_{\substack{p \text{ prime}, \\ p|n}} (1+\frac{1}{p})}. \tag{1.19}$$

It follows that

$$\dim_{\mathbb{Q}} H^*(X,\mathbb{Q})^G = \mu(G) := \frac{1}{|G|} \sum_{g \in G} \mu(\mathrm{ord}(g)) \tag{1.20}$$

[Mu, Prop. 3.4]. We remark that because G acts algebraically, we have $\dim_{\mathbb{Q}} H^*(X,\mathbb{Q})^G = \dim_{\mathbb{R}} H^*(X,\mathbb{R})^G = \dim_{\mathbb{C}} H^*(X,\mathbb{C})^G$. By definition of algebraic automorphisms $H^*(X,\mathbb{C})^G \supset H^0(X,\mathbb{C}) \oplus H^{2,0}(X,\mathbb{C}) \oplus H^{0,2}(X,\mathbb{C}) \oplus H^{2,2}(X,\mathbb{C})$, so

$$\mu(G) - 4 = \dim_{\mathbb{R}} H^{1,1}(X,\mathbb{R})^G. \tag{1.21}$$

Moreover, from theorem 1.1 we know that $\left(H^2(X,\mathbb{R})^G\right)^\perp \subset H^{1,1}(X,\mathbb{R})$ is negative definite, and because $H^{1,1}(X,\mathbb{R})$ has signature $(1,19)$, we may conclude that it contains an invariant element with positive length squared. Thus $\mu(G) \geq 5$ for every algebraic automorphism group G [Mu, Th. 1.4]. Moreover [Mu, Cor. 3.5, Prop. 3.6],

$$G \neq \{\mathbb{1}\} \Longrightarrow \mu(G) \leq 16. \tag{1.22}$$

Finally let us consider the special case of an algebraic automorphism α of order 4, which will be useful in due course. By n_k we denote the multiplicity of the eigenvalue i^k of the induced action α^* on $H^{1,1}(X,\mathbb{C})$. Because by (1.19) and (1.20) $\mu(\mathbb{Z}_4)=10$ and $\mu(\mathbb{Z}_2)=16$, using (1.21) we find $n_0=10-4=6, n_2=16-4-n_0=6$. The automorphism α^* acts on the lattice $H^2(X,\mathbb{Z})$, so it must have integer trace. On the other hand $20=\dim_{\mathbb{C}} H^{1,1}(X,\mathbb{C})=n_0+n_1+n_2+n_3$, hence

$$n_0=n_2=6, \quad n_1=n_3=4. \tag{1.23}$$

2. Special subspaces of the moduli space: Orbifold theories

This section is devoted to the study of theories which have a geometrical interpretation on an orbifold limit of $K3$. We begin by giving a short account on the relevant features of the orbifold construction, for details the reader is referred to the vast literature, e.g. [D-H-V-W, Di].

On the geometric side, the $\mathbb{Z}_l$ orbifold construction of $K3$ can be described as follows [Wa]: Consider a fourtorus T, where $T=T^2\times\widetilde{T}^2$ with two $\mathbb{Z}_l$ symmetric twotori $T^2=\mathbb{C}/L$, $\widetilde{T}^2=\mathbb{C}/\widetilde{L}$ which need not be orthogonal. Let $\zeta\in\mathbb{Z}_l$ act algebraically on $(z_1,z_2)\in T^2\times\widetilde{T}^2$ by $(z_1,z_2)\mapsto(\zeta z_1,\zeta^{-1}z_2)$. Mod out this symmetry and blow up the resulting singularities; that is, replace each singular point by a chain of exceptional divisors, which in the case of $\mathbb{Z}_l$-fixed points have as intersection matrix the Cartan matrix of A_{l-1}. In particular, the exceptional divisors themselves are rational curves, i.e. holomorphically embedded spheres with self intersection number -2. In terms of the homology of the resulting surface X these rational curves are elements of $H_2(X,\mathbb{Z})\cap H_{1,1}(X,\mathbb{C})$. To translate to cohomology we work with their Poincaré duals, which now are elements of $Pic(X)$ with length squared -2. One may check that for $l\in\{2,3,4,6\}$ this procedure changes the Hodge diamond by

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} \longmapsto \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and indeed produces a $K3$ surface X, because the automorphism we modded out was algebraic. We also obtain a rational map $\pi: T\to X$ of degree l by this procedure. To fix a hyperkähler structure we additionally need to pick the class of a Kähler metric on X. We will consider orbifold limits of $K3$ surfaces, that is use the *orbifold singular metric* on X which is induced from the flat metric on T and assigns volume zero to all the exceptional divisors. The corresponding Einstein metric is constructed by excising a sphere around each singular point of $T/\mathbb{Z}_l$ and gluing in an Eguchi Hanson sphere E_2 instead for $l=2$, or a generalized version E_l with boundary $\partial E_l=\mathbb{S}^3/\mathbb{Z}_l$ at infinity and nonvanishing Betti numbers $b_0(E_l)=1, b_2(E_l)=b_2^-(E_l)=l-1$, i.e. $\chi(E_l)=l$. The orbifold limit is the limit these Eguchi Hanson type spheres have shrunk to zero size in. The description (1.6) of the moduli space of Einstein metrics of volume 1 on $K3$ includes orbifold limits [K-T], and as was shown by Anderson [An] one can define an *extrinsic L^2-metric* on the space $\mathcal{E}$ of regular Einstein metrics of volume 1 on $K3$ such that the completion of $\mathcal{E}$ is contained in the set of regular and orbifold singular Einstein metrics.

On the conformal field theory side the orbifold construction is in total analogy to the geometric one described above. Assume we know the action of $\mathbb{Z}_l$ on the space of states $\mathcal{H}$ of a conformal field theory with geometric interpretation on the torus T we had above. To construct the orbifold conformal field theory, keep all the invariant states in $\mathcal{H}$ and then – for the sake of modular invariance, if we argue on the level of partition functions – add twisted sectors. For $\zeta \in \mathbb{Z}_l$, the ζ-twisted sector consists of states corresponding to fields φ which are only well defined up to ζ-action on the world sheet of the original theory, that is $\varphi : Z \to T, \varphi(\sigma_0+1, \sigma_1) = \zeta\varphi(\sigma_0, \sigma_1)$. Z denotes the configuration space as mentioned in the introduction and coordinates $(\sigma_0, \sigma_1), \sigma_0 \sim \sigma_0+1, (\sigma_0, \sigma_1) \sim (\sigma_0+\tau_0, \sigma_1+\tau_1)$ are chosen such that $\varphi(0,0)$ is a fixed point. In other words, the constant mode in the Fourier expansion of φ is a fixed point p_ζ of ζ. The other modes are of non integral level, so the ground state energy in the twisted sector is shifted away from zero. More precisely, the ground state $|\Sigma_{\zeta,p_\zeta}\rangle$ of the ζ-twisted sector $\mathcal{H}_{\zeta,p_\zeta}$ belongs to the Ramond sector and has dimensions $h = \overline{h} = \frac{c}{24} = \frac{1}{4}$. The corresponding field Σ_{ζ,p_ζ} introduces a cut in Z from $(0,0)$ to $(\tau_0, \tau_1) \sim (0,0)$ to establish the transformation property $\varphi(\sigma_0+1, \sigma_1) = \zeta\varphi(\sigma_0, \sigma_1)$ for $|\varphi\rangle \in \mathcal{H}_{\zeta,p_\zeta}$, often referred to as *boundary condition*. The field Σ_{ζ,p_ζ} is called a *twist field*. For explicit formulae of partition functions for $\mathbb{Z}_l$ orbifold conformal field theories see [E-O-T-Y], for the special cases $l = 2$ and $l = 4$ we are studying here see (2.3) and (2.14).

To summarize, we stress the analogy between orbifolds in the geometric and the conformal field theory sense once again; in particular, the introduction of a twist field for each fixed point and boundary condition corresponds to the introduction of an exceptional divisor in the course of blowing up the quotient singularity, if we use the metric which assigns volume zero to all the exceptional divisors.

By construction orbifold conformal field theories have a preferred geometric interpretation in the sense of section 1.2. We will now investigate this geometric interpretation for $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds, particularly taking advantage of their specific algebraic automorphisms. A program for finding a stratification of the moduli space could even be formulated as follows: Find all subspaces of theories having a geometric interpretation (Σ, V, B) with given algebraic automorphism group G. Relations between such subspaces may be described by the modding out of algebraic automorphisms. Any infinitesimal deformation of Σ by an element of $H^{1,1}(X,\mathbb{R})^G$ will preserve the symmetries in G, as well as volume deformations and B-field deformations by elements in $H^2(X,\mathbb{R})^G$. The subspace of theories with given classical symmetry group G in a geometric interpretation therefore can maximally have real dimension $3(\mu(G)-5)+1+\mu(G)-2 = 4(\mu(G)-4)$ in accord with (1.21). In particular, for the minimal value $\mu(G) = 5$, the only deformations preserving the entire symmetry are deformations of volume and those of the B-field by elements of Σ.

Of course, the above program is far from utterly realizable, even in the pure geometric context, but it might serve as a useful line of thought. $\mathbb{Z}_2$ Orbifolds actually yield the first item of this program: We can map the entire torus moduli space into the $K3$ moduli space by modding out the symmetry $z \mapsto -z$. The description is straightforward if we make use of the geometric interpretation of torus theories given by the triality automorphism (1.14), because the geometric data then turn out to translate in a simple way into the corresponding data on $K3$.

2.1. $\mathbb{Z}_2$ Orbifolds in the moduli space. Some comments on $\mathbb{Z}_2$ orbifold conformal field theories as described at the beginning of the section are due, before we can show where they are located within the moduli space $\mathcal{M}^{K3}$. We denote the $\mathbb{Z}_2$ orbifold obtained from the nonlinear σ model $\mathcal{T}(\Lambda, B_T)$ by $\mathcal{K}(\Lambda, B_T)$. If the theory on the torus has an enhanced symmetry G we frequently simply write $G/\mathbb{Z}_2$, e.g. $SU(2)_1^4/\mathbb{Z}_2$ for $\mathcal{K}(\mathbb{Z}^4, 0)$.
In the nonlinear σ model on the torus $T = \mathbb{R}^4/\Lambda$ as described in section 1.1 the current j_k generates translations in direction of coordinate x_k. This induces a natural correspondence between tangent vectors of T and fields of the nonlinear σ model which is compatible with the $so(4)$ action on the tangent spaces of T and the moduli space, respectively. After selection of an appropriate framing of $Q_l \otimes Q_r$ to identify $su(2)_{l,r}^{susy}$ with $su(2)_{l,r}$ as described in section 1 the ψ_k are the superpartners of the j_k. Hence the choice of complex coordinates $z_1 := \frac{1}{\sqrt{2}}(x_1 + ix_2)$, $z_2 := \frac{1}{\sqrt{2}}(x_3 + ix_4)$ corresponds to setting

$$\psi_\pm^{(1)} \;:=\; \tfrac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2), \quad \psi_\pm^{(2)} \;:=\; \tfrac{1}{\sqrt{2}}(\psi_3 \pm i\psi_4). \tag{2.1}$$

The holomorphic W-algebra of our theory has an $su(2)_1^2$-subalgebra generated by

$$\begin{array}{lll} J := \psi_+^{(1)}\psi_-^{(1)} + \psi_+^{(2)}\psi_-^{(2)}, & J^+ \;:=\; \psi_+^{(1)}\psi_+^{(2)}, & J^- \;:=\; \psi_-^{(2)}\psi_-^{(1)}; \\ A := \psi_+^{(1)}\psi_-^{(1)} - \psi_+^{(2)}\psi_-^{(2)}, & A^+ \;:=\; \psi_+^{(1)}\psi_-^{(2)}, & A^- \;:=\; \psi_+^{(2)}\psi_-^{(1)}. \end{array} \tag{2.2}$$

Its geometric counterpart on the torus is the Clifford algebra generated by the twoforms $dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2, dz_1 \wedge dz_2, d\overline{z}_1 \wedge d\overline{z}_2; dz_1 \wedge d\overline{z}_1 - dz_2 \wedge d\overline{z}_2, dz_1 \wedge d\overline{z}_2, dz_2 \wedge d\overline{z}_1$ upon Clifford multiplication.
The nonlinear σ model on the Kummer surface $\mathcal{K}(\Lambda)$ is the "ordinary" $\mathbb{Z}_2$ orbifold of the above, where $\mathbb{Z}_2$ acts by $j_k \mapsto -j_k, \psi_k \mapsto -\psi_k$, $k = 1, \dots, 4$. Note that the entire $su(2)_1^2$-algebra (2.2) is invariant under this action, thus any nonlinear σ model on a Kummer surface possesses an $su(2)_1^2$-current algebra. The NS-part of its partition function is

$$\begin{aligned} Z_{NS}(\tau, z) = \tfrac{1}{2}\Bigg\{ & Z_{\Lambda,B}(\tau)\left|\frac{\vartheta_3(z)}{\eta}\right|^4 + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4 \left|\frac{\vartheta_4(z)}{\eta}\right|^4 \\ & + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4 \left|\frac{\vartheta_2(z)}{\eta}\right|^4 + \left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^4 \left|\frac{\vartheta_1(z)}{\eta}\right|^4 \Bigg\}. \end{aligned} \tag{2.3}$$

Here and in the following we decompose partition functions into four parts corresponding to the four sectors $NS, \widetilde{NS}$, $R, \widetilde{R}$, i.e. with $y = exp(2\pi i z), \overline{y} = exp(-2\pi i \overline{z})$

$$\begin{aligned} Z &= \tfrac{1}{2}\left(Z_{NS} + Z_{\widetilde{NS}} + Z_R + Z_{\widetilde{R}}\right), \\ Z_{NS}(\tau, z) &= \mathrm{tr}_{NS}\left[q^{L_0 - \frac{c}{24}}\overline{q}^{\overline{L}_0 - \frac{c}{24}} y^{J_0}\overline{y}^{\overline{J}_0}\right], \\ Z_{\widetilde{NS}}(\tau, z) &= \mathrm{tr}_{NS}\left[(-1)^F q^{L_0 - \frac{c}{24}}\overline{q}^{\overline{L}_0 - \frac{c}{24}} y^{J_0}\overline{y}^{\overline{J}_0}\right] = Z_{NS}(\tau, z + \tfrac{1}{2}), \\ Z_R(\tau, z) &= \mathrm{tr}_R\left[q^{L_0 - \frac{c}{24}}\overline{q}^{\overline{L}_0 - \frac{c}{24}} y^{J_0}\overline{y}^{\overline{J}_0}\right] = (q\overline{q})^{\frac{c}{24}}(y\overline{y})^{\frac{c}{6}}\, Z_{NS}(\tau, z + \tfrac{\tau}{2}), \\ Z_{\widetilde{R}}(\tau, z) &= \mathrm{tr}_R\left[(-1)^F q^{L_0 - \frac{c}{24}}\overline{q}^{\overline{L}_0 - \frac{c}{24}} y^{J_0}\overline{y}^{\overline{J}_0}\right] = Z_R(\tau, z + \tfrac{1}{2}). \end{aligned} \tag{2.4}$$

Given Z_{NS} the entire partition function can be determined by using the above flows to find $Z_{\widetilde{NS}}, Z_R$ and $Z_{\widetilde{R}}$.
This orbifold model has an $N=(16,16)$ supersymmetry. We are interested in deformations which conserve $N=(4,4)$ subalgebras. As explained in section 1, the latter are given by chiral and antichiral $(\frac{1}{2},\frac{1}{2})$–fields. Generically, the Neveu-Schwarz sector contains 144 fields with dimensions $(h,\overline{h})=(\frac{1}{2},\frac{1}{2})$. Their quantum numbers under $(J,A;\overline{J},\overline{A})$ are $(\varepsilon_1,\varepsilon_2;\varepsilon_3,\varepsilon_4), \varepsilon_i\in\{\pm1\}$ (16 fields), $(\varepsilon_1,0;\varepsilon_3,0)$ (64 fields), and $(0,\varepsilon_2;0,\varepsilon_4)$ (64 fields). The 80 fields which are charged under $(J;\overline{J})$ yield the $N=(4,4)$ supersymmetric deformations which conserve the superalgebra that contains the J currents. The 80 fields which are charged under $(A;\overline{A})$ yield deformations conserving a different $N=(4,4)$ superalgebra. The latter corresponds to the opposite torus orientation.
Let us now focus on the description of the resulting geometric objects, namely *Kummer surfaces* denoted by $\mathcal{K}(\Lambda)$ if obtained by the $\mathbb{Z}_2$ orbifold procedure from the fourtorus $T=\mathbb{R}^4/\Lambda$. Generators of the lattice Λ are denoted by $\lambda_1,\dots,\lambda_4$. From (1.14) we obtain an associated three–plane $\Sigma_T\subset H^2(T,\mathbb{R})$, i.e. an Einstein metric on T, and we must describe how the Teichmüller space $\mathcal{T}^{3,3}$ of Einstein metrics of volume 1 on the torus is mapped into the corresponding space $\mathcal{T}^{3,19}$ for $K3$. This is best understood in terms of the lattices $H^2(T,\mathbb{Z})\cong\Gamma^{3,3}$ and $H^2(X,\mathbb{Z})\cong\Gamma^{3,19}, X=\mathcal{K}(\Lambda)$. In our notation $H^2(T,\mathbb{Z})$ is generated by $\mu_j\wedge\mu_k, j,k\in\{1,\dots,4\}$ if $(\mu_1,\dots,\mu_4)$ is the basis dual to $(\lambda_1,\dots,\lambda_4)$. Σ_T is defined by its relative position to a reference lattice $\Gamma^{3,3}\cong H^2(T,\mathbb{Z})\subset H^2(T,\mathbb{R})$. Note that in order to simplify the following argumentation we rather regard $\Sigma_T\subset H^2(T,\mathbb{Z})$ as giving the position of the lattice $H^2(T,\mathbb{Z})=\mathrm{span}_\mathbb{Z}(\mu_j\wedge\mu_k)$ relative to a fixed three–plane $\mathrm{span}_\mathbb{R}(e_1\wedge e_2+e_3\wedge e_4, e_1\wedge e_3+e_4\wedge e_2, e_1\wedge e_4+e_2\wedge e_3)$ with respect to the standard basis $(e_1,\dots,e_4)$ of $\mathbb{R}^4$.
To make contact with the theory of Kummer surfaces we pick a complex structure $\Omega_T\subset\Sigma_T$. The $\mathbb{Z}_2$ action on T has 16 fixed points $\frac{1}{2}\sum_{k=1}^4\varepsilon_k\lambda_k, \varepsilon\in\mathbb{F}_2^4$. We can therefore choose indices in $\mathbb{F}_2^4$ to label the fixed points[1]. Note that this is not only a labelling but the torus geometry indeed induces a natural affine $\mathbb{F}_2^4$-structure on the set I of fixed points [Ni1, Cor. 5]. The twoforms corresponding to the 16 exceptional divisors obtained from blowing up the fixed points are denoted by $\{E_i\mid i\in I\}$. They are elements of $Pic(X)$ no matter what complex structure we choose, because we are working in the orbifold limit, i.e. $E_i\perp\Sigma\;\forall\, i\in I$. Let $\Pi\subset Pic(X)$ denote the primitive sublattice of the Picard lattice that contains $\{E_i\mid i\in I\}$. It is called *Kummer lattice* and by [Ni1, Th. 3]:

Theorem 2.1
The Kummer lattice Π is spanned by the exceptional divisors $\{E_i\mid i\in I\}$ and $\{\frac{1}{2}\sum_{i\in H}E_i\mid H\subset I$ is a hyperplane$\}$. On the other hand, a $K3$ surface X is a Kummer surface iff $Pic(X)$ contains a primitive sublattice isomorphic to Π.

Let $\pi:T\to X$ be the degree two map from the torus to the orbifold singular Kummer surface. Using Poincaré duality, one gets maps π_* from the homology and cohomology groups of T to those of X, and π^* in the other direction. In particular, this gives the natural embedding $\pi_*: H^2(T,\mathbb{Z})(2)\hookrightarrow H^2(X,\mathbb{Z})$ (here $\Gamma(2)$ denotes Γ with quadratic form scaled by 2). The image lattice will be called K. We prefer to work with metric isomorphisms and therefore denote the image

[1] $\mathbb{F}_2$ denotes the unique finite field with two elements.

in K of an element $a \in H^2(T,\mathbb{Z})$ by $\sqrt{2}a$. In particular, we write $\sqrt{2}\mu_j \wedge \mu_k, j,k = 1,\dots,4$ for the generators of K. The lattice $H^2(X,\mathbb{Z})$ contains $K \oplus \Pi$ and is contained in the dual lattice $K^* \oplus \Pi^*$. The three–plane $\Sigma \subset H^2(X,\mathbb{R})$ which describes the location of the singular Kummer surface within the moduli space (1.6) of Einstein metrics of volume 1 on $K3$ is given by $\Sigma = \pi_* \Sigma_T$.
A description of how the lattices K and Π are embedded in $H^2(X,\mathbb{Z})$ can be found in [Ni1]. First notice $K^*/K \cong (\mathbb{Z}_2)^6 \cong \Pi^*/\Pi$, where Π^*/Π is generated by $\{\frac{1}{2}\sum_{i\in P} E_i \mid P \subset I \text{ is a plane}\}$. The isomorphism $\gamma : K^*/K \longrightarrow \Pi^*/\Pi$ is most easily understood in terms of homology by assigning the image in X of a twocycle through four fixed points in a plane $P \subset I$ to $\frac{1}{2}\sum_{i\in P} E_i$. For example, $\gamma(\frac{1}{\sqrt{2}}\mu_j \wedge \mu_k) = \frac{1}{2}\sum_{i\in P_{jk}} E_i$, $P_{jk} = \text{span}_{\mathbb{F}_2}(f_j, f_k) \subset \mathbb{F}_2^4$, $f_j \in \mathbb{F}_2^4$ the jth standard basis vector. Note that P_{jk} may be exchanged by any of its translates $l + P_{jk}, l \in \mathbb{F}_2^4$. Next check that the discriminant forms of K^*/K and Π^*/Π, i.e. the induced $\mathbb{Q}/2\mathbb{Z}$ valued quadratic forms, agree up to a sign. Then

$$H^2(X,\mathbb{Z}) \cong \{(x,y) \in K^* \oplus \Pi^* \mid \gamma(\overline{x}) = \overline{y}\}, \tag{2.5}$$

$\overline{x}, \overline{y}$ denoting the images of x, y under projection to K^*/K, Π^*/Π. The isomorphism (2.5) provides a natural primitive embedding $K \perp \Pi \hookrightarrow H^2(X,\mathbb{Z})$, which is unique up to isomorphism [Ni1, Lemma 7]. Here, $H^2(X,\mathbb{Z}) \cong \Gamma^{3,19}$ is generated by

$$M := \Big\{ \tfrac{1}{\sqrt{2}}\mu_j \wedge \mu_k + \tfrac{1}{2}\sum_{i\in P_{jk}} E_{i+l}, l \in I \Big\} \text{ and } \text{span}_{\mathbb{Z}}(E_i, i \in I). \tag{2.6}$$

Hence $\Gamma^{3,3}(2) \cong H^2(T,\mathbb{Z})(2) \hookrightarrow H^2(X,\mathbb{Z}) \cong \Gamma^{3,19}$ is naturally embedded, and in particular $\Sigma \subset H^2(X,\mathbb{R}) \cong H^2(X,\mathbb{Z}) \otimes \mathbb{R}$ is obtained directly by regarding $\Sigma_T \subset H^2(T,\mathbb{R}) \cong H^2(T,\mathbb{Z}) \otimes \mathbb{R} \hookrightarrow H^2(X,\mathbb{Z}) \otimes \mathbb{R}$ as three–plane in $H^2(X,\mathbb{R})$.
To describe where the image $\mathcal{K}(\Lambda, B_T)$ of a superconformal field theory $\mathcal{T}(\Lambda, B_T)$ under $\mathbb{Z}_2$ orbifold is located in $\mathcal{M}^{K3}$ we now generalize the above construction to the quantum level. We have to lift π_* to an embedding $\widehat{\pi}_* : H^{even}(T,\mathbb{Z})(2) \hookrightarrow H^{even}(X,\mathbb{Z})$. The image will be denoted by $\widehat{K}$. Apart from $\mu_j \wedge \mu_k$ the lattice $H^{even}(T,\mathbb{Z})$ has generators υ, υ^0 as defined in (1.5). Note that $\widehat{K}$ cannot be embedded as primitive sublattice in $\Gamma^{4,20}$ such that $\widehat{K} \perp \Pi$ because $\widehat{K}^*/\widehat{K} \cong (\mathbb{Z}_2)^8 \ncong (\mathbb{Z}_2)^6 \cong \Pi^*/\Pi$. This means that the B-field of the orbifold theory must have components in the Picard lattice.
The torus model is given by a four–plane $x_T \subset H^{even}(T,\mathbb{R})$, the corresponding orbifold model by its image $x = \widehat{\pi}_* x_T$ in $H^{even}(X,\mathbb{Z}) \otimes \mathbb{R}$. To arrive at a complete description, we must find the embedding of $H^{even}(X,\mathbb{Z})$ in $\widehat{K} \otimes \mathbb{R} + H^2(X,\mathbb{R})$. Since scalar products with elements of $\widehat{K}$ must be integral and $\sqrt{2}\upsilon^0 \in \widehat{K}$, every $a \in \Pi$ must have a lift $\frac{1}{\sqrt{2}}\upsilon + a$ or $0 + a$ in $H^{even}(X,\mathbb{Z})$. Those elements for which the lift has the form $0 + a$ must form an $O^+(H^{even}(T,\mathbb{Z}))$ invariant sublattice of Π. One may easily check that this sublattice cannot contain the exceptional divisors $E_i, i \in I$. Moreover, as unimodular lattice $H^{even}(X,\mathbb{Z})$ must contain an element of the form $\frac{1}{\sqrt{2}}\upsilon^0 + a$ with $a \in \Pi^*$. One finds that $H^{even}(X,\mathbb{Z})$ must contain the set of elements

$$\widehat{M} := M \cup \Big\{ \tfrac{1}{\sqrt{2}}\upsilon^0 - \tfrac{1}{4}\sum_{i\in I} E_i;\ -\tfrac{1}{\sqrt{2}}\upsilon + E_i, i \in I \Big\}. \tag{2.7}$$

In analogy to Nikulin's description (2.5) and (2.6) of $H^2(X,\mathbb{Z}) \cong \Gamma^{3,19}$ we now find

Lemma 2.2
The lattice Γ spanned by $\widehat{M}$ and $\{\pi \in \Pi \mid \forall\, m \in \widehat{M} : \langle \pi, m\rangle \in \mathbb{Z}\}$ is isomorphic to $\Gamma^{4,20}$.

Proof:
Define

$$\widehat{\upsilon} := \sqrt{2}\upsilon, \quad \widehat{\upsilon}^0 := \tfrac{1}{\sqrt{2}}\upsilon^0 - \tfrac{1}{4}\sum_{i\in I} E_i + \sqrt{2}\upsilon, \quad \widehat{E_i} := -\tfrac{1}{\sqrt{2}}\upsilon + E_i. \tag{2.8}$$

Then Γ is generated by $\widehat{\upsilon}, \widehat{\upsilon}^0$ and the lattice

$$\widehat{\Gamma} := \operatorname{span}_{\mathbb{Z}}\left(\tfrac{1}{\sqrt{2}}\mu_j \wedge \mu_k + \tfrac{1}{2}\sum_{i\in P_{jk}} \widehat{E_{i+l}}, l \in I;\ \ \widehat{E_i}, i \in I\right).$$

Because $\langle \widehat{E_i}, \widehat{E_j}\rangle = -2\delta_{ij}$ and upon comparison to (2.6) it is now easy to see that $\widehat{\Gamma} \cong \Gamma^{3,19}$. Moreover, $\widehat{\upsilon}, \widehat{\upsilon}^0 \perp \widehat{\Gamma}$ and $\operatorname{span}_{\mathbb{Z}}(\widehat{\upsilon}, \widehat{\upsilon}^0) \cong U$ completes the proof. □

In particular, lemma 2.2 describes a natural embedding $\Gamma^{4,4}(2) \cong H^{even}(T,\mathbb{Z})(2) \hookrightarrow H^{even}(X,\mathbb{Z}) \cong \Gamma^{4,20}$. As in the case of embedding the Teichmüller spaces $\mathcal{T}^{3,3} \hookrightarrow \mathcal{T}^{3,19}$ this enables us to directly locate the image under $\mathbb{Z}_2$ orbifold of a conformal field theory corresponding to a four–plane $x \subset H^{even}(T,\mathbb{R}) \cong \Gamma^{4,4}\otimes\mathbb{R}$ within $\mathcal{M}^{K3}$ by regarding x as four–plane in $H^{even}(X,\mathbb{R}) \cong \Gamma^{4,20}\otimes\mathbb{R}$. Note that in this geometric interpretation $\widehat{\upsilon}, \widehat{\upsilon}^0$ are the generators of $H^4(X,\mathbb{Z})$ and $H^0(X,\mathbb{Z})$.

Theorem 2.3
Let (Σ_T, V_T, B_T) denote a geometric interpretation of the nonlinear σ model $\mathcal{T}(\Lambda, B_T)$ as given by (1.14). Then the corresponding orbifold conformal field theory $\mathcal{K}(\Lambda, B_T)$ associated to the Kummer surface $X = \mathcal{K}(\Lambda)$ has geometric interpretation (Σ, V, B) where $\Sigma \in \mathcal{T}^{3,19}$ as described after theorem 2.1, $V = \frac{V_T}{2}$ and $B = \frac{1}{\sqrt{2}}B_T + \frac{1}{2}B_{\mathbb{Z}}^{(2)}$, $B_{\mathbb{Z}}^{(2)} = \frac{1}{2}\sum_{i\in I}\widehat{E_i} \in H^{even}(X,\mathbb{Z})$ with $\widehat{E_i} \in H^{even}(X,\mathbb{Z})$ of length squared -2 given in (2.8).
In particular, the $\mathbb{Z}_2$ orbifold procedure induces an embedding $\mathcal{M}^{tori} \hookrightarrow \mathcal{M}^{K3}$ as quaternionic submanifold.

Proof:
Pick a basis $\sigma_i, i \in \{1,2,3\}$ of Σ_T. Then by (1.5) the nonlinear σ model $\mathcal{T}(\Lambda, B_T)$ is given by the four–plane x with generators $\xi_i = \sigma_i - \langle \sigma_i, B_T\rangle\upsilon, i \in \{1,2,3\}$ and $\xi_4 = \upsilon^0 + B_T + \left(V_T - \frac{\|B_T\|^2}{2}\right)\upsilon$. By the embedding $\Gamma^{4,4}\otimes\mathbb{R} \cong H^{even}(T,\mathbb{R}) \hookrightarrow H^{even}(X,\mathbb{R}) \cong \Gamma^{4,20}\otimes\mathbb{R}$ given in lemma 2.2 it is now a simple task to reexpress the generators of x using the generators $\widehat{\upsilon}, \widehat{\upsilon}^0$ of $H^4(X,\mathbb{Z})$ and $H^0(X,\mathbb{Z})$:

$$\begin{aligned}
\sqrt{2}\left(\sigma_i - \langle \sigma_i, B_T\rangle\upsilon\right) &= \sqrt{2}\sigma_i - \langle \sqrt{2}\sigma_i, \tfrac{1}{\sqrt{2}}B_T\rangle\widehat{\upsilon}\\
\tfrac{1}{\sqrt{2}}\left(\upsilon^0 + B_T + \left(V_T - \tfrac{\|B_T\|^2}{2}\right)\upsilon\right) &= \widehat{\upsilon}^0 + \tfrac{1}{\sqrt{2}}B_T + \tfrac{1}{2}B_{\mathbb{Z}}^{(2)}\\
&\quad + \left(\tfrac{V_T}{2} - \tfrac{1}{2}\left\|\tfrac{1}{\sqrt{2}}B_T + \tfrac{1}{2}B_{\mathbb{Z}}^{(2)}\right\|^2\right)\widehat{\upsilon}.
\end{aligned}$$

Comparison with (1.5) directly gives the assertion of the theorem. $\square$

Theorem 2.3 makes precise how the statement that orbifold conformal field theories tend to give value $B=\frac{1}{2}$ to the B-field in direction of exceptional divisors [As2, §4] is to be understood. Note that $x^\perp \cap \Gamma^{4,20}$ does not contain vectors of length squared -2, namely $E_i \in x^\perp, ||E_i||^2=-2$ but $E_i \notin H^{even}(X,\mathbb{Z})$. In the context of compactifactions of the type IIA string on $K3$ this proves that $\mathbb{Z}_2$ orbifold conformal field theories do not have enhanced gauge symmetry. A similar statement was made in [As1] and widely spread in the literature, but we were unable to follow the argument up to our result of theorem 2.3.

2.2. T–duality and Fourier–Mukai transform. By theorem 2.3 any automorphism on the Teichmüller space $\mathcal{T}^{4,4}$ of $\mathcal{M}^{tori}$ is conjugate to an automorphism on the Teichmüller space $\mathcal{T}^{4,20}$ of $\mathcal{M}^{K3}$. In particular, nonlinear σ models on tori related by T–duality must give isomorphic theories on $K3$ under $\mathbb{Z}_2$ orbifolding. To show this explicitly and discuss the duality transformation on $\mathcal{M}^{K3}$ obtained this way is the object of this subsection.
For simplicity first assume that our σ model on the torus $T=\mathbb{R}^4/\Lambda$ has vanishing B-field, where we have chosen a geometric interpretation $(\Sigma_T, V_T, 0)$. Then T–duality acts by $(\Sigma_T, V_T, 0)\mapsto(\Sigma_T, 1/V_T, 0)$. By theorem 2.3 the corresponding $\mathbb{Z}_2$ orbifold theories have geometric interpretations $(\Sigma, V_T/2, B)$ and $(\Sigma, 1/2V_T, B)$, respectively, where Σ is obtained as image of the embedding $\Sigma_T \subset H^2(T,\mathbb{R}) \hookrightarrow H^2(X,\mathbb{R})$ and $B=\frac{1}{2}B^{(2)}_{\mathbb{Z}}=\frac{1}{4}\sum_{i\in I}\widehat{E_i}$. We will now construct an automorphism Θ of the lattice $H^{even}(X,\mathbb{Z})$ which fixes the four–plane x corresponding to the model with geometric interpretation $(\Sigma, V_T/2, B)$ and acts by $V_T/2\mapsto 1/2V_T$. In other words, we will explicitly construct the duality transformation induced by torus T–duality on $\mathcal{M}^{K3}$. Our transformation Θ below was already given in [R-W] but not with complete proof. Within the context of boundary conformal field theories, in [B-E-R] it was shown that Θ induces an isomorphism on the corresponding conformal field theories. The relation to the Fourier–Mukai transform which we will show in theorem 2.4 has not been clarified up to now.
By (1.5), the four–plane $x\subset H^{even}(X,\mathbb{Z})$ is spanned by $\widetilde{\Sigma}=\xi(\Sigma)$ and the vector $\xi_4=\widehat{\upsilon}^0+B+(\frac{V_T}{2}+1)\widehat{\upsilon}$ (notations as in theorem 2.3). Because by the above Θ fixes x and $\widetilde{\Sigma}$ pointwise, the unit vector $\xi_4/\sqrt{V_T}\in\Sigma^\perp\cap x$ must be invariant, too, i.e. invariant under the transformation $V_T\mapsto 1/V_T$. Hence

$$\tfrac{1}{\sqrt{V_T}}\widehat{\upsilon}^0+\tfrac{1}{\sqrt{V_T}}B+\left(\tfrac{1}{2}\sqrt{V_T}+\tfrac{1}{\sqrt{V_T}}\right)\widehat{\upsilon}=\sqrt{V_T}\widetilde{\upsilon}^0+\sqrt{V_T}\widetilde{B}+\left(\tfrac{1}{2\sqrt{V_T}}+\sqrt{V_T}\right)\widetilde{\upsilon}$$

for any value of V_T. We set $\widetilde{\upsilon}:=\Theta(\widehat{\upsilon}), \widetilde{\upsilon}^0:=\Theta(\widehat{\upsilon}^0)$ etc. and deduce

$$\widetilde{\upsilon}^0+\widetilde{B}+\widetilde{\upsilon}=\tfrac{1}{2}\widehat{\upsilon},\quad \widehat{\upsilon}^0+B+\widehat{\upsilon}=\tfrac{1}{2}\widetilde{\upsilon}. \tag{2.9}$$

The first equation together with $\langle\widetilde{B},\widetilde{\upsilon}\rangle=\langle\widetilde{B},\widetilde{\upsilon}^0\rangle=0, ||\widetilde{B}||^2=-2$ implies $\langle\widetilde{B},\widehat{\upsilon}\rangle=-4$ and justifies the ansatz

$$\widetilde{B}=-4\widehat{\upsilon}^0-\sum_{i\in I}\alpha_i\widehat{E_i}+a\widehat{\upsilon}\implies\sum_{i\in I}(\alpha_i-1)^2=1,\quad\sum_{i\in I}\alpha_i=8-2a.$$

The only solutions satisfying $\sum_{i\in I}\alpha_i\widehat{E_i}\in H^{even}(X,\mathbb{Z})$, which must be true by (2.9), are $\alpha_{i_0}\in\{0,2\}$ for some $i_0\in I$ and $\alpha_i=1$ for $i\neq i_0$, correspondingly $a\in\{-\frac{7}{2},-\frac{9}{2}\}$. We conclude that if the automorphism Θ exists, then it is already uniquely determined up to the choice of a and of one point $i_0\in I$. The two possible choices of a turn out to be related by the B-field shift $\widetilde{B}\mapsto\widetilde{B}-2\widetilde{B}=-\widetilde{B}$ and yield equivalent results. In the following we pick $a=-\frac{7}{2}$ and find

$$\widetilde{\upsilon}=2(\widehat{\upsilon}+\widehat{\upsilon}^0)+\tfrac{1}{2}\sum_{i\in I}\widehat{E_i},\quad \widetilde{\upsilon}^0=2(\widehat{\upsilon}+\widehat{\upsilon}^0)+\tfrac{1}{2}\sum_{i\in I}\widehat{E_i}-\widehat{E}_{i_0}. \tag{2.10}$$

One easily checks that $\widetilde{U}:=\mathrm{span}_{\mathbb{Z}}(\widetilde{\upsilon},\widetilde{\upsilon}^0)\cong U$. By $\widetilde{\Pi}$ we denote the orthogonal complement of $\widetilde{U}$ in $\mathrm{span}_{\mathbb{Z}}(\widehat{\upsilon},\widehat{\upsilon}^0)\perp\Pi\cong U\perp\Pi$, where Π is the Kummer lattice of X as introduced in theorem 2.1. Note that in I there are 15 hyperplanes $H_i, i\in I_0=I-\{i_0\}$ which do not contain i_0. The label $i\in I_0$ is understood as the vector dual to the hyperplane H_i. Since the choice of i_0 can be seen as the choice of an origin in the affine space $\mathbb{F}_2^4$, the latter can be regarded as a vector space, and we have a unique natural isomorphism $(\mathbb{F}_2^4)^*\cong\mathbb{F}_2^4$. One now checks that $\widetilde{\Pi}$ is spanned by $\widetilde{E_i}, i\in I$, with

$$\widetilde{E}_{i_0}:=\widehat{\upsilon}-\widehat{\upsilon}^0,\quad \widetilde{E_i}:=-\tfrac{1}{2}\sum_{j\in H_i}\widehat{E_j}-\widehat{\upsilon}-\widehat{\upsilon}^0\quad (i\neq i_0) \tag{2.11}$$

as well as $\frac{1}{2}\sum_{i\in H}\widetilde{E_i}$ for any hyperplane $H\subset I$. The signs of the $\widetilde{E_i}$ have been chosen such that $\widetilde{B}=\frac{1}{2}\widetilde{B}^{(2)}_{\mathbb{Z}}=\frac{1}{4}\sum_{i\in I}\widetilde{E_i}$.
Since $\langle\widetilde{E_i},\widetilde{E_j}\rangle=-2\delta_{ij}$, one has $\widetilde{\Pi}\cong\Pi$. Hence $\Theta(\widehat{E_i})=\widetilde{E_i}$ is a continuation of (2.10) to an automorphism of lattices $U\perp\Pi\cong\widetilde{U}\perp\widetilde{\Pi}$, and we find $\Theta^2=\mathbb{1}$. Note that the action of Θ can be viewed as a duality transformation exchanging vectors $i\in I$ with hyperplanes $H_i, i\in I$. Two–planes $P\subset I$ are exchanged with their duals P^* which shows that Θ can be continued to a map on the entire lattice $H^*(X,\mathbb{Z})$ consistently with (2.5). The induced action on $K=\pi_*H^2(T,\mathbb{Z})$ leaves Σ invariant. We also see that the above procedure is easily generalized to arbitrary nonlinear σ models $\mathcal{T}(\Lambda,B_T)$.
Let S denote the classical symmetry which changes the sign of $\widehat{E}_{i_0}$ and leaves the other lattice generators $\widehat{E_i},\ i\neq i_0,\ \widehat{\upsilon},\ \widehat{\upsilon}^0,\ \mu_j\wedge\mu_k$ invariant. By (2.10) and (2.11) one has $\Theta S=T_{FM}\Theta$, where T_{FM} is the Fourier-Mukai transformation which exchanges $\widehat{\upsilon}$ with $\widehat{\upsilon}^0$. Since $T_{FM}=\Theta S\Theta$, all in all we have

Theorem 2.4
Torus T–duality induces a duality transformation Θ as given by (2.10) and (2.11) on the subspace of $\mathcal{M}^{K3}$ of theories associated to Kummer surfaces in the orbifold limit (see also [R-W]). The Fourier–Mukai transform T_{FM} which exchanges $\widehat{\upsilon}$ with $\widehat{\upsilon}^0$ is conjugate to a classical symmetry S by the image Θ of the T-duality map on theories associated to the torus.

Note that by theorem 2.4 we can prove Aspinwall's and Morrison's description (1.7) of the moduli space $\mathcal{M}^{K3}$ purely within conformal field theory without recourse to Landau–Ginzburg arguments. Namely, as explained in section 1, the group $\mathcal{G}^{(16)}$ needed to project from the Teichmüller space (1.1) to the component $\mathcal{M}^{K3}$ of the moduli space contains the group $O^+(H^2(X,\mathbb{Z}))$ of classical symmetries which fix the vectors $\widehat{v}, \widehat{v}^0$ determining our geometric interpretation. Moreover, for any primitive nullvector $\widetilde{v}^0$ with $\langle \widehat{v}, \widetilde{v}^0 \rangle = 1$ there exists an element $\widetilde{g} \in \mathcal{G}^{(16)}$ such that $\widetilde{g}\widehat{v} = \widehat{v}$ and $\widetilde{g}\widehat{v}^0 = \widetilde{v}^0$. By theorem 2.4 the symmetry $T_{FM} \in O^+(H^{even}(X,\mathbb{Z}))$ which exchanges $\widehat{v}$ and $\widehat{v}^0$ and leaves x invariant also is an element of $\mathcal{G}^{(16)}$, thus $O^+(H^{even}(X,\mathbb{Z})) \subset \mathcal{G}^{(16)}$ and $O^+(H^{even}(X,\mathbb{Z})) = \mathcal{G}^{(16)}$ under the assumption that $\mathcal{M}^{K3}$ is Hausdorff, as argued in section 1.

2.3. Algebraic automorphisms of Kummer surfaces. To describe strata of the moduli space $\mathcal{M}^{K3}$ we will study subspaces of the *Kummer stratum* found above which consist of theories with enhanced classical symmetry groups in the geometric interpretation given there. Concentrating on the geometric objects first, in this subsection we investigate algebraic automorphisms of Kummer surfaces which fix the orbifold singular metric. Such an automorphism induces an automorphism of the Kummer lattice Π because by $K \cong H^2(T,\mathbb{Z})(2)$ and (2.5) all the lattice vectors of length squared -2 in $\Sigma^\perp$ belong to Π, and $\Pi \otimes \mathbb{R}$ by theorem 2.1 is spanned by the lattice vectors $E_i, i \in I$ of length squared -2. Vice versa,

Lemma 2.5
The action of an algebraic automorphism α which fixes the orbifold singular metric on a Kummer surface X is uniquely determined by its action on the set $\{E_i \mid i \in I\}$ of forms corresponding to exceptional divisors, i.e. by an affine transformation $A_\alpha \in \mathrm{Aff}(I)$.

Proof:
Let α^* denote the induced automorphism on the Kummer lattice Π. By theorem 2.1 and (2.5) the intersection form on Π is negative definite and the $\pm E_i, i \in I$ are the only lattice vectors of length squared -2. Therefore, α^* is uniquely determined by $\alpha^*(E_i) = \varepsilon_i(\alpha) e_{A_\alpha(i)}$ for $i \in I$, where $\varepsilon_i(\alpha) \in \{\pm 1\}$ and $A_\alpha \in \mathrm{Aff}(I)$. Actually, $\varepsilon_i(\alpha) = \varepsilon_i(A_\alpha)$, because $A_\alpha(i) = \mathbb{1} \Longrightarrow \varepsilon_i(\alpha) = 1$ for otherwise $E_i \in (H^2(X,\mathbb{Z})^{\alpha^*})^\perp$ with length squared -2 contradicting theorem 1.1. Assume $A_\alpha = A_{\alpha'}$ for another algebraic automorphism α' fixing the metric. Then $g := (\alpha^{-1} \circ \alpha')^*$ acts trivially on Π, and because Σ is fixed by g as well, for the group G generated by $\alpha^{-1} \circ \alpha'$ we find $\mu(G) \geq 2 + 3 + 16 = 21$. Now (1.22) shows that G is trivial, proving $\alpha = \alpha'$. □

By abuse of language in the following we will frequently use the induced action of an algebraic automorphism on Π or in $\mathrm{Aff}(I)$ as a shorthand for the entire action.

Theorem 2.6
For every Kummer surface X the group of algebraic automorphisms fixing the orbifold singular metric contains $\mathbb{F}_2^4 \subset \mathrm{Aff}(I)$, which acts by translations on I.

Proof:
Any translation $t_i \in \mathrm{Aff}(I)$ by $i \in I$ acts trivially on Π^*/Π. Thus t_i can be continued trivially to $H^2(X,\mathbb{Z})$ by (2.5). One now easily checks that the resulting automorphism of $H^2(X,\mathbb{C})$ satisfies the criteria of theorem 1.1. □

Next we will determine the group of algebraic automorphisms for the Kummer surface associated to a torus with enhanced symmetry:

Theorem 2.7
The group of algebraic automorphisms fixing the orbifold singular metric of $X = \mathcal{K}(\Lambda), \Lambda \sim \mathbb{Z}^4$ *is* $\mathcal{G}^+_{Kummer} = \mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4$. *Here,* $\mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4 \subset GL(\mathbb{F}_2^4) \ltimes \mathbb{F}_2^4 = \mathrm{Aff}(I)$ *is equipped with the standard semidirect product.*
For $\widetilde{X} = \mathcal{K}(\widetilde{\Lambda})$, *where* $\widetilde{\Lambda}$ *is generated by* $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1, 2$, *the group of algebraic automorphisms fixing the orbifold singular metric generically is* $\widetilde{\mathcal{G}}^+_{Kummer} = \mathbb{Z}_2 \ltimes \mathbb{F}_2^4$.

Proof:
To demonstrate $\mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4 \subset \mathcal{G}^+_{Kummer}$ we will show that certain algebraic automorphisms on the underlying torus $T = \mathbb{R}^4/\Lambda$ can be pushed to X and generate an additional group of automorphisms $\mathbb{Z}_2^2 \subset GL(\mathbb{F}_2^4)$ on Π. Namely, in terms of standard coordinates $(x_1, \dots, x_4)$ on T, we are looking for automorphisms which leave the forms

$$dx_1 \wedge dx_3 + dx_4 \wedge dx_2, \quad dx_1 \wedge dx_4 + dx_2 \wedge dx_3, \quad dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \quad (2.12)$$

invariant. This is true for

$$\begin{aligned} r_{12}:\ & (x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, x_4, -x_3), \\ r_{13}:\ & (x_1, x_2, x_3, x_4) \mapsto (-x_3, -x_4, x_1, x_2), \\ r_{14} = r_{12} \circ r_{13}:\ & (x_1, x_2, x_3, x_4) \mapsto (x_4, -x_3, x_2, -x_1). \end{aligned} \quad (2.13)$$

The induced action on Π is described by permutations $A_{kl} \in \mathrm{Aff}(I)$ of the $\mathbb{F}_2^4$-coordinates, namely $r_{12} \mathrel{\widehat{=}} A_{12} = (12)(34), r_{13} \mathrel{\widehat{=}} A_{13} = (13)(24)$. To visualize this action we introduce the following helpful pictures first used by H. Inose [In]: The vertical line labelled by $j \in \mathbb{F}_2^2$ symbolizes the image of the twocycle

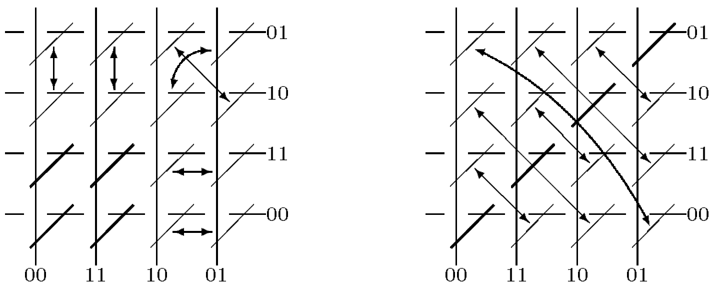


Fig. 2.1. Action of the algebraic automorphisms r_{12} (left) and r_{13} (right) on Π.

$\{x \in T \mid (x_1, x_2) = \frac{1}{2}j\}$ in X, and analogously for the horizontal line labelled by $j' \in \mathbb{F}_2^2$ we have $\{x \in T \mid (x_3, x_4) = \frac{1}{2}j'\}$. Then the diagonal lines from cycle j to cycle j' symbolize the exceptional divisor obtained from blowing up the fixed point labelled $(j, j') \in I$. Fat diagonal lines mark those exceptional divisors which are fixed by the respective automorphism.

One may now easily check that the automorphisms (2.13), viewed as automorphisms on $H^2(X,\mathbb{C})$, satisfy the criteria of theorem 1.1 and thus indeed are induced by algebraic automorphisms of X.
To see that $\mathcal{G}^+_{Kummer}$ does not contain any further elements, by lemma 2.5 it will suffice to show that no other element of $Aut(\Pi)$ can be continued to $H^2(X,\mathbb{Z})$ consistently such that it satisfies the criteria of theorem 1.1. Because all the translations of I are already contained in $\mathcal{G}^+_{Kummer}$ we can restrict our investigation to those elements $A\in GL(\mathbb{F}_2^4)\subset \mathrm{Aff}(I)$ which can be continued to $H^2(X,\mathbb{Z})$ preserving the symplectic forms on $\mathbb{F}_2^4$ that correspond to (2.12). After some calculation one finds that A must commute with all the transformations listed in (2.13). This means that A acts on I by $A'_{kl}(i)=A_{kl}(i)+|i|(1,1,1,1), |i|=\sum_k i_k\in\mathbb{F}_2$. But if any such $A'_{kl}\in\mathcal{G}^+_{Kummer}$, then also $A'\in\mathcal{G}^+_{Kummer}$, where $A'(i)=i+|i|(1,1,1,1)$. A' leaves invariant a sublattice of Π of rank 12. But then, because of (1.22) and from (1.21) A' cannot be induced by an algebraic automorphism fixing the orbifold singular metric of X.
The result for $\widetilde{\mathcal{G}}^+_{Kummer}$ follows from the above proof. Namely, if (x_1,x_2) are standard coordinates on $\Lambda_1\otimes\mathbb{R}$ and (x_3,x_4) on $\Lambda_2\otimes\mathbb{R}$, then among the automorphisms (2.13) only r_{12} is generically defined on $\widetilde{\Lambda}$. □

2.4. $\mathbb{Z}_4$ Orbifolds in the moduli space. This subsection is devoted to the study of $\mathbb{Z}_4$ orbifolds in the moduli space $\mathcal{M}^{K3}$. We first turn to some features of the $\mathbb{Z}_4$ orbifold construction on the conformal field theory side which need further discussion. From what was said at the beginning of the section, in terms of complex coordinates (2.1) on $T=\mathbb{R}^4/\Lambda$ the $\mathbb{Z}_4$ action on the nonlinear σ model is given by $(\psi^{(1)}_\pm,\psi^{(2)}_\pm)\mapsto(\pm i\psi^{(1)}_\pm,\mp i\psi^{(2)}_\pm)$. From (2.2) we readily read off that there always is a surviving $su(2)_1\oplus u(1)$ subalgebra of the holomorphic W-algebra generated by $J,J^\pm,A$. To have a $\mathbb{Z}_4$ symmetry on the entire space of states of the torus theory, the charge lattice (1.11) must obey this symmetry. So in addition to picking a $\mathbb{Z}_4$ symmetric torus, i.e. a lattice Λ generated by $\Lambda_i\cong R_i\mathbb{Z}^2, R_i\in\mathbb{R}, i=1,2$, we must have an appropriate B-field B_T in the nonlinear σ model on T which preserves this symmetry. In terms of cohomology we need $B_T\in H^2(T,\mathbb{R})^{\mathbb{Z}_4}=\mathrm{span}_\mathbb{R}(\mu_1\wedge\mu_2,\mu_3\wedge\mu_4,\mu_1\wedge\mu_3+\mu_4\wedge\mu_2,\mu_1\wedge\mu_4+\mu_2\wedge\mu_3)$. As in section 2.1 $(\mu_1,\dots,\mu_4)$ denotes a basis dual to $(\lambda_1,\dots,\lambda_4)$, λ_i being generators of Λ and $\Sigma_T\subset H^2(T,\mathbb{R})$ is regarded as giving the position of $H^2(T,\mathbb{Z})$ relative to a fixed three–plane $\mathrm{span}_\mathbb{R}(e_1\wedge e_2+e_3\wedge e_4, e_1\wedge e_3+e_4\wedge e_2, e_1\wedge e_4+e_2\wedge e_3)$. To determine the partition function, a lengthy but straightforward calculation using [E-O-T-Y, (5.2)-(5.5)] shows

$$
\begin{aligned}
&Z_{NS}(\tau,z)\\
&\quad=\tfrac12\left[\left\{\left(\tfrac12 Z_{\Lambda,B_T}(\tau)+\tfrac12\left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4+\tfrac12\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4+\tfrac12\left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^4\right)\left|\frac{\vartheta_3(z)}{\eta}\right|^4\right\}\right.\\
&\qquad\left.+\left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4\left|\frac{\vartheta_4(z)}{\eta}\right|^4+\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4\left|\frac{\vartheta_2(z)}{\eta}\right|^4+\left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^4\left|\frac{\vartheta_1(z)}{\eta}\right|^4\right],
\end{aligned}
\tag{2.14}
$$

where for $Z_{\Lambda,B_T}(\tau)$ one has to insert the expression for the specific torus T as obtained from (1.12). Comparing to (2.3) the partition function (2.14) coincides with that of the $\mathbb{Z}_2$ orbifold of a theory whose NS-partition function is the expression in curly brackets in (2.14). Indeed, the partition function of $SU(2)_1^4/\mathbb{Z}_4$, i.e. of the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\mathbb{Z}^4$ with $B_T = 0$, agrees with that of the $\mathbb{Z}_2$ orbifold $\mathcal{K}(D_4, 0)$ [E-O-T-Y]. In section 2.1 we showed that every $\mathbb{Z}_2$ orbifold conformal field theory has an $su(2)_1^2$ subalgebra of the holomorphic W-algebra. On the other hand, as demonstrated above, the $\mathbb{Z}_4$ orbifold generically only possesses an $su(2)_1 \oplus u(1)$ current algebra. For $SU(2)_1^4/\mathbb{Z}_4$ this is enhanced to $su(2)_1 \oplus u(1)^3$ which still does not agree with the one for Kummer surfaces. Hence although the theories have the same partition function, they are not isomorphic.
Similarly, the partition function of the $\mathbb{Z}_4$ orbifold of the torus model with $SO(8)_1$ symmetry agrees with that of $\mathcal{K}(\mathbb{Z}^4, 0)$ as can be seen from (3.7). In this case the theories indeed are the same as will be shown in theorem 3.9.
To have a better understanding of their location within the moduli space and their geometric properties we now construct $\mathbb{Z}_4$ orbifolds by applying another orbifold procedure to theories with enhanced symmetries which have already been located in moduli space.

Theorem 2.8
Let $\widetilde{\Lambda}$ denote a lattice generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1, 2$. Consider the $K3$ surface X obtained from the Kummer surface $\mathcal{K}(\widetilde{\Lambda})$ by modding out the algebraic automorphism $r_{12} \in \widetilde{\mathcal{G}}^+_{Kummer}$, blowing up the singularities and using the induced orbifold singular metric. Then X is the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\widetilde{\Lambda}$.

Proof:
By construction (2.13), r_{12} is induced by the automorphism $(x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, x_4, -x_3)$ with respect to standard coordinates on T. In terms of complex coordinates as in (2.1) this is just the action $\rho : (z_1, z_2) \mapsto (iz_1, -iz_2)$, and because $\mathcal{K}(\Lambda) = \widetilde{T/\rho^2}$, the assertion is clear. □

Remark:
Study figure 2.1 to see how the structure $A_1^6 \oplus A_3^4$ of the exceptional divisors in the $\mathbb{Z}_4$ orbifold comes about: Twelve of the fixed points in $\mathcal{K}(\widetilde{\Lambda})$ are identified pairwise to yield six $\mathbb{Z}_2$ fixed points in the $\mathbb{Z}_4$ orbifold, that is A_1^6. The four points labelled $i \in \{(0,0,0,0), (1,1,0,0), (0,0,1,1), (1,1,1,1)\}$ are true $\mathbb{Z}_4$ fixed points. The induced action of r_{12} on the corresponding exceptional divisor $\mathbb{CP}^1 \cong \mathbb{S}^2$ is just a 180° rotation about the north-south axis, and north and south poles are fixed points. Blow up the resulting singularities in $\mathcal{K}(\Lambda)/r_{12}$ to see how an A_3 arises from the A_1 over each true $\mathbb{Z}_4$ fixed point.

For a $\mathbb{Z}_4$ orbifold X there is an analog $\Pi\!\!\Pi$ of the Kummer lattice Π described in theorem 2.1, the primitive sublattice of $Pic(X)$ containing all the twoforms which correspond to exceptional divisors by Poincaré duality. We will give an analogous description of $\Pi\!\!\Pi$ as for Π in lemma 2.9 below. The embedding of the moduli space of $\mathbb{Z}_4$ orbifolds in $\mathcal{M}^{K3}$ then works analogously to that of $\mathbb{Z}_2$ orbifold conformal field theories as described in subsection 2.1.
Let us fix some notations. Let $\pi : T \to X$ denote the rational map of degree four. Then $K := \pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_4} = \mathrm{span}_{\mathbb{Z}}\,(2\mu_1 \wedge \mu_2, 2\mu_3 \wedge \mu_4, \mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2, \mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3)$. For the twoforms corresponding to the exceptional divisors

of the $\mathbb{Z}_4$ orbifold we adopt the labelling of fixed points by $I \cong \mathbb{F}_2^4$ as used in the $\mathbb{Z}_2$ orbifold case. Here, we have six $\mathbb{Z}_2$ fixed points labelled by $i \in I^{(2)} := \{(j_1, j_2, 1, 0), (1, 0, j_3, j_4) \mid j_k \in \mathbb{F}_2\}$. The four true $\mathbb{Z}_4$ fixed points are labelled by $i \in I^{(4)} := \{(i, i, j, j) \mid i, j \in \mathbb{F}_2\}$. The corresponding twoforms are denoted by E_i for $i \in I^{(2)}$, and for each $\mathbb{Z}_4$ fixed point $i \in I^{(4)}$ we have three exceptional divisors Poincaré dual to $E_i^{(\pm)}, E_i^{(0)}$ such that $\langle E_i^{(\pm)}, E_i^{(0)} \rangle = 1$, $\langle E_i^{(+)}, E_i^{(-)} \rangle = 0$. For ease of notation we also use the combination $E_i := 3E_i^{(+)} + 2E_i^{(0)} + E_i^{(-)}$ if $i \in I^{(4)}$.
As a first step we determine the analogs of (2.5) and (2.6) in order to describe the primitive embedding $K \perp \Pi\!\!\Pi \hookrightarrow H^2(X, \mathbb{Z})$. By (2.5) images $\kappa \in K^*$ of forms corresponding to torus cycles do not necessarily correspond to cycles in $H_2(X, \mathbb{Z})$. Namely, the Poincaré dual of a representative κ of $\overline{\kappa} \in K^*/K$ built from combinations of $\frac{1}{2}\mu_j \wedge \mu_k$ can be interpreted as the π_* image of a torus cycle which contains $\mathbb{Z}_4$ fixed points. It is not a cycle on X, since it has boundaries where the exceptional divisors were glued in instead of the fixed points by the blow up procedure. Since the discriminant forms of K^*/K and $\Pi\!\!\Pi^*/\Pi\!\!\Pi$ agree up to a sign, there is a representative η of $\overline{\eta} \in \Pi\!\!\Pi^*/\Pi\!\!\Pi$ in the image of $\overline{\kappa}$ whose Poincaré dual has the same boundary as κ but orientation reversed. We can glue a part η of a rational sphere into the boundary of κ to obtain a cycle $\kappa + \eta \in H^2(X, \mathbb{Z})$, where up to a sign κ has the same intersection number as η with every exceptional divisor.
We again adopt the notation $P_{jk} = \mathrm{span}_{\mathbb{F}_2}(f_j, f_k)$ used in subsection 2.1. Remember to count $\mathbb{Z}_2$ fixed points only once, e.g. $P_{12} = \{(0,0,0,0), (1,0,0,0), (1,1,0,0)\}$. We then have

Lemma 2.9
The lattice generated by the set M which consists of

$$\tfrac{1}{2}\mu_1 \wedge \mu_2 - \tfrac{1}{2}E_{(0,1,0,0)+\varepsilon(0,0,1,1)} - \tfrac{1}{4}\sum_{i \in P_{12} \cap I^{(4)}} E_{i+\varepsilon(0,0,1,1)}, \quad \varepsilon \in \{0,1\};$$

$$\tfrac{1}{2}\mu_3 \wedge \mu_4 + \tfrac{1}{2}E_{(0,0,0,1)+\varepsilon(1,1,0,0)} + \tfrac{1}{4}\sum_{i \in P_{34} \cap I^{(4)}} E_{i+\varepsilon(1,1,0,0)}, \quad \varepsilon \in \{0,1\};$$

$$\tfrac{1}{2}\left(\mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2\right) - \tfrac{1}{2}\sum_{i \in P_{13}} E_{i+j}, \quad j \in I^{(4)};$$

$$\tfrac{1}{2}\left(\mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3\right) - \tfrac{1}{2}\sum_{i \in P_{14}} E_{i+j}, \quad j \in I^{(4)};$$

and by $\mathcal{E} := \{E_i^{(\pm)}, E_i^{(0)}, i \in I^{(4)}; E_i, i \in I^{(2)}\}$ is isomorphic to $\Gamma^{3,19}$. In particular, $\Pi\!\!\Pi$ is generated by $\mathcal{E}$ and

$$\tfrac{1}{4}\left(E_{(0,0,0,0)} + E_{(1,1,1,1)} - E_{(0,0,1,1)} - E_{(1,1,0,0)}\right) + \tfrac{1}{2}\left(E_{(0,1,0,1)} + E_{(0,1,1,0)}\right),$$

$$\tfrac{1}{2}\left(E_{(0,0,0,0)} + E_{(0,0,1,1)} + E_{(0,1,0,0)} + E_{(0,1,1,1)} + E_{(0,1,0,1)} + E_{(0,1,1,0)}\right),$$

$$\tfrac{1}{2}\left(E_{(1,1,0,0)} + E_{(0,0,1,1)} + E_{(0,0,0,1)} + E_{(0,1,0,0)} + E_{(1,1,0,1)} + E_{(0,1,1,1)}\right).$$

This gives a natural embedding $K \perp \Pi\!\!\Pi \hookrightarrow H^2(X, \mathbb{Z})$, and $(H^2(T, \mathbb{Z}))^{\mathbb{Z}_4} \hookrightarrow H^2(X, \mathbb{Z}) \cong \Gamma^{3,19}$. Given a Kähler–Einstein metric in $\mathcal{T}^{3,3}$ defined by $\Sigma_T \subset$

$H^2(T,\mathbb{R})^{\mathbb{Z}_4}$, its image Σ under the $\mathbb{Z}_4$ orbifold procedure is read off from $\Sigma_T \subset H^2(T,\mathbb{R})^{\mathbb{Z}_4} \cong (H^2(T,\mathbb{Z}))^{\mathbb{Z}_4} \otimes \mathbb{R} \hookrightarrow H^2(X,\mathbb{Z}) \otimes \mathbb{R} \cong H^2(X,\mathbb{R})$.

In order to prove lemma 2.9 one has to show that the lattice under inspection has signature $(3,19)$ and is self dual. We omit the tedious calculation. The construction will be described in more detail in [We].
To give the location in $\mathcal{M}^{K3}$ of the image of $\mathcal{T}(\Lambda, B_T)$ under the $\mathbb{Z}_4$ orbifold we have to lift the above picture to the quantum level. As before, $H^{even}(T,\mathbb{Z}) \cong \Gamma^{4,4}$ is generated by $\mu_j \wedge \mu_k$ and υ, υ^0 defined in (1.5). As in (2.7) we extend the set M of lemma 2.9 to $\widehat{M} := M \cup \{\widehat{\upsilon}, \widehat{\upsilon}^0\}$ by

$$\widehat{\upsilon} := 2\upsilon, \quad \widehat{\upsilon}^0 := \tfrac{1}{2}\upsilon^0 - \tfrac{1}{4}\sum_{i\in I^{(2)}} E_i - \tfrac{1}{8}\sum_{i\in I^{(4)}} \left(3E_i^{(+)} + 4E_i^{(0)} + 3E_i^{(-)}\right) + 2\upsilon.$$

Defining

$$\begin{aligned} &\text{for } i \in I^{(4)}: \ \widehat{E}_i^{(\pm)} := -\tfrac{1}{2}\upsilon + E_i^{(\pm)}, \quad \widehat{E}_i^{(0)} := -\tfrac{1}{2}\upsilon + E_i^{(0)}, \\ &\text{for } i \in I^{(2)}: \ \widehat{E}_i := -\upsilon + E_i \end{aligned} \tag{2.15}$$

one now checks in exactly the same fashion as in lemma 2.2

Lemma 2.10
The lattice generated by $\widehat{M}$ and $\{\pi \in \mathrm{span}_{\mathbb{Z}}(\widehat{E}_i^{(\pm)}, \widehat{E}_i^{(0)}, i \in I^{(4)}; \widehat{E}_i, i \in I^{(2)}) \mid \forall\, m \in \widehat{M} : \langle \pi, m\rangle \in \mathbb{Z}\}$ is isomorphic to $\Gamma^{4,20}$.

The embedding $H^{even}(T,\mathbb{Z})^{\mathbb{Z}_4} \hookrightarrow H^{even}(X,\mathbb{Z})$ that is now established actually is the unique one up to lattice automorphisms (see [We], where also the other $\mathbb{Z}_M$ orbifold conformal field theories, $M \in \{3,6\}$, will be treated). Now use

$$B_{\mathbb{Z}}^{(4)} := \sum_{i\in I^{(2)}} \widehat{E}_i + \tfrac{1}{2}\sum_{i\in I^{(4)}} \left(3\widehat{E}_i^{(+)} + 4\widehat{E}_i^{(0)} + 3\widehat{E}_i^{(-)}\right) \in H^{even}(X,\mathbb{Z}) \tag{2.16}$$

to find

$$\begin{aligned} 2\left(\sigma_i - \langle \sigma_i, B_T\rangle \upsilon\right) &= 2\sigma_i - \langle 2\sigma_i, \tfrac{1}{2}B_T\rangle \widehat{\upsilon} \\ \tfrac{1}{2}\left(\upsilon^0 + B_T + \left(V - \tfrac{\|B_T\|^2}{2}\right)\upsilon\right) &= \widehat{\upsilon}^0 + \tfrac{1}{2}B_T + \tfrac{1}{4}B_{\mathbb{Z}}^{(4)} \\ &\quad + \left(\tfrac{V_T}{4} - \tfrac{1}{2}\left\| \tfrac{1}{2}B_T + \tfrac{1}{4}B_{\mathbb{Z}}^{(4)}\right\|^2\right)\widehat{\upsilon}, \end{aligned}$$

hence

Theorem 2.11
Let (Σ_T, V_T, B_T) denote a geometric interpretation of the nonlinear σ model $\mathcal{T}(\Lambda, B_T)$ as given by (1.14). Assume that Λ is generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1,2$, and $B_T \in H^2(T,\mathbb{Z})^{\mathbb{Z}_4}$ such that a $\mathbb{Z}_4$ action is well defined on $\mathcal{T}(\Lambda, B_T)$. Then the image $x \in \mathcal{T}^{4,20}$ under the $\mathbb{Z}_4$ orbifold procedure has geometric interpretation (Σ, V, B) where $\Sigma \in \mathcal{T}^{3,19}$ is found as described in lemma 2.9, $V = \frac{V_T}{4}$, and $B = \frac{1}{2}B_T + \frac{1}{4}B_{\mathbb{Z}}^{(4)}$, $B_{\mathbb{Z}}^{(4)} \in H^{even}(X,\mathbb{Z})$ as in (2.16).
In particular, the moduli space of superconformal field theories admitting an interpetation as $\mathbb{Z}_4$ orbifold is a quaternionic submanifold of $\mathcal{M}^{K3}$. Moreover,

$x^\perp \cap H^{even}(X,\mathbb{Z})$ does not contain vectors of length squared -2.

Note that from (2.16) it is easy to read off the flow of the B-field obtained from the orbifold procedure through an A_3 divisor over one of the true $\mathbb{Z}_4$ fixed points of X: On integration over any of the divisors that correspond to a $\mathbb{Z}_m$ fixed point, we get B–field flux $\frac{1}{m}$. This is also true for the other $\mathbb{Z}_M$ orbifold conformal field theories and confirms earlier results [Do,B-I] obtained in the context of brane physics.

Theorem 2.11 proves that $\mathbb{Z}_4$ orbifold conformal field theories do not correspond to string compactifications of the type IIA string on $K3$ with enhanced gauge symmetry. Concerning the algebraic automorphism group of $\mathbb{Z}_4$ orbifolds we can prove

Theorem 2.12

Let X denote the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\Lambda$. Then the group $\mathcal{G}$ of algebraic automorphisms fixing the orbifold singular metric of X consists of all the residual symmetries induced by algebraic automorphisms of $\mathcal{K}(\Lambda)$ which commute with r_{12}. Thus, generically $\mathcal{G} \cong \mathbb{F}_2^2$ is generated by the induced actions of t_{1100} and t_{0011}. If $\Lambda \sim \mathbb{Z}^4$, $\mathcal{G} \cong D_4$ is generated by the induced actions of t_{1100} and r_{13}.

If we want invariance of the conformal field theory under the entire group $\mathcal{G} \cong D_4$ of algebraic automorphisms found in theorem 2.12 we must restrict B_T to values such that $B_T \in H^2(T,\mathbb{R})^{\mathbb{Z}_4} \cap H^2(X,\mathbb{R})^{D_4} = \Sigma$ where we regard $H^2(T,\mathbb{R})^{\mathbb{Z}_4} \hookrightarrow H^2(X,\mathbb{R})$ as described in lemma 2.9. If B_T is viewed as element of $Skew(4)$ acting on $\mathbb{R}^4$ this condition is equivalent to B_T commuting with the automorphisms listed in (2.13).

2.5. Application: Fermat's description for $SU(2)_1^4/\mathbb{Z}_4$.

Theorem 2.13

The $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\mathbb{Z}^4,0)$ admits a geometric interpretation on the Fermat quartic

$$\mathcal{Q} = \left\{ (x_0,x_1,x_2,x_3) \in \mathbb{CP}^3 \,\Big|\, \sum_{i=0}^{3} x_i^4 = 0 \right\} \tag{2.17}$$

in $\mathbb{CP}^3$ with volume $V_\mathcal{Q} = \frac{1}{2}$ and B–field $B_\mathcal{Q} = -\frac{1}{2}\sigma_1^{(\mathcal{Q})}$ up to a shift in $H^2(X,\mathbb{Z})$, where $\sigma_1^{(\mathcal{Q})}$ denotes the Kähler class of $\mathcal{Q}$.

Proof:

Let $e_1,\dots,e_4$ denote the standard basis of $\mathbb{Z}^4$. Then $\mu_i = e_i$, and by theorem 2.11 with $\|B_\mathbb{Z}^{(4)}\|^2 = -32$ the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\mathbb{Z}^4,0)$ is described by the four–plane $x \in \mathcal{T}^{4,20}$ spanned by

$$\begin{aligned} \xi_1 &= \mu_1\wedge\mu_3 + \mu_4\wedge\mu_2, & \xi_2 &= \mu_1\wedge\mu_4 + \mu_2\wedge\mu_3,\\ \xi_3 &= 2(\mu_1\wedge\mu_2 + \mu_3\wedge\mu_4), & \xi_4 &= 4\widehat{\upsilon}^0 + B_\mathbb{Z}^{(4)} + 5\widehat{\upsilon}. \end{aligned}$$

To read off a different geometric interpretation, we define

$$\begin{aligned} \upsilon_\mathcal{Q} &:= \tfrac{1}{2}\left(\mu_1\wedge\mu_3 + \mu_4\wedge\mu_2 - \mu_1\wedge\mu_4 - \mu_2\wedge\mu_3\right)\\ &\quad + \tfrac{1}{2}\left(\widehat{E}_{(0,1,1,0)} - \widehat{E}_{(1,0,1,0)}\right),\\ \upsilon_\mathcal{Q}^0 &:= \mu_1\wedge\mu_3 + \mu_4\wedge\mu_2 + \mu_1\wedge\mu_2\\ &\quad + \tfrac{1}{2}\left(\widehat{E}_{(0,0,0,1)} + \widehat{E}_{(1,1,0,1)} - \widehat{E}_{(0,1,1,0)} - \widehat{E}_{(1,0,1,0)}\right). \end{aligned} \tag{2.18}$$

One checks $\upsilon_{\mathcal{Q}}, \upsilon^0_{\mathcal{Q}} \in H^{even}(X,\mathbb{Z})$ as given in lemma 2.10, $||\upsilon_{\mathcal{Q}}||^2 = ||\upsilon^0_{\mathcal{Q}}||^2 = 0$ and $\langle \upsilon_{\mathcal{Q}}, \upsilon^0_{\mathcal{Q}} \rangle = 1$ to show that $\upsilon_{\mathcal{Q}}, \upsilon^0_{\mathcal{Q}}$ is an admissible choice for nullvectors in (1.5). For the corresponding geometric interpretation $(\Sigma_{\mathcal{Q}}, V_{\mathcal{Q}}, B_{\mathcal{Q}})$ we find that $\Sigma_{\mathcal{Q}}$ is spanned by

$$\begin{aligned}
\sigma_1^{(\mathcal{Q})} &= \mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2 + \mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3 - 2\upsilon_{\mathcal{Q}},\\
\sigma_2^{(\mathcal{Q})} &= 2(\mu_1 \wedge \mu_2 + \mu_3 \wedge \mu_4) - 2\upsilon_{\mathcal{Q}},\\
\sigma_3^{(\mathcal{Q})} &= 4\widehat{\upsilon}^0 + B^{(4)}_{\mathbb{Z}} + 5\widehat{\upsilon}.
\end{aligned}$$

As complex structure $\Omega_{\mathcal{Q}} \subset \Sigma_{\mathcal{Q}}$ we pick the two–plane spanned by $\sigma_2^{(\mathcal{Q})}$ and $\sigma_3^{(\mathcal{Q})}$. Note that this plane is generated by lattice vectors, so the *Picard number* $\rho(X) := \mathrm{rk}\; Pic(X) = \mathrm{rk}\,(\Omega^\perp \cap H^2(X,\mathbb{Z}))$ of the corresponding geometric interpretation X is 20, the maximal possible value. $K3$ surfaces with Picard number 20 are called *singular* and are classified by the quadratic form on their *transcendental lattice* $Pic(X)^\perp \cap H^2(X,\mathbb{Z})$. In other words there is a one to one correspondence between singular $K3$ surfaces and even quadratic positive definite forms modulo $SL(2,\mathbb{Z})$ equivalence [Shi][2]. Because $\sigma_2^{(\mathcal{Q})}, \sigma_3^{(\mathcal{Q})}$ are primitive lattice vectors, one now easily checks that X equipped with the complex structure given by $\Omega_{\mathcal{Q}}$ has quadratic form $\mathrm{diag}(8,8)$ on the transcendental lattice. By [In] this means that our variety indeed is the Fermat quartic (2.17) in $\mathbb{CP}^3$.
Volume and B-field can now be read off using (1.5) and noting that in our geometric interpretation

$$\mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2 - \mu_1 \wedge \mu_4 - \mu_2 \wedge \mu_3 = \xi_4^{(\mathcal{Q})} \sim \upsilon^0_{\mathcal{Q}} + B_{\mathcal{Q}} + \left(V_{\mathcal{Q}} - \tfrac{1}{2}\,||B_{\mathcal{Q}}||^2\right) \upsilon_{\mathcal{Q}}.$$

$\square$

3. Special points in moduli space: Gepner and Gepner type models

Finally we discuss the probably best understood models of superconformal field theories associated to $K3$ surfaces, namely Gepner models [Ge1,Ge2]. The latter are rational conformal field theories and thus exactly solvable. For a short account on the Gepner construction and its most important features in the context of our investigations see appendix A. In this section, we explicitly locate the Gepner model $(2)^4$ and some of its orbifolds within the moduli space $\mathcal{M}^{K3}$. This is achieved by giving σ model descriptions of these models in terms of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds which we know how to locate in moduli space by the results of section 2.

3.1. Discrete symmetries of Gepner models and algebraic automorphisms of $K3$ surfaces. As argued before, a basic tool to characterize a given conformal field theory is the study of its discrete symmetry group. We will first discuss the abelian group given by phase symmetries of a Gepner model $\prod_{j=1}^r (k_j)$ with

[2] We thank Noriko Yui and Yasuhiro Goto for drawing our attention to the relevant literature concerning singular $K3$ surfaces.

central charge $c=6$ and r even [Ge1]. Recall that this theory is obtained from the fermionic tensor product of the $N=2$ superconformal minimal models $(k_j), j=1,\dots,r$, by modding out a cyclic group $\mathcal{Z}\cong\mathbb{Z}_n$, $n=\operatorname{lcm}\{2;k_i+2,i=1,\dots,r\}$. The model therefore inherits a $\mathbb{Z}_{k_j+2}$ symmetry from the parafermionic subtheories of each minimal model factor (k_j) whose generator in the bosonic sector acts by

$$\Phi^{l_j}_{m_j,s_j;\overline{m}_j,\overline{s}_j} \longmapsto e^{\frac{2\pi i}{2(k_j+2)}(m_j+\overline{m}_j)}\,\Phi^{l_j}_{m_j,s_j;\overline{m}_j,\overline{s}_j} \tag{3.1}$$

on the jth factor. The resulting abelian symmetry group of $\prod_{j=1}^r(k_j)$ is $\mathbb{Z}_2\times\mathcal{G}_{ab}$, where $\mathbb{Z}_2$ denotes charge conjugation and $\mathcal{G}_{ab}=(\prod_{j=1}^r\mathbb{Z}_{k_j+2})/\mathbb{Z}_m, m=\operatorname{lcm}\{k_i+2,i=1,\dots,r\}$. Here, $\mathbb{Z}_m$ acts by

$$\prod_{j=1}^r\mathbb{Z}_{k_j+2}\longrightarrow\prod_{j=1}^r\mathbb{Z}_{k_j+2},\quad [a_1,\dots,a_r]\longmapsto[a_1+1,\dots,a_r+1]$$

(see also [G-P]). Note that only elements of the subgroup

$$\mathcal{G}^{alg}_{ab}:=\left\{[a_1,\dots,a_r]\in\mathcal{G}_{ab}\;\middle|\;\sum_{j=1}^r\frac{a_j}{k_j+2}\in\mathbb{Z}\right\}\subset\mathcal{G}_{ab} \tag{3.2}$$

commute with spacetime supersymmetry, elements of $\mathcal{G}_{ab}-\mathcal{G}^{alg}_{ab}$ describe R-symmetries [Ge1].

Assume we can locate our Gepner model within $\mathcal{M}^{K3}$, that is we explicitly know the corresponding four–plane $x\subset H^{even}(X,\mathbb{R})$ as described in section 1. Furthermore assume that by picking a primitive nullvector $\upsilon\in H^{even}(X,\mathbb{Z})$ we have chosen a specific geometric interpretation (Σ,V,B). By construction, a Gepner model comes with a specific choice of the $N=(2,2)$ subalgebra corresponding to a specific twoplane $\Omega\subset\Sigma$. We stress that this is true for any geometric interpretation of $\prod_j(k_j)$: The choice of the $N=(2,2)$ subalgebra does *not* fix a complex structure *a priori*, it fixes a choice of complex structure *in every geometric interpretation* of our model, as was explained in section 1. Still, we now assume our $K3$ surface X to be equipped with complex structure and Kähler metric. By our discussion in section 1.2 we know that any symmetry of the Gepner model which leaves the $su(2)^{susy}_l\oplus su(2)^{susy}_r$ currents $J,J^\pm,\overline{J},\overline{J}^\pm$ and the vector υ invariant may act as an algebraic automorphism on X. Because $J^\pm=\left(\Phi^0_{\mp2,2;0,0}\right)^{\otimes r}$ and $\overline{J}^\pm=\left(\Phi^0_{0,0;\mp2,2}\right)^{\otimes r}$ (see appendix A) we conclude from (3.2) that elements of $\mathcal{G}^{alg}_{ab}$ can act as algebraic automorphisms on X fixing the B-field $B\in H^2(X,\mathbb{R})$, and vice versa. More explicitly by what was said in section 1.2, the action of such a Gepner-symmetry on the $(\frac12,\frac12)$-fields with charges, say, $Q=\overline{Q}=1$ should be identified with the induced action of an algebraic automorphism of X on $H^{1,1}(X,\mathbb{R})$. With reference to its possible geometric interpretation we call $\mathcal{G}^{alg}_{ab}$ the *abelian algebraic symmetry group of the Gepner model*.

In the following subsections we will investigate where in the moduli space of superconformal field theories associated to $K3$ surfaces to locate the Gepner model $(2)^4$ and some of its orbifolds by elements of $\mathcal{G}^{alg}_{ab}\cong(\mathbb{Z}_4)^2$. From the above discussion it is clear that given a definite geometric interpretation for $(2)^4$

the geometric interpretation of its orbifold models is obtained by modding out the corresponding algebraic automorphisms.
Apart from symmetries in $\mathbb{Z}_2 \times \mathcal{G}_{ab}$ our Gepner model will possess permutation symmetries involving identical factor theories. Their discussion is a bit more subtle, because as noted in [F-K-S] permuting fermionic fields will involve additional signs (A.9). This in particular applies to $J^\pm = \left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes r}$, meaning that odd permutations can only act algebraically when accompanied by a phase symmetry

$$[a_1,\dots,a_r] \in \mathcal{G}_{ab}: \quad \sum_{j=1}^r \frac{a_j}{k_j+2} \in \mathbb{Z} + \tfrac{1}{2}. \tag{3.3}$$

We will discuss this phenomenon in detail for the example of prime interest to us, namely the Gepner model $(2)^4$. Here $\mathcal{G}^{alg}_{ab} \cong (\mathbb{Z}_4)^2$, and the entire algebraic symmetry group is generally believed to be $\mathcal{G}^{alg} \cong (\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ [As1]. Moreover, based on Landau-Ginzburg computations and comparison of symmetries [G-V-W, G-P, F-K-S-S, As1] it is generally believed that $(2)^4$ has a geometric interpretation $(\Sigma_\mathcal{Q}, V_\mathcal{Q}, B_\mathcal{Q})$ given by the Fermat quartic (2.17) in $\mathbb{CP}^3$. Indeed, $\mathcal{Q}$ is a $K3$ surface with algebraic automorphism group $(\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ [Mu], and arguments in favour of the viewpoint that it yields a geometric interpretation of $(2)^4$ will arise from the following discussion. It is proved in corollary 3.6.
To give the action of the two generators $[1,3,0,0]$ and $[1,0,3,0]$ of $\mathcal{G}^{alg}_{ab} \cong (\mathbb{Z}_4)^2$ on the $\left(\frac{1}{2},\frac{1}{2}\right)$-fields with charges $Q = \overline{Q} = 1$ we use the shorthand notation

$$\begin{aligned} X &:= (\Phi^1_{1,0;-3,2})^{\otimes 4}, \\ Y(n_1,n_2,n_3,n_4) &:= \Phi^{n_1}_{n_1,0;n_1,0} \otimes \Phi^{n_2}_{n_2,0;n_2,0} \otimes \Phi^{n_3}_{n_3,0;n_3,0} \otimes \Phi^{n_4}_{n_4,0;n_4,0} \end{aligned} \tag{3.4}$$

($n_i \in \mathbb{N}$) and find

$[1,3,0,0] \rightarrow$ $\downarrow [1,0,3,0]$	1	-1	i	$-i$
1	$Y(1,1,1,1)$, X	$Y(0,2,0,2)$, $Y(2,0,2,0)$	$Y(1,0,1,2)$	$Y(1,2,1,0)$
-1	$Y(2,2,0,0)$, $Y(0,0,2,2)$	$Y(2,0,0,2)$, $Y(0,2,2,0)$	$Y(2,1,0,1)$	$Y(0,1,2,1)$
i	$Y(1,1,0,2)$	$Y(2,0,1,1)$	$Y(2,1,1,0)$	$Y(1,2,0,1)$
$-i$	$Y(1,1,2,0)$	$Y(0,2,1,1)$	$Y(1,0,2,1)$	$Y(0,1,1,2)$

(3.5)

Note first that by (1.20) we have $\mu(\mathbb{Z}_4 \times \mathbb{Z}_4) = 6$, in accordance with (1.21) and $2 = 6-4$ invariant fields in the above table. One moreover easily checks that the spectrum of every element $g \in \mathcal{G}^{alg}_{ab}$ of order four agrees with the one computed in (1.23) for algebraic automorphisms of order four on $K3$ surfaces. This is a strong and highly non-trivial evidence for the fact that one possible geometric interpretation of $(2)^4$ is given by a $K3$ surface whose algebraic automorphism group contains $(\mathbb{Z}_4)^2$.

As stated above, further discussion is due concerning the action of $\mathcal{S}_4$ because transpositions of fermionic modes introduce sign flips (A.9). In particular, odd elements of $\mathcal{S}_4$ do not leave $J^\pm$ invariant. To have an algebraic action of the entire group $\mathcal{S}_4$ we must therefore accompany $\sigma \in \mathcal{S}_4$ by a phase symmetry $a_\sigma = [a_1(\sigma), a_2(\sigma), a_3(\sigma), a_4(\sigma)] \in \mathcal{G}_{ab}$ which for odd σ satisfies (3.3). Thus a transposition $(\alpha, \omega) \in \mathcal{S}_4$ must be represented by $\rho((\alpha,\omega)) = (\alpha,\omega) \circ a_{(\alpha,\omega)} = a_{(\alpha,\omega)} \circ (\alpha,\omega)$ in order to have $\rho((\alpha,\omega))^2 = \mathbb{1}$. With any such choice of ρ on generators (α_j, ω_j) of $\mathcal{S}_4$ one may then check explicitly that ρ defines an algebraic action of $\mathcal{S}_4$, i.e. its spectrum on the $(\frac{1}{2}, \frac{1}{2})$-fields coincides with the spectrum of the algebraic automorphism group $\mathcal{S}_4$. Namely, any element of order two (or three, four) in $\mathcal{S}_4$ leaves $\mu(\mathbb{Z}_2) - 4 = 12$ (or $\mu(\mathbb{Z}_3) - 4 = 8, \mu(\mathbb{Z}_4) - 4 = 6$) states invariant, and elements of order four have the spectrum given in (1.23). Note in particular that by (3.3) with any consistent choice of $\sigma \mapsto a_\sigma$ the group $\mathcal{S}_4$ acts by $\sigma \mapsto \mathrm{sign}(\sigma)$ on $Y(1,1,1,1)$ and trivially on X. This leaves $X = (\Phi^1_{1,0;3,2})^{\otimes 4}$ as the unique invariant state upon the action of $(\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ in accordance with $\mu((\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4) = 5$ and (1.21).

Summarizing, we have shown that the action of the entire algebraic symmetry group $\mathcal{G}^{alg} = (\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ of $(2)^4$ as described above exhibits a spectrum consistent with its interpretation as group of algebraic automorphisms of a $K3$ surface, e.g. the Fermat quartic with geometric interpretation $(\Sigma_\mathcal{Q}, V_\mathcal{Q}, B_\mathcal{Q})$. Remember that $\mu\left(\mathcal{G}^{alg}\right) = 5$ is the minimal possible value of μ by the discussion in section 1.2. Thus by what was said in section 2 the only four invariant $(\frac{1}{2}, \frac{1}{2})$-fields $(\Phi^1_{\pm 1,0;\pm 3,2})^{\otimes 4}$, $(\Phi^1_{\pm 1,0;\pm 1,0})^{\otimes 4}$ are those corresponding to moduli of volume deformation and of B-field deformation in direction of $\Sigma_\mathcal{Q}$.

3.2. Ideas of proof: An example with $c = 3$. In this subsection we give a survey on the steps of proof we will perform to show equivalences between Gepner or Gepner type models and nonlinear σ models. As an illustration we then prove the *well known* fact that Gepner's model $(2)^2$ admits a nonlinear σ model description on the torus associated to the $\mathbb{Z}^2$ lattice.

Given two $N = 2$ superconformal field theories $\mathcal{C}^1, \mathcal{C}^2$ with central charge $c = 3d/2$ ($d = 2$ or $d = 4$) and spaces of states $\mathcal{H}^1, \mathcal{H}^2$, to prove their equivalence we show the following:

i. The partition functions of the two theories agree sector by sector in the sense of (2.4).
ii. The fields of dimensions $(h, \overline{h}) = (1, 0)$ in the two theories generate the same algebra $\mathcal{A} = \mathcal{A}_f \oplus \mathcal{A}_b$, where $\mathcal{A}_f = u(1)$ for $d = 2$, $\mathcal{A}_f = su(2)^2_1$ for $d = 4$, and $u(1)^d \subset \mathcal{A}_b$. In particular, $u(1)^c \subset \mathcal{A}$. $\mathcal{A}_f$ contains the $U(1)$-current $J^{(1)} = J$ of the $N = 2$ superconformal algebra, and a second $U(1)$-generator $J^{(2)}$ if $d = 4$. Furthermore, the fields of dimensions $(h, \overline{h}) = (0, 1)$ in both theories generate algebras isomorphic to $\mathcal{A}$ as well, such that each of the left moving $U(1)$-currents j has a right moving partner $\overline{\jmath}$.
iii. For $i = 1, 2$ define

$$\mathcal{H}^i_b := \left\{ |\varphi\rangle \in \mathcal{H}^i \;\middle|\; J^{(k)}|\varphi\rangle = 0, \quad k \in \{1, \tfrac{d}{2}\} \right\}$$

and denote the $U(1)$-currents in $u(1)^d \subset \mathcal{A}_b$ by $j^1,\dots,j^d$. We normalize them to

$$j^k(z)\, j^l(w) \sim \frac{\delta_{kl}}{(z-w)^2}. \tag{3.6}$$

Let $j^{d+k} \sim J^{(k)}, k \in \{1, \frac{d}{2}\}$ denote the remaining $U(1)$-currents when normalized to (3.6), too, and set $\mathcal{J} := (j^1,\dots,j^d;\overline{j}^1,\dots,\overline{j}^d)$. The charge lattices

$$\Gamma_b^i := \left\{ \gamma \in \mathbb{R}^{d;d} \,\middle|\, \exists\, |\varphi\rangle \in \mathcal{H}_b^i : \mathcal{J}|\varphi\rangle = \gamma|\varphi\rangle \right\}$$

of $\mathcal{H}_b^1$ and $\mathcal{H}_b^2$ with respect to $\mathcal{J}$ are isomorphic to the same self dual lattice $\Gamma_b \subset \mathbb{R}^{d;d}$; because the states in $\mathcal{H}_b^i$ are pairwise local, in order to prove this it suffices to show agreement of the $\mathcal{J}$-action on a set of states whose charge vectors generate a self dual lattice Γ_b.

Theorem 3.1
If i.-iii. are true then theories $\mathcal{C}^1$ and $\mathcal{C}^2$ are isomorphic (the converse generically is wrong, of course).

Proof:
Using i.-iii. we first show $\mathcal{H}_b^1 \cong \mathcal{H}_b^2 =: \mathcal{H}_b$. Denote by $V^i[\gamma]$ the primary field corresponding to a state in $\mathcal{H}_b^i$ with charge $\gamma = (\gamma_l;\gamma_r) \in \Gamma_b$. Notice that in both theories every charge $\gamma \in \Gamma_b$ must appear with multiplicity one, because otherwise by fusing $[V_k^i[\gamma]]\times[V_k^i[-\gamma]] = [\mathbb{1}_k^i]$ we find two states $\mathbb{1}_1^i, \mathbb{1}_2^i \in \mathcal{H}_b^i$ with vanishing charges under a total $u(1)^c \subset \mathcal{A}$ in contradiction to uniqueness of the vacuum. Now for any $\alpha = (\alpha_l;\alpha_r), \beta = (\beta_l;\beta_r) \in \Gamma_b$ we have

$$V^i[\alpha](z)\; V^i[\beta](w) \sim c^i_{\alpha,\beta}(z-w)^{\alpha_l\beta_l}(\overline{z}-\overline{w})^{\alpha_r\beta_r}\; V^i[\alpha+\beta](w) + \cdots,$$

so it remains to be shown that we can arrange $c^1_{\alpha,\beta} = c^2_{\alpha,\beta}$ for all $\alpha,\beta \in \Gamma_b$ by normalizing the primary fields appropriately. In other words, we must find constants $d_\gamma \in \mathbb{R}$ for any $\gamma \in \Gamma_b$ such that $\forall\, \alpha,\beta \in \Gamma_b : c^2_{\alpha,\beta} = d_\alpha d_\beta c^1_{\alpha,\beta}$. This is possible, because having fixed $d_\alpha, d_\beta, d_\gamma, d_\delta \in \mathbb{R}$ such that

$$c^2_{\alpha,\beta} = d_\alpha d_\beta c^1_{\alpha,\beta},\; c^2_{\alpha,\gamma} = d_\alpha d_\gamma c^1_{\alpha,\gamma},\; c^2_{\alpha,\delta} = d_\alpha d_\delta c^1_{\alpha,\delta},\; c^2_{\beta,\gamma} = d_\beta d_\gamma c^1_{\beta,\gamma}$$

for four nonzero twopoint functions $c^i_{\alpha,\beta}, c^i_{\alpha,\gamma}, c^i_{\alpha,\delta}, c^i_{\beta,\delta}$, by the crossing symmetries

$$\frac{c^1_{\alpha,\beta}c^1_{\gamma,\delta}}{c^1_{\alpha,\gamma}c^1_{\beta,\delta}} = \frac{c^2_{\alpha,\beta}c^2_{\gamma,\delta}}{c^2_{\alpha,\gamma}c^2_{\beta,\delta}} \quad\text{and}\quad \frac{c^1_{\alpha,\gamma}c^1_{\beta,\delta}}{c^1_{\alpha,\delta}c^1_{\beta,\gamma}} = \frac{c^2_{\alpha,\gamma}c^2_{\beta,\delta}}{c^2_{\alpha,\delta}c^2_{\beta,\gamma}}$$

etc. we automatically have $c^2_{\gamma,\delta} = d_\gamma d_\delta c^1_{\gamma,\delta}$ and $c^2_{\beta,\delta} = d_\beta d_\delta c^1_{\beta,\delta}$. If more than two of the six twopoint functions vanish, then by similar arguments the normalization of one of the primaries is independent of the three others and a consistent choice of $d_\alpha, d_\beta, d_\gamma, d_\delta \in \mathbb{R}$ is therefore possible, too. The proof of $\mathcal{H}_b^1 \cong \mathcal{H}_b^2 \cong \mathcal{H}_b$ is now complete.
Because Γ_b is self dual, for any state $|\varphi\rangle \in \mathcal{H}^i$ carrying charge γ with respect to $\mathcal{J}$ we have $\gamma \in \Gamma_b$ and thus find vertex operators $V^i[\pm\gamma] \in \mathcal{H}_b^i$. By ii. and iii. $T := \frac{1}{2}\sum_{k=1}^c (j^k)^2$ acts as Virasoro field T^i on each of the theories (check that $T - T^i$ has dimensions $h = \overline{h} = 0$ with respect to T^i). Thus the restriction of the Virasoro field T^i to $\mathcal{H}_b^i$ is given by $T_b^i := \frac{1}{2}\sum_{k=1}^d (j^k)^2$, and by picking suitable

combinations P of descendants j_{-n}^k and $\widetilde{P}$ of ascendants $j_n^k, n \geq 0, k \in \{1, \ldots, d\}$, we find $|\psi\rangle := P\, V^i[-\gamma] |\varphi\rangle$ such that

$$|\varphi\rangle = |\psi\rangle \otimes V^i[\gamma]\, \widetilde{P}\, |0\rangle_b \quad \text{and} \quad |\psi\rangle \in \mathcal{H}_f^i := \left\{ |\chi\rangle \in \mathcal{H}^i \;\middle|\; T_b^i |\chi\rangle = 0 \right\}.$$

This shows $\mathcal{H}^i \cong \mathcal{H}_f^i \otimes \mathcal{H}_b$ for $i = 1, 2$. $\mathcal{H}_f^1$ and $\mathcal{H}_f^2$ are representations of $\mathcal{A}_f = u(1)$ (for $d = 2$) or $\mathcal{A}_f = su(2)_1^2$ (for $d = 4$) which are completely determined by charge and dimension of the lowest weight states. Because by ii. $\mathcal{A}_f$ contains the $U(1)$-current J of the total $N = 2$ superconformal algebra, the partition functions of our theories agree by i., and we already know $\mathcal{H}^i \cong \mathcal{H}_f^i \otimes \mathcal{H}_b$ for $i = 1, 2$, we may conclude $\mathcal{H}_f^1 \cong \mathcal{H}_f^2$. $\square$

Let's see how the procedure described above works:

Theorem 3.2

Gepner's model $\mathcal{C}^1 = (2)^2$ has a nonlinear σ model description $\mathcal{C}^2$ on the two dimensional torus $T_{SU(2)_1^2}$ with $SU(2)_1^2$ lattice $\Lambda = \mathbb{Z}^2$ and B-field $B = 0$.

Proof:

If we can prove i.-iii. in the above list, by theorem 3.1 we are done.

i. Using (A.10) for computing the partition function of $(2)^2$ on one hand and (1.12) for the partition function of the σ model on $T_{SU(2)_1^2}$ with $B = 0$ on the other, we find

$$Z_{NS}(\tau, z) = \frac{1}{2} \left[\left| \frac{\vartheta_2}{\eta} \right|^4 + \left| \frac{\vartheta_3}{\eta} \right|^4 + \left| \frac{\vartheta_4}{\eta} \right|^4 \right] \left| \frac{\vartheta_3(z)}{\eta} \right|^2$$

for both theories.

ii. The nonlinear σ model on $T_{SU(2)_1^2}$ has two rightmoving abelian currents j_1, j_2 which we normalize to

$$j_\alpha(z)\, j_\beta(w) \sim \frac{\frac{1}{2}\delta_{\alpha\beta}}{(z-w)^2}.$$

Their superpartners are free Majorana fermions ψ_1, ψ_2 with coupled boundary conditions. By e_1, e_2 we denote the generators of the lattice $\Lambda = \Lambda^* = \mathbb{Z}^2$ which defines our torus. Then the $(1,0)$-fields in the nonlinear σ model are given by the three abelian currents $J = i\psi_2\psi_1$ (the $U(1)$ current of the $N = 2$ superconformal algebra), $Q = j_1 + j_2$, $R = j_1 - j_2$, and the four vertex operators $V_{\pm e_i, \pm e_i}, i = 1, 2$.

In the Gepner model $(2)^2$ we have an abelian current j, j' from each minimal model factor along with Majorana fermions ψ, ψ', where by (A.8) $\psi\psi' = \Phi^0_{4,2;0,0} \otimes \Phi^0_{4,2;0,0}$. The $U(1)$ current of the total $N = 2$ superconformal algebra is $J = j + j'$, and comparing J, Q, R-charges we can make the following identifications:

$$\begin{aligned}
i\psi_2\psi_1 = J \;=\; j + j', \quad j_1 + j_2 \;=\; Q \;=\; j - j', \quad j_1 - j_2 \;=\; R \;=\; i\psi\psi', \\
V_{e_1,e_1} = \Phi^0_{2,0;0,0} \otimes \Phi^0_{2,2;0,0} + \Phi^0_{-2,0;0,0} \otimes \Phi^0_{-2,2;0,0}, \\
V_{e_2,e_2} = \Phi^0_{2,0;0,0} \otimes \Phi^0_{2,2;0,0} - \Phi^0_{-2,0;0,0} \otimes \Phi^0_{-2,2;0,0}, \\
V_{-e_1,-e_1} = \Phi^0_{2,2;0,0} \otimes \Phi^0_{2,0;0,0} + \Phi^0_{-2,2;0,0} \otimes \Phi^0_{-2,0;0,0}, \\
V_{-e_2,-e_2} = \Phi^0_{2,2;0,0} \otimes \Phi^0_{2,0;0,0} - \Phi^0_{-2,2;0,0} \otimes \Phi^0_{-2,0;0,0}.
\end{aligned}$$

Thus the $(1,0)$-fields in the two theories generate the same algebra $\mathcal{A} = u(1) \oplus su(2)_1^2 = \mathcal{A}_f \oplus \mathcal{A}_b$. Obviously, the same structure arises on the right handed sides.

iii. The space $\mathcal{H}_b^1$ for the σ model is just the bosonic part of the theory. The charge lattice Γ_b with respect to the currents $\mathcal{J} := (Q,R;\overline{Q},\overline{R}) = (j_1+j_2, j_1-j_2; \overline{\jmath}_1+\overline{\jmath}_2, \overline{\jmath}_1-\overline{\jmath}_2)$ thus contains the charges $M := \left\{\frac{1}{2}(\varepsilon;\pm\varepsilon), \varepsilon\in\{\pm1\}^2\right\}$, carried by vertex operators $V_{\pm e_i,0}, V_{0,\pm e_i}$, $i=1,2$. M generates the self dual lattice $\left\{\frac{1}{2}(a+b;a-b) \mid a,b\in\mathbb{Z}^2, \sum_{k=1}^2 a_k \equiv \sum_{k=1}^2 b_k \equiv 0\,(2)\right\} = \Gamma_b$.
To complete the proof of iii. we observe that in the Gepner model the fields $\Phi^1_{n,0;n,0}\otimes\Phi^1_{-n,0;-n,0} \pm \Phi^1_{-3n,2;n,0}\otimes\Phi^1_{3n,2;-n,0}$ and $\Phi^1_{n,0;-n,0}\otimes\Phi^1_{3n,2;n,0}$ $\pm\Phi^1_{-3n,2;-n,0}\otimes\Phi^1_{-n,0;n,0}$, $n\in\{\pm1\}$, are uncharged with respect to J and carry $\mathcal{J} = (j-j', i\psi\psi'; \overline{\jmath}-\overline{\jmath}', i\overline{\psi}\,\overline{\psi}')$-charges $M = \left\{\frac{1}{2}(\varepsilon;\pm\varepsilon), \varepsilon\in\{\pm1\}^2\right\}$ generating Γ_b.

$\square$

3.3. Gepner type description of $SU(2)_1^4/\mathbb{Z}_2$.

Theorem 3.3
Let $\mathcal{C}^1 = \widehat{(2)}^4$ denote the Gepner type model which is obtained as orbifold of $(2)^4$ by the group $\mathbb{Z}_2 \cong \langle[2,2,0,0]\rangle \subset \mathcal{G}_{ab}^{alg}$. Then $\mathcal{C}^1$ admits a nonlinear σ model description $\mathcal{C}^2 = \mathcal{K}(\mathbb{Z}^4,0)$ on the Kummer surface $\mathcal{K}(\Lambda)$ associated to the torus $T_{SU(2)_1^4}$ with $SU(2)_1^4$ lattice $\Lambda=\mathbb{Z}^4$ and vanishing B-field.

Proof:
We prove conditions i.-iii. of section 3.2 and then use theorem 3.1.

i. From (1.12) one finds

$$Z_{\Lambda=\mathbb{Z}^4,B_T=0}(\tau) = \left[\frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|^4 + \left|\frac{\vartheta_3}{\eta}\right|^4 + \left|\frac{\vartheta_4}{\eta}\right|^4\right)\right]^2. \tag{3.7}$$

Applying the orbifold procedure for the $\mathbb{Z}_2$-action of $[2,2,0,0]\in\mathcal{G}_{ab}^{alg}$ to the partition function (A.10) of the Gepner model $(2)^4$ [F-K-S-S] one checks that $\mathcal{C}^1$ and $\mathcal{C}^2$ have the same partition function obtained by inserting (3.7) into (2.3).

ii. In the nonlinear σ model $\mathcal{C}^2$ the current algebra (2.2) is enhanced to $u(1)^4\oplus su(2)_1^2$. The additional $U(1)$-currents are $U_i := V_{e_i,e_i} + V_{-e_i,-e_i}$, $i=1,\dots,4$, where the e_i are the standard generators of $\Lambda=\Lambda^*=\mathbb{Z}^4$.
In the Gepner type model $\mathcal{C}^1 = \widehat{(2)}^4$, apart from the $U(1)$-currents $J_1,\dots,J_4$ from the factor theories, where $J=J_1+\dots+J_4$, we find four additional fields with dimensions $(h,\overline{h}) = (1,0)$; comparing the respective operator product

expansions the following identifications can be made:

$$\begin{array}{rcl} J = J_1 + J_2 + J_3 + J_4\,, & J^{\pm} = & \left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes 4}\,; \\ A = J_1 + J_2 - J_3 - J_4\,, & A^{\pm} = & \left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes 2} \otimes \left(\Phi^0_{\pm 2,2;0,0}\right)^{\otimes 2}\,; \\ \hline \end{array}$$

$$\begin{array}{rcl} \frac{1}{2}(U_1+U_2) = P & = & J_1 - J_2; \\ \frac{1}{2}(U_3+U_4) = Q & = & J_3 - J_4; \\ \frac{1}{2}(U_1-U_2) = R & = & i\left(\Phi^0_{4,2;0,0}\right)^{\otimes 2} \otimes \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2}\,; \\ \frac{1}{2}(U_3-U_4) = S & = & i\left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \otimes \left(\Phi^0_{4,2;0,0}\right)^{\otimes 2}\,. \end{array} \tag{3.8}$$

Thus the $(1,0)$-fields in the two theories generate the same algebra $\mathcal{A} = su(2)_1^2 \oplus u(1)^4 = \mathcal{A}_f \oplus \mathcal{A}_b$. Obviously, the same structure arises on the right handed sides.

iii. We show that $\mathcal{H}_b^1$ and $\mathcal{H}_b^2$ both have self dual $\mathcal{J} := (P,Q,R,S;\overline{P},\overline{Q},\overline{R},\overline{S})$-charge lattice[3]

$$\Gamma_b = \left\{ (x+y; x-y) \,\middle|\, x \in \tfrac{1}{2}D_4, y \in D_4^* \right\}, \tag{3.9}$$

generated by

$$\begin{aligned} M_{tw} := \Big\{ \tfrac{1}{2}(x;x) \in \mathbb{R}^{4,4} \Big|\, x \in \{ &(\varepsilon_1,\varepsilon_2,0,0),(0,0,\varepsilon_1,\varepsilon_2), \\ &(0,\varepsilon_1,\varepsilon_2,0),(\varepsilon_1,0,0,\varepsilon_2), \varepsilon_i \in \{\pm 1\}\}\Big\} \\ \text{and } M_{inv} := \left\{ (\varepsilon;0) \,\middle|\, \varepsilon \in \{\pm1\}^4 \right\}. & \end{aligned}$$

In the σ model $\mathcal{C}^2$ we denote by $\Sigma_\delta, \delta \in \mathbb{F}_2^4$ the twist field corresponding to the fixed point $p_\delta = \frac{1}{2}\sum_{i=1}^4 \delta_i e_i$ of the $\mathbb{Z}_2$ orbifold. To determine the action of U_i on twist fields notice that by definition, Σ_δ introduces a cut on the configuration space Z to establish the boundary condition $\varphi(\sigma_0+1,\sigma_1) = -\varphi(\sigma_0,\sigma_1)$ for fields φ in the corresponding twisted sector, i.e. $\varphi(0,0) = p_\delta$ (see section 2). Action of a vertex operator with winding mode λ will shift the constant mode p_δ of each twisted field by $\frac{\lambda}{2}$ [H-V]. Hence,

$$U_i(z)\ \Sigma_\delta(w) \sim \frac{1/2}{z-w}\ \Sigma_{\delta+e_i}(w), \tag{3.10}$$

where the factor $\frac{1}{2}$ is determined up to phases by observing $T_f^2|\Sigma_\delta\rangle = 0, T_b^2 = \frac{1}{4}\sum_{i=1}^4 (U_i)^2$, and $h = \overline{h} = \frac{1}{4}$ for twist fields. The phases are fixed by appropriately normalizing the twist fields. One now checks that

$$\forall\, \varepsilon \in \{\pm1\}^4: \quad s_\varepsilon := \sum_{\delta\in\mathbb{F}_2^4} \prod_{i=1}^4 (\varepsilon_i)^{\delta_i}\ \Sigma_\delta$$

are uncharged under $(J;\overline{J})$ and $(A;\overline{A})$ and carry $\mathcal{J}$-charges M_{tw}. For $\varepsilon,\delta \in \{\pm1\}$ and $k,l \in \{1,\dots,4\}$ we define

$$E_{kl}^{\varepsilon\delta} := \left(j_k - \tfrac{\delta}{2}\left(V_{e_k,e_k} - V_{-e_k,-e_k}\right)\right)\left(j_l - \tfrac{\varepsilon}{2}\left(V_{e_l,e_l} - V_{-e_l,-e_l}\right)\right).$$

[3] In our coordinates $D_4 = \{x \in \mathbb{Z}^4 \mid \sum_{i=1}^4 x_i \equiv 0\,(2)\}$ and $D_4^* = \mathbb{Z}^4 + (\mathbb{Z}+1/2)^4$.

Then $E_{13}^{\varepsilon\delta}, E_{14}^{\varepsilon\delta}, E_{23}^{\varepsilon\delta}, E_{24}^{\varepsilon\delta}$ are $(J,A;\overline{J},\overline{A})$-uncharged and carry $\mathcal{J}$-charges M_{inv}.
In the Gepner model, introducing $\mathcal{O}(n_1) := \left(\Phi^1_{2,1;2n_1,n_1}\right)^{\otimes 2}$, $\mathcal{P}(n_2) := \Phi^0_{n_2,n_2;n_2,n_2} \otimes \Phi^0_{-n_2,-n_2;-n_2,-n_2}$ $(n_i \in \{\pm 1\})$ as shorthand notation we find $(J,A;\overline{J},\overline{A})$-uncharged fields $\mathcal{O}(n_1)\otimes\mathcal{O}(n_2)$, $\mathcal{O}(n_1)\otimes\mathcal{P}(n_2)$, $\mathcal{P}(n_1)\otimes\mathcal{O}(n_2)$, $\mathcal{P}(n_1)\otimes\mathcal{P}(n_2)$ which after diagonalization with respect to the $\mathcal{J}$-action carry charges M_{tw}.
Similarly, setting $\mathcal{Q}(n,s) := \Phi^0_{2n,s;0,0} \otimes \Phi^0_{2n,s+2;0,0}$, the fields $\mathcal{Q}(n_1,s_1)\otimes\mathcal{Q}(n_2,s_2)$, $n_i \in \{\pm 1\}$, $s_i \in \{0,2\}$ after diagonalization have charges M_{inv}.

For later reference we note that by what was said in section 1 there are eight more fields in the Ramond sector with dimensions $h = \overline{h} = \frac{1}{4}$. Each of them is uncharged under $\mathcal{J}$ and either $(A;\overline{A})$ or $(J;\overline{J})$. We denote by $W^J_{\varepsilon_1,\varepsilon_2}, W^A_{\varepsilon_1,\varepsilon_2}, \varepsilon_i \in \{\pm 1\}$ the fields corresponding to the lowest weight states of $su(2)_1 \cong \langle J, J^\pm\rangle$ or $su(2)_1 \cong \langle A, A^\pm\rangle$, with $(J;\overline{J})$ or $(A;\overline{A})$-charge $(\varepsilon_1;\varepsilon_2)$ respectively and identify

$$\begin{aligned} W^J_{\varepsilon_1,\varepsilon_2} &= \left(\Phi^0_{-\varepsilon_1,-\varepsilon_1;-\varepsilon_2,-\varepsilon_2}\right)^{\otimes 4} \\ W^A_{\varepsilon_1,\varepsilon_2} &= \left(\Phi^0_{-\varepsilon_1,-\varepsilon_1;-\varepsilon_2,-\varepsilon_2}\right)^{\otimes 2} \otimes \left(\Phi^0_{\varepsilon_1,\varepsilon_1;\varepsilon_2,\varepsilon_2}\right)^{\otimes 2}. \end{aligned} \tag{3.11}$$

In σ model language and by the discussion in section 1, by applying left and right handed spectral flow to the J-uncharged $W^A_{\varepsilon_1,\varepsilon_2}$ we obtain $(\frac{1}{2},\frac{1}{2})$-fields in $\mathcal{F}_{1/2}$, the real and imaginary parts of whose $(1,1)$-superpartners describe infinitesimal deformations of the torus $T_{SU(2)_1^4}$ our Kummer surface is associated to.
Summarizing, we can now obtain a list of all fields needed to generate $\mathcal{H}^1$ and $\mathcal{H}^2$ as well as a complete field by field identification by comparison of charges; for the resulting list of $(\frac{1}{4},\frac{1}{4})$-fields see appendix B. □

Note that because $D_4 \cong \sqrt{2}D_4^*$ for the $\mathcal{J}$-charge lattice (3.9)

$$\Gamma_b \cong \left\{ \tfrac{1}{\sqrt{2}}(\mu+\lambda,\mu-\lambda) \,\middle|\, \mu \in D_4^*, \lambda \in D_4 \right\}.$$

Thus Γ_b is the charge lattice of the bosonic part of the σ model $\mathcal{C}^3 = \mathcal{T}(D_4,0)$. Theory $\mathcal{C}^1$ was obtained by taking the ordinary $\mathbb{Z}_2$ orbifold of the torus model on $T_{SU(2)_1^4}$, but as pointed out in [K-S], for the bosonic part of the theory this is equivalent to taking the $\mathbb{Z}_2$ orbifold associated to a shift $\delta = \frac{1}{2\sqrt{2}}(\mu_0;\mu_0)$, $\mu_0 = \sum_i e_i \in \Lambda^*$ on the charge lattice of $T_{SU(2)_1^4}$. Under this *shift orbifold*, the lattices $\Lambda = \Lambda^* = \mathbb{Z}^4$ are transformed by

$$\Lambda^* \mapsto \Lambda^* + \left(\Lambda^* + \tfrac{1}{2}\mu_0\right) = D_4^*, \quad \Lambda \mapsto \{\lambda \in \Lambda \,|\langle \mu_0, \lambda\rangle \equiv 0\,(2)\,\} = D_4,$$

so the bosonic part of the resulting theory indeed is that of $\mathcal{C}^3$. The entire bosonic sector of $\mathcal{C}^1 = \mathcal{C}^2$ agrees with that of $\mathcal{C}^3$, because the shift acts trivially on fermions, and the ordinary $\mathbb{Z}_2$ orbifold just interchanges twisted and untwisted boundary conditions of the fermions in time direction. The difference between the theories merely amounts in opposite assignments of Ramond and Neveu-Schwarz sector on the twisted states resulting in different elliptic genera for the $K3$-model $\mathcal{C}^1 = \mathcal{C}^2$ and the torus model $\mathcal{C}^3$. The fact that the partition functions actually do not agree before projection onto even fermion numbers is not relevant here because locality is violated before the projection is carried out. So, on the level of conformal field theory:

Remark 3.4
The Gepner type model $\mathcal{C}^1 = (\widehat{2})^4$ viewed as nonlinear σ model $\mathcal{C}^2$ on the Kummer surface $\mathcal{K}(\mathbb{Z}^4, 0)$ is located at a meeting point of the moduli spaces of theories associated to $K3$ surfaces and tori, respectively. Namely, its bosonic sector is identical with that of the nonlinear σ model $\mathcal{C}^3 = \mathcal{T}(D_4, 0)$.

This property does not translate to the stringy interpretation of our conformal field theories, though. When we take external degrees of freedom into account, spin statistics theorem dictates in which representations of $SO(4)$ the external free fields may couple to internal Neveu-Schwarz or Ramond fields, respectively. The theories $\mathcal{C}^1 = \mathcal{C}^2$ and $\mathcal{C}^3$ therefore correspond to different compactifications of the type IIA string.

3.4. Gepner's description for $SU(2)_1^4/\mathbb{Z}_4$.

Theorem 3.5
The Gepner model $\mathcal{C}^I = (2)^4$ admits a nonlinear σ model description $\mathcal{C}^{II}$ on the $\mathbb{Z}_4$ orbifold of the torus $T_{SU(2)_1^4}$ with $SU(2)_1^4$-lattice $\Lambda = \mathbb{Z}^4$ and vanishing B-field.

Proof:
It is clear that $\mathcal{C}^I = (2)^4$ can be obtained from $\mathcal{C}^1 = (\widehat{2})^4$, for which we already have a σ model description by theorem 3.3, by the $\mathbb{Z}_2$ orbifold procedure which revokes the orbifold used to construct $\mathcal{C}^1$. The corresponding action is multiplication by -1 on $\langle[2,2,0,0]\rangle$-twisted states, i.e.

$$[2',2',0,0]: \quad \bigotimes_{i=1}^{4} \Phi^{l_i}_{m_i,s_i;\overline{m}_i,\overline{s}_i} \longmapsto e^{\frac{2\pi i}{8}(\overline{m}_1 - m_1 - \overline{m}_3 + m_3)} \bigotimes_{i=1}^{4} \Phi^{l_i}_{m_i,s_i;\overline{m}_i,\overline{s}_i}. \tag{3.12}$$

Among the $(1,0)$-fields the following are invariant under $[2',2',0,0]$ (use (2.2) and (3.8)):

$$\begin{aligned} &J = \psi_+^{(1)}\psi_-^{(1)} + \psi_+^{(2)}\psi_-^{(2)},\ J^+ = \psi_+^{(1)}\psi_+^{(2)}, \qquad J^- = \psi_-^{(2)}\psi_-^{(1)};\\ &A = \psi_+^{(1)}\psi_-^{(1)} - \psi_+^{(2)}\psi_-^{(2)};\quad P = \tfrac{1}{2}(U_1+U_2);\quad Q = \tfrac{1}{2}(U_3+U_4). \end{aligned} \tag{3.13}$$

Hence we have a surviving $su(2)_1 \oplus u(1)^3$ subalgebra of our holomorphic W-algebra. In appendix B we give a list of all $(\frac{1}{4},\frac{1}{4})$-fields in $\mathcal{C}^1 = (\widehat{2})^4$ together with their description in the σ model $\mathcal{C}^2$ on the $\mathbb{Z}_2$ orbifold $\mathcal{K}(\mathbb{Z}^4,0)$. A similar list can be obtained for the $(2,0)$-fields as discussed in the proof of theorem 3.3. From these lists and (3.13) one readily reads off that the states invariant under (3.12) coincide with those invariant under the automorphism r_{12} on $\mathcal{K}(\mathbb{Z}^4,0)$ (see theorem 2.7) which is induced by the $\mathbb{Z}_4$ action $(j_1,j_2,j_3,j_4) \mapsto (-j_2,j_1,j_4,-j_3)$, i.e. $(\psi_\pm^{(1)},\psi_\pm^{(2)}) \mapsto (\pm i\psi_\pm^{(1)}, \mp i\psi_\pm^{(2)})$ on the underlying torus $T_{SU(2)_1^4}$. The appertaining permutation of exceptional divisors in the $\mathbb{Z}_2$ fixed points is depicted in figure 2.1. The action of r_{12} and that induced by (3.12) agree on the algebra $\mathcal{A}$ of $(1,0)$-fields and a set of states generating the entire space of states, thus they are the same. Because of $\mathcal{C}^1 = \mathcal{C}^2$ (theorem 3.3) and the fact that $\mathcal{C}^I = (2)^4$ is obtained from $\mathcal{C}^1$ by modding out (3.12), it is clear that modding out $\mathcal{K}(\mathbb{Z}^4,0)$

by the algebraic automorphism r_{12} will lead to a σ model description of $(2)^4$. As shown in theorem 2.8 the result is the $\mathbb{Z}_4$ orbifold $\mathcal{C}^{II}$ of $T_{SU(2)_1^4}$. $\square$

Theorem 3.5 has been conjectured in [E-O-T-Y] because of agreement of the partition functions of $\mathcal{C}^I$ and $\mathcal{C}^{II}$. This of course is only part of the proof as can be seen from our argumentation in section 2.4. There we showed that $SU(2)_1^4/\mathbb{Z}_4$ does not admit a σ model description on a Kummer surface although its partition function by [E-O-T-Y] agrees with that of $\mathcal{K}(D_4, 0)$, too.
From theorem 2.13 and theorem 3.5 we conclude:

Corollary 3.6
The Gepner model $(2)^4$ admits a geometric interpretation on the Fermat quartic (2.17) in $\mathbb{CP}^3$ with volume $V_{\mathcal{Q}} = \frac{1}{2}$.

Let (Σ, V, B) denote the geometric interpretation of $(2)^4$ we gain from theorem 3.5. By the proof of theorem 3.3 we know the moduli $V^\pm_{\delta,\varepsilon} + V^\pm_{-\delta,-\varepsilon}$ and $i(V^\pm_{\delta,\varepsilon} - V^\pm_{-\delta,-\varepsilon}), \delta, \varepsilon \in \{\pm 1\}$ for volume and B-field deformation in direction of Σ of the underlying torus $T_{SU(2)_1^4}$ of our $\mathbb{Z}_4$ orbifold: We apply left and right handed spectral flows to $W^A_{1,1}, W^A_{-1,-1}$ as given in (3.11) and then compute the corresponding $(1,1)$-superpartners. In terms of Gepner fields this means

$$\begin{aligned} V^+_{\delta,\varepsilon} &= \Phi^2_{2\delta,2;2\varepsilon,2} \otimes \Phi^2_{2\delta,0;2\varepsilon,0} \otimes \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \\ &\quad + \Phi^2_{2\delta,0;2\varepsilon,0} \otimes \Phi^2_{2\delta,2;2\varepsilon,2} \otimes \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} , \\ V^-_{\delta,\varepsilon} &= \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \otimes \Phi^2_{2\delta,2;2\varepsilon,2} \otimes \Phi^2_{2\delta,0;2\varepsilon,0} \\ &\quad + \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \otimes \Phi^2_{2\delta,0;2\varepsilon,0} \otimes \Phi^2_{2\delta,2;2\varepsilon,2} . \end{aligned} \tag{3.14}$$

Indeed, $V^\pm_{\delta,\varepsilon}$ are uncharged under J and A as they should, because both $U(1)$-currents must survive deformations within the moduli space of $\mathbb{Z}_4$ orbifold conformal field theories. On the other hand by our discussion in section 3.1 the $(1,1)$-superpartners of $(\Phi^1_{\pm 1,0;\pm 3,2})^{\otimes 4}$, $(\Phi^1_{\pm 1,0;\pm 1,0})^{\otimes 4}$, which carry $(A;\overline{A})$-charges $\mp(1;1)$, give the moduli of volume and corresponding B-field deformation if we choose the quartic hypersurface (2.17) as geometric interpretation of Gepner's model $(2)^4$. Hence along the "quartic line" we generically only have an $su(2)_1$-algebra of $(1,0)$-fields. This agrees with the analogous picture for $c = 9$ and the Gepner model $(3)^5$ where all additional $U(1)$-currents vanish upon deformation along the quintic line [D-G].

Symmetries and algebraic automorphisms revised: $(2)^4$ *and* $(\widehat{2})^4$. Among the algebraic symmetries $\mathbb{Z}_4^2 \rtimes \mathcal{S}_4$ of the Gepner model $(2)^4$ all the phase symmetries $\mathbb{Z}_4^2$ commute with the action of $[2,2,0,0]$ which we mod out to obtain $(\widehat{2})^4$. The residual $\mathbb{Z}_2 \times \mathbb{Z}_4$ has a straightforward continuation to $(\widehat{2})^4$ (i.e. to the twisted states). Moreover, $[2',2',0,0]$ as given in (3.12) which reverts the orbifold with respect to $[2,2,0,0]$ must belong to the algebraic symmetry group $\widehat{\mathcal{G}}^{alg}$ of $(\widehat{2})^4$. Nevertheless, one notices that $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle [2',2',0,0], [1,3,0,0] \rangle$ leaves $6 \neq 8 = \mu(\mathbb{Z}_2 \times \mathbb{Z}_2) - 4$ states invariant and thus does not act algebraically by (1.21). We temporarilly leave the symmetry $[1,3,0,0]$ out of discussion, because then by the methods described in section 3.1 we find a consistent algebraic

action of $(\mathbb{Z}_2\times\mathbb{Z}_4)\rtimes D_4$ on $(\widehat{2})^4$, where $\mathbb{Z}_2\times\mathbb{Z}_4=\langle[2',2',0,0],[1,0,3,0]\rangle$ and $D_4=\langle(12),(13)(24)\rangle\subset\mathcal{S}_4$ is the commutant of $[2,2,0,0]$.
Let us compare to the σ model description $\mathcal{K}(\mathbb{Z}^4,0)$ of $(\widehat{2})^4$: In theorem 2.7 the group of algebraic automorphisms of $\mathcal{K}(\mathbb{Z}^4,0)$ which leave the orbifold singular metric invariant was determined to $\mathcal{G}^+_{Kummer}=\mathbb{Z}_2^2\ltimes\mathbb{F}_2^4$. Although it is isomorphic to the algebraic symmetry group $(\mathbb{Z}_2\times\mathbb{Z}_4)\rtimes D_4$ of $(\widehat{2})^4$ found so far, $\mathcal{G}^+_{Kummer}$ must act differently on $(\widehat{2})^4$. Namely, from the proof of theorem 3.5 we know that the σ model equivalent of $[2',2',0,0]$ is $r_{12}\in\mathcal{G}^+_{Kummer}$. Thus only the commutant $\mathcal{H}\subset\mathcal{G}^+_{Kummer}$ of r_{12} can comprise residual symmetries descending from the $\mathbb{Z}_4$ orbifold description on $(2)^4$. This is no contradiction, because by the discussion in section 1.2 different subgroups of the entire algebraic symmetry group of $(\widehat{2})^4$ may leave the respective nullvector υ invariant which defines the geometric interpretation. By what was said in section 1 it is actually no surprise to find symmetries of conformal field theories which do not descend to classical symmetries of a given geometric interpretation. The Gepner type model $(\widehat{2})^4$ is an example where the existence of such symmetries can be checked explicitly.
By the results of section 2.3 we find $\mathcal{H}=\mathbb{Z}_2\times D_4=\langle r_{12},r_{13},t_{1100}\rangle$ (see also theorem 2.12). We now use our state by state identification obtained in the proof of theorem 3.3 (see appendix B) to determine the corresponding elements of $\widehat{\mathcal{G}}^{alg}$ and find

$$\begin{aligned} r_{13}&=(13)(24) &&\in\mathcal{S}_4\\ t_{1100}&=\xi\circ[1,3,0,0]=:[1',3',0,0]. \end{aligned}\tag{3.15}$$

Here ξ acts by multiplication with -1 on those Gepner states corresponding to the 16 twist fields Σ_δ of the Kummer surface and trivially on all the other generating fields of the space of states we discussed in the proof of theorem 3.3. Note that ξ is a symmetry of the theory because by the selection rules for amplitudes of twist fields any n-point function containing an odd number of twist fields will vanish. The geometric interpretation tells us that modding out $(\widehat{2})^4$ by ξ will revoke the ordinary $\mathbb{Z}_2$ orbifold procedure i.e. produce $\mathcal{T}(\mathbb{Z}^4,0)$. We conclude remarking that by the modification (3.15) of the $[1,3,0,0]$-action the full group $\widehat{\mathcal{G}}^{alg}=(\mathbb{Z}_2^2\times\mathbb{Z}_4)\rtimes D_4$ acts algebraically on $(\widehat{2})^4$. The subgroup $\mathcal{H}$ consists of all the residual symmetries of $(2)^4$ surviving both deformations along the quartic and the $\mathbb{Z}_4$ orbifold line and acting classically in both geometric interpretations of $(2)^4$ known so far, the $\mathbb{Z}_4$ orbifold and the quartic one.

3.5. Gepner type description of $SO(8)_1/\mathbb{Z}_2$.

Theorem 3.7
Let $\widetilde{\mathcal{C}}^1=(\widetilde{2})^4$ denote the Gepner type model which is obtained as orbifold of $(2)^4$ by the group $\mathbb{Z}_2\times\mathbb{Z}_2\cong\langle[2,2,0,0],[2,0,2,0]\rangle\subset\mathcal{G}^{alg}_{ab}$. This model admits a nonlinear σ model description $\widetilde{\mathcal{C}}^2$ on the Kummer surface $\mathcal{K}(\frac{1}{\sqrt{2}}D_4,B^*)$ associated to the torus $T_{SO(8)_1}$ with $SO(8)_1$-lattice $\Lambda=\frac{1}{\sqrt{2}}D_4$ and B-field value B^* for which the theory has enhanced symmetry by the Frenkel-Kac mechanism.

Proof:
Let $e_1,\dots,e_4$ denote the standard basis of $\mathbb{Z}^4$. With respect to this basis the

B-field which leads to a full $SO(8)_1$ symmetry for the σ model on $T_{SO(8)_1}$ is

$$B^* = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array}\right) : \Lambda \otimes \mathbb{R} \longrightarrow \Lambda^* \otimes \mathbb{R}\,, \qquad (3.16)$$

a twotorsion point in $H^2(T_{SO(8)_1}, \mathbb{R})/H^2(T_{SO(8)_1}, \mathbb{Z})$.
We are now ready to use theorem 3.1 if we can prove i.-iii. of section 3.2.

i. From (1.12) we find

$$Z_{\frac{1}{\sqrt{2}}D_4,B^*}(\tau) = \frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|^8 + \left|\frac{\vartheta_3}{\eta}\right|^8 + \left|\frac{\vartheta_4}{\eta}\right|^8\right). \qquad (3.17)$$

Applying the orbifold procedure for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action of $\langle[2,2,0,0],[2,0,2,0]\rangle \subset \mathcal{G}_{ab}^{alg}$ to the partition function (A.10) of the Gepner model $(2)^4$ [F-K-S-S] one checks that $\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ have the same partition function obtained by inserting (3.17) into (2.3).

ii. We have an enhancement of the current algebra (2.2) of the nonlinear σ model $\widetilde{\mathcal{C}}^2$ to $su(2)_1^6$. The 12 additional $(1,0)$-fields are $U_\alpha := \frac{1}{\sqrt{2}}\left(V_{\alpha,\alpha+B^*\alpha} + V_{-\alpha,-\alpha-B^*\alpha}\right)$, where α belongs to the D_4 rootsystem $\{\pm\frac{1}{\sqrt{2}}e_i \pm \frac{1}{\sqrt{2}}e_j\}$. We set

$$W_{i,j}^{\pm} := \tfrac{1}{2}\left(U_{\frac{1}{\sqrt{2}}(e_i+e_j)} \pm U_{\frac{1}{\sqrt{2}}(e_i-e_j)}\right)$$

to see that upon a consistent choice of cocycle factors for the vertex operators these fields indeed comprise an extra $su(2)_1^4$:

$$\begin{aligned} P &:= W_{1,4}^+ + W_{2,3}^+, \quad P^{\pm} := \tfrac{1}{\sqrt{2}}\left(W_{1,2}^+ + W_{3,4}^+\right) \pm \tfrac{1}{\sqrt{2}}\left(W_{2,4}^+ + W_{1,3}^+\right),\\ Q &:= W_{1,2}^+ - W_{3,4}^+, \quad Q^{\pm} := \tfrac{1}{\sqrt{2}}\left(W_{1,3}^+ - W_{2,4}^+\right) \pm \tfrac{1}{\sqrt{2}}\left(W_{1,4}^+ - W_{2,3}^+\right),\\ R &:= iW_{2,4}^- - iW_{1,3}^-, \, R^{\pm} := \tfrac{1}{\sqrt{2}}\left(W_{1,4}^- - W_{2,3}^-\right) \pm \tfrac{1}{\sqrt{2}}\left(W_{1,2}^- - W_{3,4}^-\right),\\ S &:= W_{1,4}^- + W_{2,3}^-, \quad S^{\pm} := \tfrac{1}{\sqrt{2}}\left(W_{1,2}^- + W_{3,4}^-\right) \pm \tfrac{1}{\sqrt{2}}\left(W_{2,4}^- + W_{1,3}^-\right). \end{aligned} \qquad (3.18)$$

For the Gepner type model $\widetilde{\mathcal{C}}^2 = (\widetilde{2})^4$ we use X_{ij} as a shorthand notation for the field having factors $\Phi^0_{4,2;0,0}$ in the ith and jth position and factors $\Phi^0_{0,0;0,0}$ otherwise, and Y_{ij} for the field having factors $\Phi^0_{-2,2;0,0}$ in the ith and jth position and factors $\Phi^0_{2,2;0,0}$ otherwise. By comparison of operator product expansions one then checks that the following identifications can be made:

$$\begin{array}{ll} J = J_1 + J_2 + J_3 + J_4, & J^{\pm} = \left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes 4};\\ A = J_1 + J_2 - J_3 - J_4, & A^+ = Y_{12}, \quad A^- = Y_{34};\\ \hline P = \frac{1}{\sqrt{2}}\left(J_1 - J_2 + J_3 - J_4\right), & P^+ = Y_{13}, \quad P^- = Y_{24};\\ Q = \frac{1}{\sqrt{2}}\left(J_1 - J_2 - J_3 + J_4\right), & Q^+ = Y_{14}, \quad Q^- = Y_{23};\\ R = \frac{i}{\sqrt{2}}\left(X_{13} - X_{24}\right), & R^{\pm} = \mp\frac{1}{2}\left(X_{12} + X_{34}\right) + \frac{i}{2}\left(X_{14} + X_{23}\right);\\ S = \frac{i}{\sqrt{2}}\left(X_{13} + X_{24}\right), & S^{\pm} = \pm\frac{1}{2}\left(X_{12} - X_{34}\right) + \frac{i}{2}\left(X_{14} - X_{23}\right). \end{array}$$

Thus the $(1,0)$-fields in the two theories generate the same algebra $\mathcal{A} = su(2)_1^{\,2} \oplus su(2)_1^{\,4} = \mathcal{A}_f \oplus \mathcal{A}_b$. Obviously, the same structure arises on the right handed sides.

iii. We will show that the spaces of states $\widetilde{\mathcal{H}}_b^1$ and $\widetilde{\mathcal{H}}_b^2$ of $\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ both have self dual $\mathcal{J} := (P,Q,R,S;\overline{P},\overline{Q},\overline{R},\overline{S})$-charge lattice

$$\widetilde{\Gamma}_b = \left\{ \tfrac{1}{\sqrt{2}}(x+y; x-y) \,\middle|\, x,y \in \mathbb{Z}^4 \right\}. \tag{3.19}$$

In the Gepner type model $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$ we find 16 fields with dimensions $h = \overline{h} = \frac{1}{4}$ which are uncharged under $(J,A;\overline{J},\overline{A})$; diagonalizing them with respect to the $\mathcal{J}$-action for $j \in \{P,Q,R,S\}$ we obtain fields $E_j^\pm, F_j^\pm$ uncharged under all $U(1)$-currents apart from j and with $(j,\overline{\jmath})$-charge $\frac{1}{\sqrt{2}}(\pm1,\pm1)$ and $\frac{1}{\sqrt{2}}(\pm1,\mp1)$, respectively. Namely,

$$\begin{aligned}
E_P^\pm &= \Phi^0_{\mp1,\mp1;\mp1,\mp1} \otimes \Phi^0_{\pm1,\pm1;\pm1,\pm1} \otimes \Phi^0_{\mp1,\mp1;\mp1,\mp1} \otimes \Phi^0_{\pm1,\pm1;\pm1,\pm1},\\
F_P^\pm &= \Phi^0_{\mp1,\mp1;\pm1,\pm1} \otimes \Phi^0_{\pm1,\pm1;\mp1,\mp1} \otimes \Phi^0_{\mp1,\mp1;\pm1,\pm1} \otimes \Phi^0_{\pm1,\pm1;\mp1,\mp1},\\
E_Q^\pm &= \Phi^0_{\mp1,\mp1;\mp1,\mp1} \otimes \Phi^0_{\pm1,\pm1;\pm1,\pm1} \otimes \Phi^0_{\pm1,\pm1;\pm1,\pm1} \otimes \Phi^0_{\mp1,\mp1;\mp1,\mp1},\\
F_Q^\pm &= \Phi^0_{\mp1,\mp1;\pm1,\pm1} \otimes \Phi^0_{\pm1,\pm1;\mp1,\mp1} \otimes \Phi^0_{\pm1,\pm1;\mp1,\mp1} \otimes \Phi^0_{\mp1,\mp1;\pm1,\pm1},
\end{aligned}$$

and with $\varepsilon_R := -1, \varepsilon_S := 1$ for $j \in \{R,S\}$

$$\begin{aligned}
E_j^\pm = \left(\Phi^1_{2,1;2,1}\right)^{\otimes4} &+ \varepsilon_j \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes4}\\
&\pm \big[\Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1}\\
&\qquad +\varepsilon_j\, \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1}\big],\\
F_j^\pm = \left(\Phi^1_{2,1;2,1}\right)^{\otimes2} &\otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes2} + \varepsilon_j \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes2} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes2}\\
&\pm \big[\Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1}\\
&\qquad +\varepsilon_j\, \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1}\big].
\end{aligned}$$

Among the corresponding charges under $\mathcal{J}$ we find $\frac{1}{\sqrt{2}}(e_i;\pm e_i)$ generating $\widetilde{\Gamma}_b$. In the sigma model $\widetilde{\mathcal{C}}^1$ we set

$$\begin{aligned}
\alpha_1 &:= \tfrac{1}{\sqrt{2}}\left(e_1+e_2\right), \quad & \alpha_2 &:= \tfrac{1}{\sqrt{2}}\left(e_2-e_1\right),\\
\alpha_3 &:= \tfrac{1}{\sqrt{2}}\left(e_1+e_3\right), \quad & \alpha_4 &:= \tfrac{1}{\sqrt{2}}\left(e_4-e_2\right).
\end{aligned}$$

Let Σ_δ with $\delta \in \mathbb{F}_2^4$ denote the twist field corresponding to the fixed point $\frac{1}{2}\sum_{i=1}^4 \delta_i\alpha_i$. The action of P,Q,R,S and their right handed partners is determined as in (3.10). Then by normalizing appropriately and matching $(\mathcal{J},\overline{\mathcal{J}})$-charges we find that the following identifications can be made (sums run over $\delta \in \mathbb{F}_2^4$ with the indicated restrictions):

$$E_P^\pm = \sum_{\delta_1=\delta_2,\delta_3=\delta_4} \Sigma_\delta \pm \sum_{\delta_1\neq\delta_2,\delta_3\neq\delta_4} \Sigma_\delta,$$

$$
\begin{aligned}
F_P^{\pm} &= \sum_{\delta_1\neq\delta_2,\delta_3=\delta_4} \Sigma_\delta \pm \sum_{\delta_1=\delta_2,\delta_3\neq\delta_4} \Sigma_\delta,\\
E_Q^{\pm} &= \sum_{\delta_1=\delta_2,\delta_3=\delta_4} (-1)^{\delta_4}\Sigma_\delta \pm \sum_{\delta_1\neq\delta_2,\delta_3=\delta_4} (-1)^{\delta_4}\Sigma_\delta,\\
F_Q^{\pm} &= \sum_{\delta_1\neq\delta_2,\delta_3\neq\delta_4} (-1)^{\delta_3}\Sigma_\delta \pm \sum_{\delta_1=\delta_2,\delta_3\neq\delta_4} (-1)^{\delta_3}\Sigma_\delta,\\
E_R^{\pm} &= \sum_{\delta_1=\delta_2,\delta_3=\delta_4} (-1)^{\delta_1}\Sigma_\delta \pm \sum_{\delta_1=\delta_2,\delta_3\neq\delta_4} (-1)^{\delta_1}\Sigma_\delta,\\
F_R^{\pm} &= \sum_{\delta_1\neq\delta_2,\delta_3\neq\delta_4} (-1)^{\delta_2}\Sigma_\delta \pm \sum_{\delta_1\neq\delta_2,\delta_3=\delta_4} (-1)^{\delta_2}\Sigma_\delta,\\
E_S^{\pm} &= \sum_{\delta_1=\delta_2,\delta_3=\delta_4} (-1)^{\delta_2+\delta_3}\Sigma_\delta \pm \sum_{\delta_1\neq\delta_2,\delta_3\neq\delta_4} (-1)^{\delta_2+\delta_3}\Sigma_\delta,\\
F_S^{\pm} &= \sum_{\delta_1\neq\delta_2,\delta_3=\delta_4} (-1)^{\delta_2+\delta_3}\Sigma_\delta \pm \sum_{\delta_1=\delta_2,\delta_3\neq\delta_4} (-1)^{\delta_2+\delta_3}\Sigma_\delta.
\end{aligned}
$$

In particular, the corresponding $(\mathcal{J},\overline{\mathcal{J}})$–charges generate $\widetilde{\Gamma}_b$.

□

Recall the Greene-Plesser construction for mirror symmetry [G-P] to observe that the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold $\widetilde{(2)}^4$ of $(2)^4$ is invariant under mirror symmetry. This can be regarded as an explanation for the high degree of symmetry found for $\widetilde{(2)}^4=\widetilde{\mathcal{C}}^1$.
In view of (3.19) it is clear that the same phenomenon as described in remark 3.4 appears for the theory discussed above:

Remark 3.8
The Gepner type model $\widetilde{\mathcal{C}}^1=\widetilde{(2)}^4$, or equivalently the nonlinear σ model $\widetilde{\mathcal{C}}^2=\mathcal{K}(\frac{1}{\sqrt{2}}D_4,B^*)$, B^* given by (3.16), is located at a meeting point of the moduli spaces of theories associated to $K3$ surfaces and tori, respectively. Namely, its bosonic sector is identical with that of the nonlinear σ model $\widetilde{\mathcal{C}}^3$ on the $SU(2)_1^4$-torus with vanishing B-field.

This again can be deduced from the results in [K-S] once one observes that the lattice denoted by $\Lambda_{O(n)\times O(n)}$ there in the case $n=4$ is isomorphic to $\widetilde{\Gamma}_b$ as defined in (3.19). The relation between the two meeting points $(2)^4=\mathcal{C}^1=\mathcal{C}^2\cong\mathcal{C}^3$ and $\widetilde{(2)}^4=\widetilde{\mathcal{C}}^1=\widetilde{\mathcal{C}}^2\cong\widetilde{\mathcal{C}}^3$ of the moduli spaces found so far is best understood by observing that $\widetilde{\mathcal{C}}^1=\widetilde{(2)}^4$ can be constructed from $\mathcal{C}^1=\widehat{(2)}^4$ by modding out $\mathbb{Z}_2\cong\langle[2,0,2,0]\rangle\subset\mathcal{G}_{alg}^{ab}$. If we formulate the orbifold procedure in terms of the charge lattice Γ_b of $\mathcal{C}^1=\widehat{(2)}^4$ as described in [G-P], this amounts to a shift orbifold by the vector $\delta=\frac{1}{2}(-1,1,0,0;1,-1,0,0)$ on Γ_b. Indeed, this shift simply reverts the shift we used to explain remark 3.4 and brings us back onto the torus $T_{SU(2)_1^4}$. But as for $\mathcal{C}^1=\mathcal{C}^2$ and $\mathcal{C}^3$, $\widetilde{\mathcal{C}}^1=\widetilde{\mathcal{C}}^2$ and $\widetilde{\mathcal{C}}^3$ will correspond to different compactifications of the type IIA string.
From (3.15) we are able to determine the geometric counterpart of $[2,0,2,0]$ on $\mathcal{K}(\mathbb{Z}^4,0)$: It is the unique nontrivial central element t_{1111} of the algebraic automorphism group $\mathcal{G}_{Kummer}^{+}$ depicted in figure 3.1. Hence the commutant of

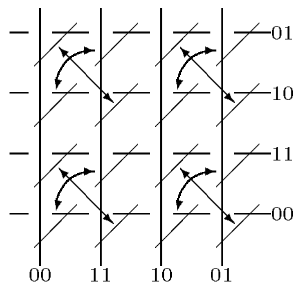


Fig. 3.1. Action of the algebraic automorphism t_{1111} on the Kummer lattice Π.

t_{1111} is the entire $\mathcal{G}^+_{Kummer}$, but it is not clear so far how to continue the residual $\mathcal{G}^+_{Kummer}/\mathbb{Z}_2$ algebraically to the twisted sectors in $(\widetilde{2})^4$ with respect to the t_{1111} orbifold.

We remark that conformal field theory also helps us to draw conclusions on the geometry of the Kummer surfaces under inspection: $\mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$ is obtained from $\mathcal{K}(\mathbb{Z}^4, 0)$ by modding out the classical symmetry t_{1111}, so in terms of the decomposition (1.4) we stay in the same "chart" of $\mathcal{M}^{K3}$, i.e. choose the same nullvector υ for both theories. This means that we can explicitly relate the respective geometric data. For both Kummer surfaces we choose the complex structures induced by the $N=(2,2)$ algebra in the corresponding Gepner models $(\widehat{2})^4$ and $(\widetilde{2})^4$. Thus we identify $J^{\pm} = \left(\Phi^0_{\mp 2,2;\mp 2,2}\right)^{\otimes 4}$ in both theories with the twoforms $\pi_*(dz_1 \wedge dz_2)$, $\pi_*(d\overline{z}_1 \wedge d\overline{z}_2)$ defining the complex structure of $\mathcal{K}(\Lambda)$. Here $\pi: T_\Lambda \to \mathcal{K}(\Lambda)$ is the rational map of degree two, $\Lambda = \mathbb{Z}^4$ or $\Lambda = \frac{1}{\sqrt{2}}D_4$, respectively. Then both $\mathcal{K}(\Lambda)$ are *singular* $K3$ surfaces (see section 2.5). Given the lattices of the underlying tori one can compute the intersection form for real and imaginary part of the above twoforms defining the complex structure. One finds that they span sublattices of the transcendental lattices with forms diag$(4,4)$ for $\mathcal{K}(\mathbb{Z}^4)$ and diag$(8,8)$ for $\mathcal{K}(\frac{1}{\sqrt{2}}D_4)$, respectively. The factor of two difference was to be expected, because t_{1111} has degree two. Nevertheless, one may check that the transcendental lattices themselves for both surfaces have quadratic form diag$(4,4)$. Note that for a given algebraic automorphism in general it is hard to decide how the transcendental lattices transform under modding out [In, Cor. 1.3.3]. In our case, we could read it off thanks to the Gepner type descriptions of our conformal field theories.

3.6. Gepner type description of $SO(8)_1/\mathbb{Z}_4$.

Theorem 3.9
The Gepner type model $\mathcal{C}^1 = (\widehat{2})^4$ which agrees with $\mathcal{C}^2 = \mathcal{K}(\mathbb{Z}^4, 0)$ by theorem 3.3 admits a nonlinear σ model description as $\mathbb{Z}_4$ orbifold of the torus model $\mathcal{T}(\frac{1}{\sqrt{2}}D_4, B^*)$ with $SO(8)_1$ symmetry.

Proof:
The proof works analogously to that of theorem 3.5. From theorem 2.8 it follows that the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\frac{1}{\sqrt{2}}D_4, B^*)$ with B^* defined by (3.16) is obtained from

$\widetilde{\mathcal{C}}^2 = \mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$ by modding out the automorphism r_{12} as depicted in figure 2.1. Thus we should work with the models $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$ and $\widetilde{\mathcal{C}}^2 = \mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$ which are isomorphic by theorem 3.7. We use the notations introduced there. Then r_{12} is induced by $e_1 \mapsto e_2, e_2 \mapsto -e_1, e_3 \mapsto -e_4, e_4 \mapsto e_3$. Of the $su(2)_1^6$ current algebra of $\widetilde{\mathcal{C}}^2$ we find a surviving $su(2)_1^2 \oplus u(1)^4$ current algebra on the $\mathbb{Z}_4$ orbifold generated by $J, J^\pm, A; P, P^\pm, Q, R, S$ (see equations (2.2) and (3.18)). The action on the generators $E_j^\pm, F_j^\pm; j \in \{P, Q, R, S\}$ is already diagonalized. All the $E_j^\pm$ are invariant as well as $F_P^\pm$. On the fermionic part of the space of states of $\widetilde{\mathcal{C}}^2$ the identifications (3.11) hold. The fields $W^J_{\varepsilon_1,\varepsilon_2}$ and $W^A_{\varepsilon_1,\varepsilon_1}, \varepsilon_i \in \{\pm 1\}$ are those invariant under the $\mathbb{Z}_4$ action. Our field by field identifications of theorem 3.7 now allow us to read off the induced action on the Gepner type model $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$. One checks that it agrees with the symmetry $[2', 2', 0, 0]$ defined in (3.12) which revokes the orbifold by the $\mathbb{Z}_2$ action of $[2, 2, 0, 0]$. Because $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$ was constructed from the Gepner model $(2)^4$ by modding out $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle [2, 2, 0, 0], [2, 0, 2, 0] \rangle \subset \mathcal{G}^{alg}_{ab}$, it follows that the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\frac{1}{\sqrt{2}}D_4, B^*)$ agrees with the Gepner type model obtained from $(2)^4$ by modding out $\mathbb{Z}_2 \cong \langle [2, 0, 2, 0] \rangle$. This clearly is isomorphic to $(\widehat{2})^4$ by a permutation of the minimal model factors. □

4. Conclusions: A panoramic picture of the moduli space

We conclude by joining the information we gathered so far to a panoramic picture of those strata of the moduli space we have fully under control now (figure 4.1).

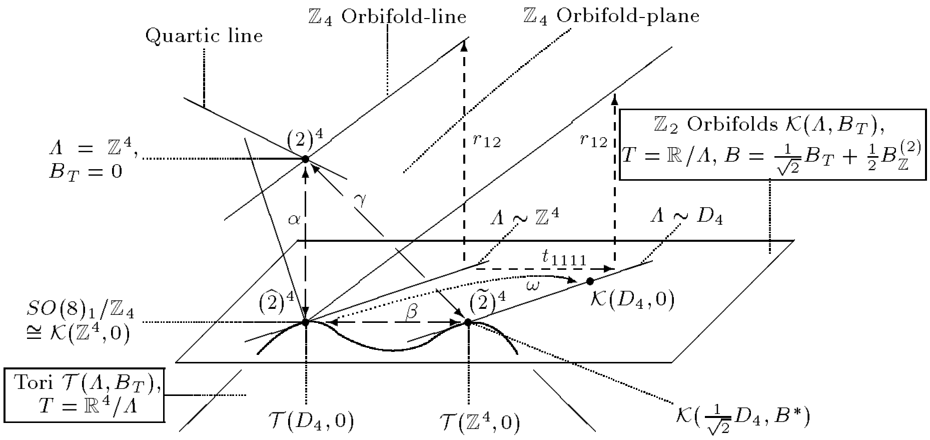


Fig. 4.1. Strata of the moduli space.

The rest of this section is devoted to a summary of what we have learned about the various components depicted in figure 4.1. All the strata are defined as quaternionic submanifolds of the moduli space $\mathcal{M}^{K3}$ consisting of theories which admit certain restricted geometric interpretations. In other words, a suitable choice of υ as described in section 1 yields (Σ, V, B) such that Σ, B have the respective properties. In the following we will always tacitly assume that an appropriate choice of υ has been performed already.

Figure 4.1 contains two strata of real dimension 16, depicted as a horizontal plane and a mexican hat like object, respectively. The horizontal plane is the *Kummer stratum*, the subspace of the moduli space consisting of all theories which admit a geometric interpretation on a Kummer surface X in the orbifold limit. In other words, it is the 16 dimensional moduli space of all theories $\mathcal{K}(\Lambda, B_T)$ obtained from a nonlinear σ model on a torus $T = \mathbb{R}^4/\Lambda$ by applying the ordinary $\mathbb{Z}_2$ orbifold procedure; the B-field takes values $B = \frac{1}{\sqrt{2}}B_T + \frac{1}{2}B_{\mathbb{Z}}^{(2)}$, where $B_T \in H^2(T,\mathbb{R}) \hookrightarrow H^2(X,\mathbb{R})$ (see the explanation after theorem 2.1), and $B_{\mathbb{Z}}^{(2)} \in H^{even}(X,\mathbb{Z})$ as described in theorem 2.3. We have an embedding $\mathcal{M}^{tori} \hookrightarrow \mathcal{M}^{K3}$ as quaternionic submanifold, and we know how to locate this stratum within $\mathcal{M}^{K3}$. Kummer surfaces in the orbifold limit have a generic group $\mathbb{F}_2^4$ of algebraic automorphisms which leave the metric invariant. Any conformal field theory associated to such a Kummer surface possesses an $su(2)_1^2$ subalgebra (2.2) of the holomorphic W-algebra.

The mexican hat like object in figure 4.1 depicts the moduli space (1.17) of theories associated to tori. Two meeting points with the Kummer stratum have been determined so far, namely $(\widehat{2})^4$ and $(\widetilde{2})^4$ (see remarks 3.4 and 3.8). We found $(\widehat{2})^4 = \mathcal{K}(\mathbb{Z}^4, 0) = \mathcal{T}(D_4, 0)$ and $(\widetilde{2})^4 = \mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*) = \mathcal{T}(\mathbb{Z}^4, 0)$, where B^* was defined in (3.16).

The vertical plane in figure 4.1 depicts a stratum of real dimension 8, namely the moduli space of theories admitting a geometric interpretation as $\mathbb{Z}_4$ orbifold of a nonlinear σ model on $T = \mathbb{R}^4/\Lambda$. In order for the orbifold procedure to be well defined we assume Λ to be generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1, 2$ (Λ_1 is not necessarily orthogonal to Λ_2) and $B_T \in H^2(T,\mathbb{R})^{\mathbb{Z}_4} \hookrightarrow H^2(X,\mathbb{R})$ (see lemma 2.9). The B-field then takes values $B = \frac{1}{2}B_T + \frac{1}{4}B_{\mathbb{Z}}^{(4)}$ as described in theorem 2.11, where the embedding of this stratum in $\mathcal{M}^{K3}$ is also explained. The generic group of algebraic automorphisms for $\mathbb{Z}_4$ orbifolds is $\mathbb{Z}_2 \ltimes \mathbb{F}_2^4$. By theorem 3.9 there is a meeting point with the Kummer stratum in the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\frac{1}{\sqrt{2}}D_4, B^*)$, where B^* is given by (3.16), which agrees with $\mathcal{K}(\mathbb{Z}^4, 0) = (\widehat{2})^4$.

The four lines in figure 4.1 are strata of real dimension 4 which are defined by restriction to theories admitting a geometric interpretation (Σ, V, B) with fixed Σ and allowed B-field values $B \in \Sigma$. Thus the volume is the only geometric parameter along the lines and we can associate a fixed hyperkähler structure on $K3$ to each of them. For all four lines it turns out that one can choose a complex structure such that the respective $K3$ surface is singular. Hence Σ can be described by giving the quadratic form on the transcendental lattice and the Kähler class for this choice of complex structure. Specifically we have:

- $\mathbb{Z}^4$*-line:* The subspace of the Kummer stratum given by theories $\mathcal{K}(\Lambda, B_T)$ with $\Lambda \sim \mathbb{Z}^4$ and $B_T \in \Sigma$, which is marked by $\Lambda \sim \mathbb{Z}^4$ in figure 4.1.
- $\mathbb{Z}_4$ *Orbifold-line:* The moduli space of all theories which admit a geometric interpretation on a $K3$ surface obtained from the nonlinear σ model on a torus

$T = \mathbb{R}^4/\Lambda$, $\Lambda \sim \mathbb{Z}^4$ with B-field B_T commuting with the automorphisms listed in (2.13).

- *Quartic line:* Though well established in the context of Landau-Ginzburg theories, this stratum has been somewhat conjectural up to now. We describe it as the moduli space of theories admitting a geometric interpretation $(\Sigma_{\mathcal{Q}}, V_{\mathcal{Q}}, B_{\mathcal{Q}})$ on the Fermat quartic (2.17) equipped with a Kähler metric in the class of the Fubini-Study metric, in order for $\Sigma_{\mathcal{Q}}$ to be invariant under the algebraic automorphism group $G = \mathbb{Z}_4^2 \rtimes \mathcal{S}_4$. The B-field is restricted to values $B_{\mathcal{Q}} \in \Sigma_{\mathcal{Q}}$, because $\mu(G) = 5$ and therefore $H^2(X,\mathbb{R})^G = \Sigma_{\mathcal{Q}}$.
- *D_4-line:* The moduli space of theories $\mathcal{K}(\Lambda, B_T)$, $\Lambda \sim D_4$ admitting as geometric interpretation a Kummer surface $\mathcal{K}(\Lambda)$ and $B_T \in \Sigma$. This line is labelled by $\Lambda \sim D_4$ in figure 4.1.

The four lines are characterized by the following data[4]:

name of line	associated form on the transcendental lattice	group of algebraic automorphisms leaving the metric invariant	generic $(1,0)$–current algebra
$\mathbb{Z}^4$-line	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\mathcal{G}^+_{Kummer} = \mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4$ $\cong (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes D_4$	$su(2)_1^2$
$\mathbb{Z}_4$ orbifold-line	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	D_4	$su(2)_1 \oplus u(1)$
quartic line	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathcal{S}_4$	$su(2)_1$
D_4-line	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\mathbb{Z}_2 \ltimes \mathbb{F}_2^4$	$su(2)_1^2$

In figure 4.1 we have two different shortdashed arrows indicating relations between lines. Consider the Kummer surface $\mathcal{K}(\mathbb{Z}^4)$ associated to the $\mathbb{Z}^4$-line. As demonstrated in theorem 2.8, the group $\mathcal{G}^+_{Kummer}$ of algebraic automorphisms of $\mathcal{K}(\mathbb{Z}^4)$ which leave the metric invariant contains the automorphism r_{12} of order two (see figure 2.1) which upon modding out produces the $\mathbb{Z}_4$ orbifold-line. The entire moduli space of $\mathbb{Z}_4$ orbifold conformal field theories is obtained this way from $\mathbb{Z}_2$ orbifold theories $\mathcal{K}(\Lambda, B_T)$, where Λ is generated by $\Lambda_i \cong R_i\mathbb{Z}^2$, $R_i \in \mathbb{R}, i = 1, 2$ and $B_T \in H^2(T,\mathbb{R})^{\mathbb{Z}_4}$.
Modding out $t_{1111} \in \mathcal{G}^+_{Kummer}$ (see figure 3.1) on the $\mathbb{Z}^4$-line produces the D_4-line, as argued at the end of section 3.5. Note that the $K3$ surfaces associated to $\mathbb{Z}^4$- and D_4-lines have the same quadratic form on their transcendental lattices and hence are identical as algebraic varieties. Still, the corresponding lines in moduli space are different because different Kähler classes are fixed. In our terminology this is expressed by the change of lattices of the underlying tori on transition from one line to the other. The D_4-line can also be viewed as the image of the $\mathbb{Z}^4$-line upon shift orbifold on the underlying torus.
Finally, we list the zero dimensional strata shown in figure 4.1.
To construct $\mathcal{K}(D_4, 0)$ on the D_4-line, we may as well apply the ordinary $\mathbb{Z}_2$ orbifold procedure to the D_4-torus theory in the meeting point $(\widehat{2})^4$ (the arrow

[4] The quadratic form for the transcendental lattice of quartic and the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\mathbb{Z}^4$ can be found in [In, Shi].

with label ω in figure 4.1). We stress that in contrast to what was conjectured in [E-O-T-Y] this is not a meeting point with the $\mathbb{Z}_4$ orbifold-plane.
As demonstrated in theorem 3.5 and also conjectured in [E-O-T-Y], Gepner's model $(2)^4$ is the point of enhanced symmetry $\Lambda = \mathbb{Z}^4, B_T = 0$ on the $\mathbb{Z}_4$ orbifold-line. In section 3.1 we have studied the algebraic symmetry group of $(2)^4$ and in corollary 3.6 proved that it admits a geometric interpretation with Fermat quartic target space, too. In terms of the Gepner model, the moduli of infinitesimal defomation along the $\mathbb{Z}_4$ orbifold and the quartic line are real and imaginary parts of $V^{\pm}_{\delta,\varepsilon}(\delta,\varepsilon \in \{\pm 1\})$ as in (3.14) and of the $(1,1)$-superpartners of $(\Phi^1_{\pm 1,0;\pm 3,2})^{\otimes 4}$, $(\Phi^1_{\pm 1,0;\pm 1,0})^{\otimes 4}$, respectively (see section 3.4).
The Gepner type models $\widehat{(2)}^4$ and $\widetilde{(2)}^4$ which are meeting points of torus and $K3$ moduli spaces have been mentioned above. For all the longdash arrowed correspondences $(2)^4 \overset{\alpha}{\longleftrightarrow} \widehat{(2)}^4 \overset{\beta}{\longleftrightarrow} \widetilde{(2)}^4 \overset{\gamma}{\longleftrightarrow} (2)^4$ in figure 4.1 we explicitly know the symmetries to be modded out from the Gepner (type) model as well as the corresponding algebraic automorphisms on the geometric interpretations. For instance, $\widetilde{(2)}^4 \xrightarrow{r_{12}} \widehat{(2)}^4 \xrightarrow{r_{12}} (2)^4$. Hence for these examples we know precisely how to continue geometric symmetries to the quantum level.

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A. Minimal models and Gepner models

The $N = 2$ minimal superconformal models form the discrete series $(k), k \in \mathbb{N}$ of unitary representations of the $N = 2$ superconformal algebra with central charges $c = 3k/(k+2)$. For constructing the model (k) we may start from a $\mathbb{Z}_k$ parafermion theory and add a free bosonic field. More precisely, (k) is the coset model

$$\frac{SU(2)_k \otimes U(1)_2}{U(1)_{k+2,diag}}\,. \tag{A.1}$$

The primary fields are denoted by $\Phi^l_{m,s;\overline{m},\overline{s}}(z,\overline{z})$, where $l \in \{0,\dots,k\}$ is twice the spin of the corresponding field in the affine $SU(2)_k$ and we have tacitly specialized to the diagonal invariant by imposing $l = \overline{l}$. The remaining quantum numbers $m, \overline{m} \in \mathbb{Z}_{2(k+2)}$ and $s, \overline{s} \in \mathbb{Z}_4$ label the representations of $U(1)_{k+2,diag}$ and $U(1)_2$ in the decomposition (A.1), respectively, and must obey $l \equiv m + s \equiv \overline{m} + \overline{s}$ (2). Here, the fields with even (odd) s create states in the lefthanded Neveu-Schwarz (Ramond) sector, and analogously for $\overline{s}$ and the righthanded sectors. Moreover the identification

$$\Phi^l_{m,s;\overline{m},\overline{s}}(z,\overline{z}) \sim \Phi^{k-l}_{m+2+k,s+2;\overline{m}+2+k,\overline{s}+2}(z,\overline{z}) \tag{A.2}$$

holds. By (A.1), the corresponding characters $X^l_{m,s;\overline{m},\overline{s}}$ can be obtained from the level k string functions $c^l_j, l \in \{0,\dots,k\}, j \in \mathbb{Z}_{2k}$ of $SU(2)_k$ and classical theta functions $\Theta_{a,b}, a \in \mathbb{Z}_{2b}$ of level $b = 2k(k+2)$ by [Ge2, R-Y, Qi]

$$\begin{aligned} X^l_{m,s;\overline{m},\overline{s}}(\tau,z) &= \chi^l_{m,s}(\tau,z)\cdot\chi^l_{\overline{m},\overline{s}}(\overline{\tau},\overline{z}), \\ \chi^l_{m,s}(\tau,z) &= \sum_{j=1}^{k} c^l_{4j+s-m}(\tau)\Theta_{2m-(k+2)(4j+s),2k(k+2)}(\tau,\frac{z}{k+2})\,. \end{aligned} \tag{A.3}$$

Modular transformations act by

$$\begin{aligned} \chi^l_{m,s}(\tau+1,z) &= exp\left[2\pi i\left(\frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8}-\frac{c}{24}\right)\right]\chi^l_{m,s}(\tau,z) \\ \chi^l_{m,s}\left(-\tfrac{1}{\tau},\tfrac{z}{\tau}\right) &= \kappa(k)\sum_{l',m',s'}\sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right)e^{\pi i\frac{mm'}{(k+2)}}e^{-\pi i\frac{ss'}{2}}\chi^{l'}_{m',s'}(\tau,z), \end{aligned} \tag{A.4}$$

where $\kappa(k)$ is a constant depending only on k and the summation runs over $l' \in \{0,\dots,k\}, m' \in \{-k-1,\dots,k+2\}, s' \in \{-1,\dots,2\}, l'+m'+s' \equiv 0\ (2)$.
Let $\psi^l_{m,s}$ denote a lowest weight state in the irreducible representation of the $N=2$ superconformal algebra with character $\chi^l_{m,s}$. Conformal dimension and charge of $\psi^l_{m,s}$ then are

$$h^l_{m,s} = \frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8}\bmod 1, \quad Q^l_{m,s} = \frac{m}{k+2}-\frac{s}{2}\bmod 2. \tag{A.5}$$

The fusion–algebra is

$$\left[\psi^l_{m,s}\right]\times\left[\psi^{l'}_{m',s'}\right] = \sum_{\substack{\overline{l}=|l-l'| \\ \overline{l}\equiv l+l'(2)}}^{\min(l+l',2k-l-l')}\left[\psi^{\overline{l}}_{m+m',s+s'}\right]. \tag{A.6}$$

Note that by (A.5) and (A.6) the operators of left and right handed spectral flow are associated to the fields $\Phi^0_{-1,-1;0,0} = \psi^0_{-1,-1}$ and $\Phi^0_{0,0;-1,-1} = \overline{\psi^0_{-1,-1}}$, respectively.
The NS-part of our modular invariant partition function is now given by

$$Z_{NS}(\tau,z) = \tfrac{1}{2}\sum_{\substack{l=0,\dots,k \\ m=-k-1,\dots,k+2 \\ l+m\equiv 0(2)}}\left(\chi^{l,0}_m(\tau,z)+\chi^{l,2}_m(\tau,z)\right)\left(\chi^{l,0}_m(\overline{\tau},\overline{z})+\chi^{l,2}_m(\overline{\tau},\overline{z})\right), \tag{A.7}$$

and expressions for the other three parts $Z_{\widetilde{NS}}, Z_R, Z_{\widetilde{R}}$ are obtained by flows as described in (2.4).
In the case $k=2$ which we employ in this paper, the parafermion algebra is nothing but the algebra satisfied by the Majorana fermion ψ of the Ising model. By inspection of the charge lattice one may confirm that the minimal model (2) can readily be constructed by tensoring the Ising model with the one dimensional free theory which describes a bosonic field φ compactified on a circle of radius $R=2$. The primary fields decompose as

$$\begin{aligned} \Phi^l_{m,s;\overline{m},\overline{s}}(z,\overline{z}) &= \Xi^l_{m-s;\overline{m}-\overline{s}}(z,\overline{z})\; e^{\frac{i}{2\sqrt{2}}(-m+2s)\varphi}(z)\; e^{\frac{i}{2\sqrt{2}}(-\overline{m}+2\overline{s})\overline{\varphi}}(\overline{z}) \\ \Xi^0_{j;\overline{j}}(z,\overline{z}) &= \Xi^2_{j\pm 2;\overline{j}\pm 2}(z,\overline{z}) \;=\; \xi^0_j(z)\xi^0_{\overline{j}}(\overline{z}), \quad \xi^0_0 = \mathbb{1},\ \xi^0_2 = \psi, \end{aligned} \tag{A.8}$$

and $\Xi^1_{1,1} = \Xi^1_{-1,-1}$, $\Xi^1_{1,-1} = \Xi^1_{-1,1}$ denote the ground states of the two $h = \overline{h} = \frac{1}{16}$ representations of the Ising model. Indeed, the level 2 string functions are obtained from the characters of lowest weight representations in the Ising model by dividing by the Dedekind eta function.

To construct a Gepner model with central charge $c = 3d/2, d \in \{2,4,6\}$, one first takes the (fermionic) tensor product of r minimal models $\otimes_{i=1}^r (k_i)$ such that the central charges add up to $\sum_{i=1}^r 3k_i/(k_i+2) = 3d/2$. The bosonic modes acting on different theories commute and the fermionic modes anticommute. More concretely [F-K-S, (4.5)],

$$\Phi^{l_1}_{m_1,s_1;\overline{m}_1,\overline{s}_1} \otimes \Phi^{l_2}_{m_2,s_2;\overline{m}_2,\overline{s}_2} = (-1)^{\frac{1}{4}(s_1-\overline{s}_1)(s_2-\overline{s}_2)} \Phi^{l_2}_{m_2,s_2;\overline{m}_2,\overline{s}_2} \otimes \Phi^{l_1}_{m_1,s_1;\overline{m}_1,\overline{s}_1}. \tag{A.9}$$

The diagonal sums $T, J, G^\pm$ of the fields which generate the $N=2$ algebras of the factor theories (k_i) then comprise a total $N=2$ superconformal algebra of central charge $c = 3d/2$. Denote by $\mathcal{Z}$ the cyclic group generated by $e^{2\pi i J_0}$, then $\mathcal{Z} \cong \mathbb{Z}_n$ with $n = \mathrm{lcm}\,\{2; k_i+2, i=1,\dots,r\}$. Now the Gepner model $\prod_{i=1}^r (k_i)$ is the orbifold of $\otimes_{i=1}^r (k_i)$ with respect to $\mathcal{Z}$. Effectively this means that $\prod_{i=1}^r (k_i)$ is obtained from $\otimes_{i=1}^r (k_i)$ by projecting onto integer left and right charges in the $(NS + \widetilde{NS})$-sector, onto integer or half integer left and right charges in the $(R+\widetilde{R})$-sector according to c being even or odd, and adding twisted sectors for the sake of modular invariance. In particular, the so constructed model describes an $N=(2,2)$ superconformal field theory with central charge $c = 3d/2$ and (half) integer charges. For $d=4$ the Gepner model is thus associated to a $K3$ surface or a torus, as discussed in the introduction. We again decompose the partition function as in (2.4) and find

$$\begin{aligned} Z_{NS}(\tau,z) = \sum_{b=0}^{n} \sideset{}{'}\sum_{(\boldsymbol{l},\boldsymbol{m})} \prod_{j=1}^{r} \Big(\tfrac{1}{2} \Big(\chi^{l_j,0}_{m_j}(\tau,z) + \chi^{l_j,2}_{m_j}(\tau,z) \Big) \cdot \\ \cdot \Big(\chi^{l_j,0}_{m_j+2b}(\overline{\tau},\overline{z}) + \chi^{l_j,2}_{m_j+2b}(\overline{\tau},\overline{z}) \Big) \Big), \end{aligned} \tag{A.10}$$

where $\sum'_{(\boldsymbol{l},\boldsymbol{m})}$ denotes the sum over all values $(\boldsymbol{l},\boldsymbol{m}) \in \mathbb{Z}^{2r}$ with $l_j \in \{0,\dots,k_j\}$, $m_j \in \{-k_j-1,\dots,k_j+2\}, l_j + m_j \equiv 0\ (2)$ and $\sum_{j=1}^r \frac{m_j}{k_j+2}, \sum_{j=1}^r \frac{\overline{m}_j}{k_j+2} \in \mathbb{Z}$. We note that the field $\prod_{j=1}^r \Phi^{l_i}_{m_j,s_j;\overline{m}_j,\overline{s}_j}$ of the resulting Gepner model belongs to the bth twisted sector with respect to the orbifold by $\mathcal{Z}$ iff $2b \equiv (\overline{m}_j - m_j) \bmod n$ for $j=1,\dots,r$. This means that the $(b+1)$st twisted sector is obtained from the bth twisted sector by applying the twofold right handed spectral flow which itself is associated to the primary field $\left(\Phi^0_{0,0;-2,2}\right)^{\otimes r}$ of our theory. We explicitly see that for $c=6$ the fields $\left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes r}$ belonging to the operators of twofold lefthanded spectral flow are nothing but the $SU(2)$-currents $J^\pm$ which extend the $N=2$ superconformal algebra to an $N=4$ superconformal algebra, and analogously for the righthanded algebra. Moreover, to calculate $Z_{NS}(\tau,z;\overline{\tau},\overline{z})$ instead of using the closed formula (A.10) one may proceed as follows: Start by multiplying the NS-parts of the partition functions of the minimal models $(k_i), i=1,\dots,r$. Keep only the $\mathcal{Z}$-invariant i.e. integrally charged part of this function; let us denote the result by $F(\tau,z;\overline{\tau},\overline{z})$. Add the bth twisted sectors, $b=1,\dots,n$, by performing a $2b$-fold righthanded spectral flow, i.e. by adding

$\overline{q}^{db^2/4}\overline{y}^{db/2}F(\tau, z; \overline{\tau}, \overline{z} + b\overline{\tau})$. This way calculations get extremely simple as soon as the characters of the minimal models are written out in terms of classical theta functions.

We further note that to accomplish Gepner's actual construction of a consistent theory of superstrings in $10 - d$ dimensions we would firstly have to take into account $8 - d$ additional free superfields representing flat (10-d)-dimensional Minkowski space in light-cone gauge, secondly perform the GSO projection onto odd integer left and right charges and thirdly convert the resulting theory into a heterotic one. However, at the stage described above we have constructed a consistent conformal field theory with central charge $c = 3d/2$ which for $d = 4$ is associated to a $K3$ surface or a torus, so we may and will omit these last three steps of Gepner's construction.

B. Explicit field identifications: $(\widehat{2})^4 = \mathcal{K}(\mathbb{Z}^4, 0)$

In this appendix, we give a complete list of $(\frac{1}{4}, \frac{1}{4})$-fields in $(\widehat{2})^4$ (see theorem 3.3) together with their equivalents in the nonlinear σ model on $\mathcal{K}(\mathbb{Z}^4, 0)$. As usual, $\varepsilon, \varepsilon_i \in \{\pm 1\}$ and we use notations as in (3.10) and (3.11).

Untwisted $(\frac{1}{4}, \frac{1}{4})$-fields with respect to the $\langle[2, 2, 0, 0]\rangle$-orbifold:

$$\begin{aligned}
\left(\Phi^0_{-\varepsilon_1,-\varepsilon_1;-\varepsilon_2,-\varepsilon_2}\right)^{\otimes 4} &= W^J_{\varepsilon_1,\varepsilon_2}\\
\left(\Phi^0_{-\varepsilon,-\varepsilon;-\varepsilon,-\varepsilon}\right)^{\otimes 2} \otimes \left(\Phi^0_{\varepsilon,\varepsilon;\varepsilon,\varepsilon}\right)^{\otimes 2} &= W^A_{\varepsilon,\varepsilon}\\
\left(\Phi^1_{2,1;2,1}\right)^{\otimes 4} &= \Sigma_{0000} - \Sigma_{1100} + \Sigma_{1111} - \Sigma_{0011}\\
\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 4} &= \Sigma_{1010} + \Sigma_{0101} - \Sigma_{0110} - \Sigma_{1001}
\end{aligned}$$

$$\left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1}$$
$$= \Sigma_{0000} - \Sigma_{1100} - \Sigma_{1111} + \Sigma_{0011} + \Sigma_{0010} + \Sigma_{0001} - \Sigma_{1101} - \Sigma_{1110}$$
$$\left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1}$$
$$= \Sigma_{0000} - \Sigma_{1100} - \Sigma_{1111} + \Sigma_{0011} - \Sigma_{0010} - \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}$$
$$\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2}$$
$$= \Sigma_{0000} + \Sigma_{1100} - \Sigma_{1111} - \Sigma_{0011} + \Sigma_{1000} + \Sigma_{0100} - \Sigma_{1011} - \Sigma_{0111}$$
$$\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2}$$
$$= \Sigma_{0000} + \Sigma_{1100} - \Sigma_{1111} - \Sigma_{0011} - \Sigma_{1000} - \Sigma_{0100} + \Sigma_{1011} + \Sigma_{0111}$$
$$\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1}$$
$$= (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) + (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011})$$
$$+ (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) + (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001})$$
$$\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1}$$
$$= (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) + (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011})$$
$$- (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) - (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001})$$
$$\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1}$$
$$= (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) - (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011})$$
$$- (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) + (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001})$$
$$\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1}$$
$$= (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) - (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011})$$
$$+ (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) - (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001})$$

Twisted $(\frac{1}{4}, \frac{1}{4})$-fields with respect to the $\langle[2,2,0,0]\rangle$-orbifold:

$$\left(\Phi^0_{-\varepsilon,-\varepsilon;\varepsilon,\varepsilon}\right)^{\otimes 2} \otimes \left(\Phi^0_{\varepsilon,\varepsilon;-\varepsilon,-\varepsilon}\right)^{\otimes 2} = W^A_{\varepsilon,-\varepsilon}$$
$$\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} = \Sigma_{1000} - \Sigma_{0100} + \Sigma_{0111} - \Sigma_{1011}$$
$$\left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} = \Sigma_{0010} - \Sigma_{0001} + \Sigma_{1101} - \Sigma_{1110}$$

$$\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1}$$
$$= \Sigma_{1000} - \Sigma_{0100} + \Sigma_{1011} - \Sigma_{0111} + \Sigma_{1010} - \Sigma_{0101} + \Sigma_{1001} - \Sigma_{0110}$$
$$\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1}$$
$$= \Sigma_{1000} - \Sigma_{0100} + \Sigma_{1011} + \Sigma_{0111} - \Sigma_{1010} + \Sigma_{0101} - \Sigma_{1001} + \Sigma_{0110}$$
$$\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2}$$
$$= \Sigma_{0010} - \Sigma_{0001} - \Sigma_{1101} + \Sigma_{1110} + \Sigma_{1010} - \Sigma_{0101} - \Sigma_{1001} + \Sigma_{0110}$$
$$\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2}$$
$$= \Sigma_{0010} - \Sigma_{0001} - \Sigma_{1101} + \Sigma_{1110} - \Sigma_{1010} + \Sigma_{0101} + \Sigma_{1001} - \Sigma_{0110}$$

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