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Theoretische Physik

Exercise 1
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Crash Course in Theoretical Particle Physics

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–Class Exercises–

1.1 Derivation and Representations of Gamma Matrices

In the lecture you have seen that one way to describe relativistic particles is the Dirac equation, which uses a set of four matrices which fulfill

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1}$$

In this exercise we will see why that is and that there are different ways of choosing these γ^μ.

(a) If we try to "linearise" the Klein Gordon equation $(\partial_\mu\partial^\mu + m^2)\psi = 0$, we can do a "Schrödinger"-like ansatz

$$\left(i\alpha_i\partial^i + \beta m\right)\psi = i\partial_t\psi \tag{2}$$

Squaring both sides of Eq. (2) should give the Klein-Gordon equation. Show that this leads to the following requirements for α and β,

$$\beta^2 = \alpha_i^2 = \mathbb{1}\,, \qquad \{\beta, \alpha_i\} = \{\alpha_i, \alpha_j\} = 0, \qquad i \neq j. \tag{3}$$

(b) Why can't the α_i and β be just numbers and why do they have to be at least of dimension 4×4?

(c) Show that multiplying Eq. (2) with β yields the Dirac equation if we define the Dirac gamma matrices γ^μ, where $\mu = 0, \ldots, 3$ as follows,

$$\gamma^0 = \beta\,, \qquad \gamma^i = \beta\alpha^i\,, \quad i = 1, 2, 3\,. \tag{4}$$

Show that these fulfill the known anticommutaton relations,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}\,, \tag{5}$$

by combining their definitions and properties from Eq. (3) and Eq. (4) .

(d) One possible representation for the γ^μ is the so–called "Dirac Representation"

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where $\sigma^i (i = 1, 2, 3)$ is Pauli matrix, $\mathbb{1}$ denotes the 2×2 unit matrix, and 0 is the 2×2 matrix of zeros.

Another possible representation is the so–called "Weyl basis", which defines γ^0 differently:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Think about why sometimes the one, sometimes the other representation can be advantageous to use!

(e) Show that they both fulfill the identity $\gamma^{\mu,\dagger} = \gamma^0\gamma^\mu\gamma^0$!

1.2 The Dirac Equation

The Dirac equation is given by

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

and its plane wave solutions are

$$\psi = \omega e^{-ip\cdot x}$$

where $p^\mu = (E, \vec{p})$, $\omega = N\begin{pmatrix}\phi\\ \chi\end{pmatrix}$.

ω is the four-component Dirac spinor which in the Dirac representation can conveniently be decomposed into two-component spinors ϕ and χ. N is a normalization constant.

(a) Define the conjugate spinor

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$$

and use the covariant form of the Dirac equation to derive the adjoint Dirac equation

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

.

(b) Show that the Dirac probability current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, i.e. $\partial_\mu j^\mu = 0$.

(c) Use the Dirac equation to find a coupled system of equations for ϕ and χ. Use them to find the "particle spinor" $u(p)$, with $E > 0$, and the "antiparticle spinor" $v(p') \equiv v(-p)$, with $E < 0$ as follows:

$$u(p) = N\begin{pmatrix}\phi\\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi\end{pmatrix}, \qquad v(-p) = N\begin{pmatrix}\frac{-\vec{p}\cdot\vec{\sigma}}{-E+m}\chi\\ \chi\end{pmatrix} \tag{6}$$

(d) Prove the Dirac equations for the free spinors $(\not{p} - m)u(p) = 0, (\not{p} + m)v(p) = 0$