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Theoretische Physik

Exercise 2
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Crash Course in Theoretical Particle Physics

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–Class Exercises–

2.1 Fun with equations of motion

(a) Calculate the equation of motion for a free scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{1}$$

and show that it reproduces the Klein-Gordon equation!

(b) Calculate the equation of motions for a free Dirac fermion

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2}$$

and show that it reproduces the Dirac equations for ψ and $\bar{\psi}$!

2.2 Identifying Particles in the Lagrangian
Fields that appear in the Lagrangian do not necessarily correspond to physically observable particles with a well defined mass! In this exercise we will discuss this and understand the concept of "mass eigenstates". Consider the following Lagrangian

$$\mathcal{L} = \bar{\psi}_1 i\gamma^\mu \partial_\mu \psi_1 + \bar{\psi}_2 i\gamma^\mu \partial_\mu \psi_2 - m_{11}\bar{\psi}_1\psi_1 - m_{12}(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1) - m_{22}\bar{\psi}_2\psi_2 \tag{3}$$

(a) Find the equations of motion for ψ_1 and ψ_2. Do they correspond to free Dirac equations of two particles ψ_1 and ψ_2?

(b) Write the mass terms in Eq. (3) in the form of a matrix $\mathcal{M}$ like $\mathcal{L}_\mathcal{M} = -\bar{\boldsymbol{\psi}}\mathcal{M}\boldsymbol{\psi}$ with $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$.

(c) Diagnoalize $\mathcal{L}$, that is: find the eigenvalues M_1 and M_2 of $\mathcal{M}$ and use an orthogonal transformation

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{4}$$

to find a Lagrangian in which no mixed term $m\bar{\chi}_1\chi_2$ occurs.

(d) Calculate the equations of motion for χ_1 and χ_2. What do you conclude?

(e) Imagine the original ψ_1 had an interaction term with a scalar field ϕ of the form $\kappa\phi\bar{\psi}_1\psi_1$ and ψ_2 did not couple to ϕ. What does this mean for the particles χ_1 and χ_2? Which one of them interacts with ϕ? What does the interaction strength depend on?

2.3 Noether's theorem
Consider the following Lagrangian density with two *real* scalar fields $\phi_i(x)$ $(i = 1, 2)$

$$\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \phi_i \partial^\mu \phi_i - m^2\phi_i^2).$$

(a) Obtain the Euler-Lagrange equation of motion for the two fields.

(b) Show that this Lagrangian is invariant under the following transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{5}$$

The 2×2 matrix appearing in this transformation can be interpreted as an element of the group $SO(2)$ of 2-dimensional rotations.

(c) Invariance under the transformation (5) means that this is a symmetry of the Lagrangian. Obtain the Noether current that corresponds to this symmetry.

(d) Check that the Noether current is indeed conserved.

(e) Rewrite the Lagrangian in terms of a complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$.

(f) Rewrite the transformation (5) in terms of the complex field ϕ.

(g) Find the Noether current for this rewritten transformation. Check that it is the same as above.