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Exercise 3
7th October 2015
WS 15/16

# Crash Course in Theoretical Particle Physics

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–Homework Exercises–

### 3.3 Higgs boson decay

The decay width of the Higgs boson (into two particles) is given by

$$\Gamma = \frac{1}{2m_h} \int \frac{d^3p}{2E_p(2\pi)^3} \frac{d^3k}{2E_k(2\pi)^3} |F|^2 (2\pi)^4 \delta^{(4)}(p_h - (p+k)), \tag{1}$$

where $p_h$ is the Higgs 4-momentum and $p$ and $k$ denote the 4-momenta of the final state particles.

(a) We consider two final state particles with equal masses $m$, in this case $F$ does not depend on $p$ and $k$. To simplify the integration, show that

$$\delta^{(4)}(p_h - (p+k)) = \frac{E}{2|\vec{p}|} \delta\left(|\vec{\boldsymbol{p}}| - \frac{m_h}{2}\sqrt{1 - \frac{4m^2}{m_h^2}}\right) \delta^{(3)}(\vec{\boldsymbol{p}} + \vec{\boldsymbol{k}}). \tag{2}$$

(b) Using your result from (a), show that the Lorentz invariant phase space (LIPS) for two particles is given by

$$\begin{aligned} \text{LIPS} &= \int \frac{d^3p}{2E_p(2\pi)^3} \frac{d^3k}{2E_k(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_h - (p+k)) \\ &= \frac{1}{8\pi}\sqrt{1 - \frac{4m^2}{m_h^2}}. \end{aligned}$$

With this result, Eq. (1) simplifies to

$$\Gamma = \frac{1}{2m_h} |F|^2 \text{LIPS}.$$

### 3.4 Muon lifetime

*Beware: This exercise is tough so don't worry if you don't manage to get to the very end. However, you should try to solve as many subexercises as you can! (In most cases, intermediate results that we give allow you to continue calculations even if you are stuck in the middle!*

The muon is the lightest unstable particle in the SM. The goal of this exercise is to compute its lifetime. The decay rate $\Gamma$ is the decay probability per unit time, e.g.

$$\frac{\mathrm{d}N}{N} = -\Gamma\,\mathrm{d}t. \tag{3}$$

The lifetime $\tau = 1/\Gamma$, is the time it takes for a sample to decay until one is left with the fraction $1/e$ of the original amount. The muon decays via $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ and we can consider the final state particles to be massless[1]. As you will learn to calculate in the TPP lecture, in the rest frame of the muon, the Feynman amplitude takes the form

$$|F|^2 = \frac{g^4}{M_W^4} m_\mu^2 E_{\bar{\nu}_e}(m_\mu - 2E_{\bar{\nu}_e}). \tag{4}$$

(a) We now want to evaluate the phase space integral similarly to before, but now with three final state particles:

$$\mathrm{d}\Gamma = \frac{1}{2m_\mu}\left(\frac{\mathrm{d}^3 p_{\bar{\nu}_e}}{(2\pi)^3 E_{\bar{\nu}_e}}\right)\left(\frac{\mathrm{d}^3 p_{\nu_\mu}}{(2\pi)^3 E_{\nu_\mu}}\right)\left(\frac{\mathrm{d}^3 p_e}{(2\pi)^3 E_e}\right) \times (2\pi)^4 \delta^4\left(p_\mu - p_{\bar{\nu}_e} - p_{\nu_\mu} - p_e\right)|F|^2. \tag{5}$$

As a first step, perform the integration over $\mathrm{d}^3 p_{\nu_\mu}$, you should obtain

$$\mathrm{d}\Gamma = \frac{|F|^2}{16(2\pi)^5 m_\mu}\left(\frac{\mathrm{d}^3 p_{\bar{\nu}_e}\,\mathrm{d}^3 p_e}{E_{\bar{\nu}_e} E_{\nu_\mu} E_e}\right) \times \delta\left(m_\mu - E_{\bar{\nu}_e} - E_{\nu_\mu} - E_e\right). \tag{6}$$

(b) Recall that we have assumed that all the decay products are massless, show that in this approximation the following relation holds

$$E_{\nu_\mu}^2 = E_{\bar{\nu}_e}^2 + E_e^2 + 2E_{\bar{\nu}_e} E_e \cos\theta, \tag{7}$$

where $\theta$ is the angle between $\vec{\boldsymbol{p}}_{\bar{\nu}_e}$ and $\vec{\boldsymbol{p}}_e$. Next write $\mathrm{d}^3 p_e$ in spherical coordinates and aided by the above relation show that it can be written as

$$\mathrm{d}^3 p_{\bar{\nu}_e} = -\frac{E_{\bar{\nu}_e} E_{\nu_\mu}}{E_e}\,\mathrm{d}E_{\bar{\nu}_e}\,\mathrm{d}E_{\nu_\mu}\,\mathrm{d}\phi\,, \tag{8}$$

with $\phi$ being the polar angle, i.e. $\phi \in [0, 2\pi)$.

(c) Use the previous relations to show that after integrating out the angular dependencies $\mathrm{d}\Gamma$ takes the form

$$\mathrm{d}\Gamma = \frac{|F|^2}{16(2\pi)^4 m_\mu}\frac{\mathrm{d}^3 p_e\,\mathrm{d}E_{\bar{\nu}_e}}{E_e^2}\int_{|E_{\bar{\nu}_e}-E_e|}^{|E_{\bar{\nu}_e}+E_e|}\mathrm{d}E_{\nu_\mu}\,\delta\left(m_\mu - E_{\bar{\nu}_e} - E_{\nu_\mu} - E_e\right). \tag{9}$$

(d) Show that, according to this expression, $\mathrm{d}\Gamma$ vanishes unless

$$\frac{1}{2}\left(|E_{\bar{\nu}_e} - E_e| + E_{\bar{\nu}_e} + E_e\right) \le \frac{m_\mu}{2} \le \left(E_{\bar{\nu}_e} + E_e\right). \tag{10}$$

The right handed side implies that the combined energy of the electron and the neutrino must be at least half of the muon masses. The left handed side stipulates that the most energy either of these particles can have is half of the muon rest energy.

(e) Use the previous result and the explicit form of the matrix element to perform the integration over $E_{\bar{\nu}_e}$.

(f) Finally perform the $\mathrm{d}^3 p_e$ integration in order to obtain the decay rate. Make a numerical estimation of the muon lifetime. If you actually manage to do the full calculation and reach this point with a final answer of $\tau = \mathcal{O}(10^{-6})$ s, you'll earn our full respect!

[1] Not doing that will change the difficulty of this calculation from "Nightmare" to "Hell"!