Exercise-sheet 1 (19th - 21st of October 2009)

1 In class exercise:

1.1 Potentialbarrier (revision of Theoretical Physics III)

A particlebeam of energy $0 < E < V_0$ is originating at $-\infty$ and approching a potential barrier:
$V(x) = \begin{cases} V_0 & |x| < a \\ 0 & otherwise \end{cases}$

1.1. Write down the Schrödingerequation and the Ansatz for the wave-function for the three areas. The incoming wave has an amplitude of 1.

1.2. What are the connection conditions of the wavefunction?

1.3. Calculate the transmissioncoefficient T and the reflectioncoefficient R and verify $T + R = 1$

1.4. Show that for the case $E << V_0$ the following is valid:
$T \approx (\frac{4k\mu}{k^2+\mu^2})^2 exp(-2S_0)$ with
$k^2 = \frac{2mE}{\hbar^2}$, $\mu^2 = \frac{2m(V_0-E)}{\hbar}$ and $S_0 = \frac{1}{\hbar}\int_{-a}^{a} dx\sqrt{2m(V(x)-E)}$

1.2 Besselfunctions

A plane wave can be expaned as Besselfunctions:
$e^{i\vec{k}\vec{r}} = \sum_{l=0}^{\infty} c_l j_l(kr) P_l(\cos\theta)$
here θ is the angle between $\vec{k}$ and $\vec{r}$. Calculate the coefficients c_l.
Hint: Derive the following equation with respect to kr and use the recursive relations:

$$\begin{aligned} (2l+1)j_l'(x) &= lj_{l-1}(x) - (l+1)j_{l+1}(x) \\ (2l+1)xP_l(x) &= (l+1)P_{l+1}(x) + lP_{l-1}(x) \end{aligned}$$

2 Homework - due date: 26th of october 2009 at 16:00.

2.1 Basics about scattering theory - 1-d example (21 points)

consider a potential which vanishes for $x \geq 0$ and doesnt for $x < 0$.

2.1. Show that $\psi_k(x,t) = e^{-\frac{i}{\hbar}E(k)t}\phi_k(x)$, $\phi_k(x) = A(k)e^{ikx} + B(k)e^{-ikx}$, $E(k) = \frac{\hbar^2 k^2}{2m}$ is a solution for the Schödingerequation for $x \geq 0$.

2.2. Why one can assume $\phi_k(x)$ to be real? Show $A(k) = B(-k)$.

2.3. Define the Jostfunction as $F(k) = 2ikA(k)$. Show that $F^*(k) = F(-k)$.

2.4. We write $F(k) = |F(k)|e^{i\delta(k)}$, with $\delta(k)$ as scattering phase. Show $\phi_k(x) = \frac{1}{k}|F(k)|\sin(kx + \delta(k))$.

2.5. Show that $\phi_k(x)$ can be written with the scattering function $S(k)$ as: $\phi_k(x) = -\frac{1}{2ik}F(-k)(e^{-ikx} - S(k)e^{ikx})$, with $S(k) := \frac{F(k)}{F(-k)}$. What is the physical meaning of $S(k)$? Show that $S(k) = e^{2i\delta(k)}$implying $|S(k)|^2 = 1$.

2.6. Show:
$S(k) = \frac{1+ik\frac{\phi_k(0)}{\phi'_k(0)}}{1-ik\frac{\phi_k(0)}{\phi'_k(0)}}$
The term $R(E) := \frac{\phi_k(0)}{\phi'_k(0)}$ is called R-function and because of the logarithmic derivation, it does not depend on the normalization of the wave-function. Why R is only a function of k^2 and therefore of E?

2.7. The scattering length is defined as $a = \lim_{k\to 0}\frac{\delta(k)}{k}$. One can show, that this limit indeed exists. The scattering amplitute $f(k)$ is defined as $f(k) = \frac{S(k)-1}{2ik}$. Show $|f(0)|^2 = a^2$ and $|f(k)|^2 \leq \frac{1}{k^2}$.

2.2 Spherically symmetric and localized potential (9 points)

Consider the elastic scattering of a plane wave e^{ikr}at a localized pherical potential $V(r)$. Because of the boundary conditions (resulting from the set up of the experiment) the problem is rotationally symmetric. Outside ($V(r) = 0$) the solution of this problem is given through a linear combination of the solutions of the hamiltonian for the free particle:

$$\psi_k(r) = \sum_{l=0}^{\infty} c_l R_l(kr) P_l(\cos\theta)$$

with $R_l(kr) = h_l^*(kr) + S_l h_l(kr)$.

2.1. Why the radial componente of the current density has to vanish?

2.2. Why does this condition hold for each partial wave?

2.3. Show that $|S_l| = 1$ for all l.