Exercise-sheet 12 (18th - 20th of January 2010)

1 In class exercise:

1.1 Schrödinger-, Heisenberg- and interaction-picture

Naturally the subscript "S/H/I" are for states or operators in the Schrödinger/Heisenberg/Interaction-picture.

1. Repeat the main ideas of the Schödinger-, Heisenberg- and Interaction(=Dirac) picture.

2. When are the state-vectors for all three pictures the same?

3. Show $\frac{\partial}{\partial t}|\psi_H(t)\rangle = 0$.

4. Calculate the Matrix elements of $A_H(t) = e^{\frac{i}{\hbar}Ht} A_s e^{-\frac{i}{\hbar}Ht}$.

5. Show $\frac{d}{dt}A_H(t) = \frac{i}{\hbar}[H, A_H(t)]$. (equation of motion!)

6. Show $H_I(t)|\psi_I(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle$

7. Show $i\hbar\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0)$ and give a formal solution.

8. Compare the equation of motion of the interaction and Heisenberg-picture.

9. Consider now the so called 'Hubbard-model' in one dimension. This is a strongly simplified model of electrons on a chain with the following Hamiltonian:

$H = -t\sum_{i,\alpha=\uparrow,\downarrow}(c^\dagger_{i+1,\alpha}c_{i,\alpha} + c^\dagger_{i,\alpha}c_{i+1,\alpha}) + U\sum_i c^\dagger_{i,\uparrow}c_{i,\uparrow}c^\dagger_{i,\uparrow}c_{i,\downarrow}$

a) Calculate the commutators of $c^\dagger_{i,\alpha}(t)$ and $c_{i,\alpha}(t)$ with H.
b) Obtain the equation of motion of the ladder operators in the Heisenberg-picture.

2 Bonus Homework - due date: 25.01.2010 at 16:00 (40 bonus points).

If you are slightly below the 50 % criteria for the exercises 1-11, you have here the chance to earn 'bonus points' to qualify for the exam (however if you are far below 50 % then that won't save you). For all others the following exercises can be considered as further examples to prepare for the exam.

2.1 Scattering theory (9 points)

A particle of mass m is scattered at a sphercially symmetric potential of the form:
$V(r) = V_0\Theta(R - r)$
We consider s-wave scattering only!

1. Calculate the scattering phase $\delta_{l=0}(k)$ by solving the Schrödingerequation exactly.

2. Discuss the dependence of $\delta_{l=0}(k)$ from the sign of V_0.

3. Find the scattering amplitute and the differential cross section.

2.2 Vibrational Modes of Benzene (12 points)

Consider the molecule of Benzene C_6H_6, where the carbon-atoms are arranged on a hexagon and each of the hydrogen-atoms are attached to the carbon-atoms. The symmetry group for Benzene is D_{6h} for which the character table is given below.

1. Sketch the molecule, and find how many degrees of freemdom are there for vibrations.

2. Find the characters of the vibrational modes.

3. Use the character table below to decompose the representation into irreducible representations to classify the vibrational modes. Point out whether there is degeneracy.

D_{6h}	1	C_2	$2C_3$	$2C_6$	$3U_2$	$3U_2'$	I	S_2	$2S_3$	$2S_6$	$3\sigma_d$	$3\sigma_d'$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	1	1	1	-1	-1	1	1	1	1	-1	-1
B_{1g}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
B_{2g}	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1
E_{2g}	2	2	-1	-1	0	0	2	2	-1	-1	0	0
E_{1g}	2	-2	-1	1	0	0	2	-2	-1	1	0	0
A_{1u}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
A_{2u}	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
B_{1u}	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
B_{2u}	1	-1	1	-1	1	1	-1	1	-1	1	1	-1
E_{2u}	2	2	-1	-1	0	0	-2	-2	1	1	0	0
E_{1u}	2	-2	-1	1	0	0	-2	2	1	-1	0	0

2.3 Wigner-Eckart Theorem (10 points)

1. Calculate the reduced matrix element $\langle \tau' j' || J^{(1)} || \tau j \rangle$ of the angular momentum operator $\vec{J}$ in a standard basis $|\tau j m\rangle$.
You may look up the needed Clebsch-Gordon coefficient below.

2. Proof the following: Be $\vec{A}$ a vectoroperator to the angular momentum $\vec{J}$, then the following is valid for a standard-basis:
$\langle \tau j m | A_z | \tau j m \rangle = \frac{m\hbar}{j(j+1)\hbar^2} \langle \tau j m | \vec{J}\vec{A} | \tau j m \rangle$
Hint: consider $\vec{J}\vec{A} = \sum_{i=1}^{3} A_i V_i = \sum_{q=1,0,-1} (-1)^q A_q^{(1)} B_{-q}^{(1)}$. You may look up the needed Clebsch-Gordon coefficient below.

$\langle j_1 1, m - m_2, m_2 | j m \rangle$

	$m_2 = 1$	$m_2 = 0$	$m_2 = 1$
$j = j_1 + 1$	$\sqrt{\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+1)}}$
$j = j_1$	$-\sqrt{\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}}$	$\frac{m}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}}$
$j = j_1 - 1$	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}}$

2.4 Second quantization (9 points)

Give the following quantities in second quantization:

1. particle density $\rho(x)$ and the fouriertransform $\rho(q) = \sum_x e^{-iqx} \rho(x)$

2. current density $j(x)$ and the fouriertransform $j(q) = \sum_x e^{-iqx} j(x)$

3. kinetic energy E