Exercise-sheet 9 (14th - 16th of December 2009)

1 In class exercise:

1.1 Second quantization

1. Consider a system of L harmonic oscilators, which are placed on lattice positions of a periodic chain. For this Bose-operators the commutor is $[a_n, a_m^\dagger] = \delta_{n,m}$. Be $\tilde{a}_q = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{-iqn} a_n$ the fouriertransform of this operators. What is $[\tilde{a}_p, \tilde{a}_q^\dagger]$?

2. Consider now the ladder operators for Fermions. For these:
$\{c_{n,\alpha}, c_{m,\beta}^\dagger\} = c_{n,\alpha} c_{m,\beta}^\dagger + c_{m,\beta}^\dagger c_{n,\alpha} = \delta_{n,m} \delta_{\alpha,\beta}$
$\{c_{n,\alpha}, c_{m,\beta}\} = \{c_{n,\alpha}^\dagger, c_{m,\beta}^\dagger\} = 0$
Here $\{,\}$ is the anti-commutator. α and β are the spinindices. These can take the values $\frac{1}{2} :=\uparrow$ and $-\frac{1}{2} :=\downarrow$.
Consider again the fouriertransform and calculate $\{\tilde{c}_{p,\alpha}, \tilde{c}_{q,\beta}^\dagger\}$, $\{\tilde{c}_{p,\alpha,} \tilde{c}_{q,\beta}\}$ and $\{\tilde{c}_{p,\alpha,}^\dagger \tilde{c}_{q,\beta}^\dagger\}$.

3. The spin-operators are defined as:
$S_n^i = \frac{\hbar}{2} \sum_{\alpha,\beta} c_{n,\alpha}^\dagger \sigma_{\alpha,\beta}^i c_{n,\beta}$.
Are those operators consistent with an angular-momentum-algebra?

2 Homework - due date: 21st of December 2009 at 16:00 (20 points).

2.1 Tensoroperators and Wigner-Eckart Theorem (20 points)

Given are the following operators:
$R^{(1)}_{\pm} = \mp\frac{1}{\sqrt{2}}(x_1 \pm ix_2) = \sqrt{\frac{4\pi}{3}} r Y_{1,\pm 1}(\Omega)$
$R^{(1)}_0 = x_3 = \sqrt{\frac{4\pi}{3}} r Y_{1,0}(\Omega)$
Consider the angular momentum operators ($L_\pm = L_1 \pm iL_2$) and remember that $[x_j, p_k] = i\hbar\delta_{jk}$ (*).

1. Use (*) to show that:
$[L_\pm, R^{(1)}_q] = \sqrt{2 - q(q \pm 1)}\hbar R^{(1)}_{q\pm 1}$
$[L_3, R^{(1)}_q] = q\hbar R^{(1)}_q$
with $q = 1, 0, -1$
This shows that $\vec{R}$ is a Tensoroperator of rank 1 (=Vektoroperator).

2. Consider $\vec{R}^2 = x_1^2 + x_2^2 + x_3^2$. Show that $[L_j, \vec{R}^2] = 0$.

3. Consider the space of a particle with three degrees of freedom ($\vec{r} = (x_1, x_2, x_3)$). Suppose the state $|nlm\rangle$: for a fixed n this state is in the stanard-basis for angular-momentum (in other words l, m are the quantumnumbers for $\vec{L}^2$ and L_z). The wave-function in position-space is then $\langle\vec{r}|nlm\rangle = u_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\Omega)$. Calculate the reduced Matrixelement $\langle n'l'||R^{(1)}||nl\rangle$ with $R^{(1)}_q$ as defined above, and the help of the given wave-function.
Hint: $Y_{\lambda\mu}(\Omega)Y_{lm}(\Omega) = \sum_{L=|l-\lambda|}^{l+\lambda} \sqrt{\frac{(2l+1)(2\lambda+1)}{4\Pi(2L+1)}}\langle l\lambda m\mu|L, m+\mu\rangle\langle l\lambda 00|L0\rangle Y_{L,m+\mu}(\Omega)$

4. What is the relation between l and l' so that the reduced Matrixelement in (3.) is non-zero?

5. Now consider a $T^{(2)}_q$ for $q = 2, 1, 0, -1, -2$ again composed out of the spherical standard-components of an irreducible Tensoroperator with respect to the operator $\vec{J}$. $|\tau jm\rangle$ be now a state where for a fixed τ this gives a standard basis for $\vec{J}^2$ and J_z. Show that the expectation value of $T^{(2)}_0$ is zero for $j = 0$ and $j = \frac{1}{2}$.