Identities for Pauli–matrices

Define the "generalized Pauli–matrices" as

$$\sigma^\mu \equiv (I, \sigma^p)\,; \tag{1a}$$

$$\overline{\sigma}^\mu \equiv (I, -\sigma^p)\,. \tag{1b}$$

where σ^p stands for the three ordinary Pauli–matrices, and I is the 2–dimensional unit matrix. σ^μ and $\overline{\sigma}^\mu$ have one dotted and one undotted index each. These can be manipulated using the "metric in spinor space" ϵ, with $\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1$, $\epsilon^{AA} = \epsilon_{AA} = 0$:

$$\bar{\sigma}^{\mu \dot{A} B} = \epsilon^{\dot{A}\dot{C}} \epsilon^{BD} \sigma^\mu_{D\dot{C}}\,; \tag{2a}$$

$$\sigma^\mu_{A\dot{B}} = \epsilon_{AC} \epsilon_{\dot{B}\dot{D}} \bar{\sigma}^{\mu \dot{D} C}\,. \tag{2b}$$

We also need

$$\sigma^{\mu\nu} = \frac{i}{4} \left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu \right)\,; \qquad \bar{\sigma}^{\mu\nu} = \frac{i}{4} \left(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \right) = \left(\sigma^{\mu\nu} \right)^\dagger\,. \tag{3}$$

The following identities follow directly from the properties of the Pauli–matrices:

$$\sigma^\mu_{A\dot{B}} \sigma_{\mu C\dot{D}} = 2 \epsilon_{AC} \epsilon_{\dot{B}\dot{D}}\,; \tag{4a}$$

$$\bar{\sigma}^{\mu \dot{A} B} \bar{\sigma}_\mu^{\ \dot{C} D} = 2 \epsilon^{BD} \epsilon^{\dot{A}\dot{C}}\,; \tag{4b}$$

$$\sigma^\mu_{\ A\dot{B}} \bar{\sigma}_\mu^{\ \dot{C} D} = 2 \delta_A^{\ D} \delta^{\dot{C}}_{\ \dot{B}}\,; \tag{4c}$$

$$\left(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu \right)_A^{\ B} = 2 g^{\mu\nu} \delta_A^{\ B}\,; \tag{4d}$$

$$\left(\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu \right)^{\dot{A}}_{\ \dot{B}} = 2 g^{\mu\nu} \delta^{\dot{A}}_{\ \dot{B}}\,; \tag{4e}$$

$$\sigma^\mu \bar{\sigma}^\nu \sigma^\rho + \sigma^\rho \bar{\sigma}^\nu \sigma^\mu = 2 (g^{\mu\nu} \sigma^\rho + g^{\nu\rho} \sigma^\mu - g^{\mu\rho} \sigma^\nu)\,; \tag{4f}$$

$$\bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho + \bar{\sigma}^\rho \sigma^\nu \bar{\sigma}^\mu = 2 (g^{\mu\nu} \bar{\sigma}^\rho + g^{\nu\rho} \bar{\sigma}^\mu - g^{\mu\rho} \bar{\sigma}^\nu)\,; \tag{4g}$$

$$\mathrm{tr} \left(\sigma^\mu \bar{\sigma}^\nu \right) = 2 g^{\mu\nu}\,; \tag{4h}$$

$$\mathrm{tr} \left(\sigma^\mu \bar{\sigma}^\nu \sigma^\rho \bar{\sigma}^\tau \right) = 2 (g^{\mu\nu} g^{\rho\tau} + g^{\mu\tau} g^{\nu\rho} - g^{\mu\rho} g^{\nu\tau} - i \epsilon^{\mu\nu\rho\tau})\,; \tag{4i}$$

$$\mathrm{tr} \left(\sigma^{\mu\nu} \sigma^{\alpha\beta} \right) = \frac{1}{2} \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha} + i \epsilon^{\mu\nu\alpha\beta} \right)\,. \tag{4j}$$

Identities for spinors

Let $\xi, \chi, \ldots$ be 2–component Weyl spinors in the $(\frac{1}{2}, 0)$ representation of the homogeneous Lorentz group. The corresponding hermitean conjugated spinors $\bar{\xi}, \bar{\chi}, \ldots$ then belong to the $(0, \frac{1}{2})$ representation. The contraction of spinor indices is defined by

$$\xi\chi \equiv \xi^A \chi_A = -\xi_A \chi^A; \quad \bar{\xi}\bar{\chi} \equiv \bar{\xi}_{\dot{A}} \bar{\chi}^{\dot{A}} = -\bar{\xi}^{\dot{A}} \bar{\chi}_{\dot{A}} \,. \tag{5}$$

If the components of these spinors are anti–commuting (fermionic) field operators, we have

$$\xi\chi = \chi\xi = (\bar{\xi}\bar{\chi})^\dagger = (\bar{\chi}\bar{\xi})^\dagger \,. \tag{6}$$

Moreover, let $\theta_A, \bar{\theta}_{\dot{A}}$ be anti–commuting Grassmann variables or coordinates. They can also be collected in 2–component spinors. We then have the following identities:

$$\xi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\xi \,; \tag{7a}$$

$$\xi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\xi, \ \bar{\xi}\bar{\sigma}^{\mu\nu}\bar{\chi} = -\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\xi} \,; \tag{7b}$$

$$\theta^A\theta^B = -\frac{1}{2}\epsilon^{AB}\theta\theta \,; \tag{7c}$$

$$\theta_A\theta_B = \frac{1}{2}\epsilon_{AB}\theta\theta \,; \tag{7d}$$

$$\bar{\theta}_{\dot{A}}\bar{\theta}_{\dot{B}} = -\frac{1}{2}\epsilon_{\dot{A}\dot{B}}\bar{\theta}\bar{\theta} \,; \tag{7e}$$

$$\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}} = \frac{1}{2}\epsilon^{\dot{A}\dot{B}}\bar{\theta}\bar{\theta} \,; \tag{7f}$$

$$\theta\xi \ \theta\chi = -\frac{1}{2}\xi\chi \ \theta\theta \,; \tag{7g}$$

$$\bar{\theta}\bar{\xi} \ \bar{\theta}\bar{\chi} = -\frac{1}{2}\bar{\xi}\bar{\chi} \ \bar{\theta}\bar{\theta} \,; \tag{7h}$$

$$\xi\zeta \ \bar{\chi}\bar{\tau} = \frac{1}{2}\xi\sigma^\mu\bar{\chi} \ \zeta\sigma_\mu\bar{\tau} \,; \tag{7i}$$

$$\bar{\xi}\bar{\zeta} \ \chi\tau = \frac{1}{2}\bar{\xi}\bar{\sigma}^\mu\chi \ \bar{\zeta}\bar{\sigma}_\mu\tau \,; \tag{7j}$$

$$\theta\sigma^\mu\bar{\theta} \ \theta\sigma^\nu\bar{\theta} = \frac{1}{2}g^{\mu\nu}\theta\theta \ \bar{\theta}\bar{\theta} \,; \tag{7k}$$

$$\zeta\xi \ \chi\sigma^\mu\bar{\tau} = -\frac{1}{2}\zeta\chi \ \xi\sigma^\mu\bar{\tau} + \zeta\sigma^{\mu\nu}\chi \ \xi\sigma_\nu\bar{\tau} \,; \tag{7l}$$

$$\zeta\xi \ \bar{\chi}\bar{\sigma}^\mu\tau = -\frac{1}{2}\tau\xi \ \bar{\chi}\bar{\sigma}^\mu\zeta - \tau\sigma^{\mu\nu}\xi \ \bar{\chi}\bar{\sigma}_\nu\zeta \,; \tag{7m}$$

$$(\sigma^\mu\bar{\theta})_A\theta\sigma^\nu\bar{\theta} = \bar{\theta}\bar{\theta}\left[\frac{1}{2}g^{\mu\nu}\theta_A - i(\sigma^{\mu\nu}\theta)_A\right] \,; \tag{7n}$$

$$(\theta\sigma^\mu)_{\dot{A}}\bar{\theta}\bar{\sigma}^\nu\theta = -\theta\theta\left[\frac{1}{2}\bar{\theta}_{\dot{A}}g^{\mu\nu} + i(\bar{\theta}\bar{\sigma}^{\mu\nu})_{\dot{A}}\right] \,. \tag{7o}$$

(7 g-o) are Fierz identities for 2–component spinors.

Relations involving 4–component spinors

Later we'll want to write Feynman rues using the usual 4–component (Dirac– or Majorana)–spinors. We'll use the "chiral" or "Weyl" representation of the Dirac matrices, defined by:

$$\delta_{ab} = \begin{pmatrix} \delta_A^{\ B} & 0 \\ 0 & \delta^{\dot{A}}_{\ \dot{B}} \end{pmatrix}, \ (\gamma^\mu_W a_\mu)_{ab} = \begin{pmatrix} 0 & \sigma^\mu_{A\dot{B}} \\ \bar{\sigma}^{\mu\dot{A}B} & 0 \end{pmatrix} a_\mu \,. \tag{8}$$

Here a_μ is an arbitrary 4–vector, and die indices a, b run from 1 to 4. Indices $A, \dot{A}, \dots$ run from 1 to 2, as before. γ_5 then has the form

$$\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} . \tag{9}$$

A Dirac spinor in this representation can be written as

$$\psi = \begin{pmatrix} \xi_{+A} \\ \\ \bar{\xi}_-^{\dot{A}} \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \xi_-^A & \bar{\xi}_{+\dot{A}} \end{pmatrix} , \tag{10}$$

i.e. the $(\frac{1}{2}, 0)$ spinor ξ_+ is the left–handed component of the Dirac spinor ψ, while the $(0, \frac{1}{2})$ spinor $\bar{\xi}_-$ forms the right–handed component. In a Majorana spinor these components are by definition related by hermitean conjugation, i.e. a Majorana spinor can be written in terms of a single Weyl spinor:

$$\lambda_M = \begin{pmatrix} \lambda_A \\ \\ \bar{\lambda}^{\dot{A}} \end{pmatrix}, \ \bar{\lambda}_M = \begin{pmatrix} \lambda^A & \bar{\lambda}_{\dot{A}} \end{pmatrix} . \tag{11}$$

This yields the following identities for products of Dirac spinors:

$$\bar{\psi}_1 \psi_2 = \xi_{1-}\xi_{2+} + \bar{\xi}_{1+}\bar{\xi}_{2-} \,; \tag{12a}$$

$$\bar{\psi}_1 \gamma_5 \psi_2 = -\xi_{1-}\xi_{2+} + \bar{\xi}_{1+}\bar{\xi}_{2-} \,; \tag{12b}$$

$$\bar{\psi}_1 \gamma^\mu \psi_2 = \xi_{1-}\sigma^\mu\bar{\xi}_{2-} + \bar{\xi}_{1+}\bar{\sigma}^\mu\xi_{2+} \,; \tag{12c}$$

$$\bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2 = \xi_{1-}\sigma^\mu\bar{\xi}_{2-} - \bar{\xi}_{1+}\bar{\sigma}^\mu\xi_{2+} \,; \tag{12d}$$

$$\bar{\psi}_1 \Sigma^{\mu\nu} \psi_2 = \xi_{1-}\sigma^{\mu\nu}\xi_{2+} + \bar{\xi}_{1+}\bar{\sigma}^{\mu\nu}\bar{\xi}_{2-} \,; \tag{12e}$$

$$\bar{\psi}_1 \psi_{2L} = \xi_{1-}\xi_{2+} \,; \tag{12f}$$

$$\bar{\psi}_1 \psi_{2R} = \bar{\xi}_{1+}\bar{\xi}_{2-} \,; \tag{12g}$$

$$\bar{\psi}_1 \gamma^\mu \psi_{2L} = \bar{\xi}_{1+}\bar{\sigma}^\mu\xi_{2+} \,; \tag{12h}$$

$$\bar{\psi}_1 \gamma^\mu \psi_{2R} = \xi_{1-}\sigma^\mu\bar{\xi}_{2-} \,; \tag{12i}$$

$$\overline{\psi_1^C} \psi_2 = \xi_{1+}\xi_{2+} + \xi_{1-}\xi_{2-} \,; \tag{12j}$$

$$\overline{\psi_1^C} \gamma_5 \psi_2 = -\xi_{1+}\xi_{2+} + \xi_{1-}\xi_{2-} \,. \tag{12k}$$

The analogous results for two Majorana 4–spinors λ_{1M} and λ_{2M} are:

$$\bar{\lambda}_{1M}\lambda_{2M} = \lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2 = \bar{\lambda}_{2M}\lambda_{1M}\,; \tag{13a}$$

$$\bar{\lambda}_{1M}\gamma_5\lambda_{2M} = -\lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2 = \bar{\lambda}_{2M}\gamma_5\lambda_{1M}\,; \tag{13b}$$

$$\bar{\lambda}_{1M}\gamma^\mu\lambda_{2M} = \bar{\lambda}_1\bar{\sigma}^\mu\lambda_2 + \lambda_1\sigma^\mu\bar{\lambda}_2 = -\bar{\lambda}_{2M}\gamma^\mu\lambda_{1M}\,; \tag{13c}$$

$$\bar{\lambda}_{1M}\gamma^\mu\gamma_5\lambda_{2M} = \bar{\lambda}_1\bar{\sigma}^\mu\lambda_2 - \lambda_1\sigma^\mu\bar{\lambda}_2 = \bar{\lambda}_{2M}\gamma^\mu\gamma_5\lambda_{1M}\,; \tag{13d}$$

$$\bar{\lambda}_{1M}\Sigma^{\mu\nu}\lambda_{2M} = \lambda_1\sigma^{\mu\nu}\lambda_2 + \bar{\lambda}_1\bar{\sigma}^{\mu\nu}\bar{\lambda}_2 = -\bar{\lambda}_{2M}\Sigma^\mu\nu\lambda_{1M}\,; \tag{13e}$$

$$\bar{\lambda}_{1M}\lambda_{2M(L,R)} = \bar{\lambda}_{2M}\lambda_{1M(L,R)}\,; \tag{13f}$$

$$\bar{\lambda}_{1M}\gamma^\mu\lambda_{2M(L,R)} = -\bar{\lambda}_{2M}\gamma^\mu\lambda_{1M(L,R)}\,; \tag{13g}$$

$$\bar{\lambda}_M\gamma^\mu\lambda_M = 0\,. \tag{13h}$$

Finally there are "mixed" identities:

$$\bar{\lambda}_M\psi_L = \lambda\xi_+\,; \tag{14a}$$

$$\bar{\lambda}_M\psi_R = \bar{\lambda}\bar{\xi}_-\,; \tag{14b}$$

$$\bar{\psi}_L\lambda_M = \bar{\lambda}\bar{\xi}_+\,; \tag{14c}$$

$$\bar{\psi}_R\lambda_M = \lambda\xi_-\,. \tag{14d}$$

Another useful identity for the product of 4 Majorana spinors is:

$$\bar{\lambda}_M\gamma^\mu\gamma_5\lambda_M\ \bar{\lambda}_M\gamma^\nu\gamma_5\lambda_M = g^{\mu\nu}(\bar{\lambda}_M\lambda_M)^2 = -g^{\mu\nu}(\bar{\lambda}_M\gamma_5\lambda_M)^2\,, \tag{15}$$

which holds for Majorana spinors λ_M at a *fixed* space–time point x.

Calculus with Grassmann variables

Derivatives w.r.t. a Grassmann variable are defined as $\partial_A \equiv \dfrac{\partial}{\partial\theta^A}$, $\partial^A \equiv \dfrac{\partial}{\partial\theta_A}$, $\bar{\partial}^{\dot{A}} \equiv \dfrac{\partial}{\partial\bar{\theta}_{\dot{A}}}$, $\bar{\partial}_{\dot{A}} \equiv \dfrac{\partial}{\partial\bar{\theta}^{\dot{A}}}$. This immediately leads to the following identities:

$$\partial_A\theta^B = \delta_A^{\ B}\,; \tag{16a}$$

$$\partial^A\theta_B = \delta^A_{\ B}\,; \tag{16b}$$

$$\bar{\partial}_{\dot{A}}\bar{\theta}^{\dot{B}} = \delta_{\dot{A}}^{\ \dot{B}}\,; \tag{16c}$$

$$\bar{\partial}^{\dot{A}}\bar{\theta}_{\dot{B}} = \delta^{\dot{A}}_{\ \dot{B}}\,; \tag{16d}$$

$$\partial_A\theta_B = -\epsilon_{AB}\,; \tag{16e}$$

$$\partial^A\theta^B = -\epsilon^{AB}\,; \tag{16f}$$

$$\bar{\partial}^{\dot{A}}\bar{\theta}^{\dot{B}} = -\epsilon^{\dot{A}\dot{B}}\,; \tag{16g}$$

$$\bar{\partial}_{\dot{A}}\bar{\theta}_{\dot{B}} = -\epsilon_{\dot{A}\dot{B}}\,. \tag{16h}$$

Moreover, we obviously have $0 = \partial_A \bar\theta_{\dot B} = \partial^A \bar\theta^{\dot B} = \partial_A \bar\theta^{\dot B} = \partial^A \bar\theta_{\dot B} = \bar\partial^{\dot A} \theta_B = \bar\partial^{\dot A} \theta^B = \bar\partial_{\dot A} \theta_B = \bar\partial_{\dot A} \theta^B$.

Note that raising or lowering of an index gives an extra minus sign in case of Grassmann derivatives. The following identities hold for Grassmann derivatives of an arbitrary function of θ and $\bar\theta$:

$$\epsilon^{AB} \partial_B = -\partial^A \,; \tag{17a}$$

$$\epsilon_{AB} \partial^B = -\partial_A \,; \tag{17b}$$

$$\epsilon_{\dot A \dot B} \bar\partial^{\dot B} = -\bar\partial_{\dot A} \,; \tag{17c}$$

$$\epsilon^{\dot A \dot B} \bar\partial_{\dot B} = -\bar\partial^{\dot A} \,. \tag{17d}$$

All components of ∂, $\bar\partial$ anti–commute, i.e. $0 = \{\partial_A, \partial_B\} = \{\bar\partial_{\dot A}, \bar\partial_{\dot B}\} = \{\partial_A, \bar\partial_{\dot B}\}$ etc. Grassmann derivatives of products of fermionic fields ψ, χ etc. and/or Grassmann coordinates can be evaluated using the chain rule, which however contains an additional minus sign. E.g. $\partial(\psi\chi) = (\partial\psi)\chi - \psi(\partial\chi)$ etc. This leads to the following identities for second Grassmann derivatives $\partial\partial \equiv \partial^A \partial_A$ and $\bar\partial\bar\partial = \bar\partial_{\dot A} \bar\partial^{\dot A}$:

$$\partial\partial(\theta\theta) = \bar\partial\bar\partial(\bar\theta\bar\theta) = 4 \,. \tag{18}$$

Clearly the product of three or more derivatives w.r.t. θ or $\bar\theta$ vanishes.

An integral over a Grassmann variable is almost the same as a derivative w.r.t. this variable. In particular:

$$\int d\theta_A \theta_B = \delta_{AB} \,. \tag{19}$$

An analogous relation holds for the integral over $\bar\theta_{\dot A}$. When generalizing to higher–dimensional integrals it is convenient to require $\int d^2\theta \theta^2 = 1$. This leads to the following definition of the measure of integration:

$$d^2\theta = -\frac{1}{4} d\theta^A d\theta_A \,; \tag{20a}$$

$$d^2\bar\theta = -\frac{1}{4} d\bar\theta_{\dot A} d\bar\theta^{\dot A} \,; \tag{20b}$$

$$d^4\theta = d^2\bar\theta d^2\theta \,. \tag{20c}$$

Moreover, the integral over "unsaturated" Grassmann variables vanishes, i.e. $\int d\theta_A f = 0$ if f does not depend on θ_A. This implies:

$$\int d^2\theta = \int d^2\bar\theta = \int d^2\theta \theta^A = \int d^2\bar\theta \bar\theta_{\dot A} = 0 \,. \tag{21}$$

These defining properties lead to the following identities:

$$\int d^2\theta \theta^A \theta^B = -\frac{1}{2} \epsilon^{AB} \,; \tag{22a}$$

$$\int d^2\bar\theta \bar\theta_{\dot A} \bar\theta_{\dot B} = -\frac{1}{2} \epsilon_{\dot A \dot B} \,; \tag{22b}$$

$$\int d^2\theta \theta\theta = \int d^2\bar\theta \bar\theta\bar\theta = 1 \,; \tag{22c}$$

$$\int d^4\theta\theta\theta\ \bar{\theta}\bar{\theta} = 1\,. \tag{22d}$$

Occasionally it is useful to introduce $\delta-$functions of Grassmann variables:

$$\int d^2\theta\delta^{(2)}(\theta) = \int d^2\bar{\theta}\delta^{(2)}(\bar{\theta}) = 1\,; \tag{23a}$$

$$\delta^{(2)}(\theta) = \theta\theta,\ \ \delta^{(2)}(\bar{\theta}) = \bar{\theta}\bar{\theta}\,. \tag{23b}$$

Finally, the following identities involving integration and differentiation can be shown to hold:

$$\int d^2\theta f(\theta,\bar{\theta}) = \frac{1}{4}\partial\partial f(\theta,\bar{\theta})\,; \tag{24a}$$

$$\int d^2\bar{\theta} f(\theta,\bar{\theta}) = \frac{1}{4}\bar{\partial}\bar{\partial} f(\theta,\bar{\theta})\,; \tag{24b}$$

$$\int d^2\theta\ \partial_A f(\theta,\bar{\theta}) = \int d^2\bar{\theta}\ \bar{\partial}^{\dot{A}} f(\theta,\bar{\theta}) = 0\,; \tag{24c}$$

$$\int d^4\theta\ f(\theta,\bar{\theta}) = \frac{1}{16}\partial\partial\ \bar{\partial}\bar{\partial} f(\theta,\bar{\theta})\,. \tag{24d}$$