Freeze-Out of Particles

Abtin Narimani

University of Bonn

s6abnari@uni-bonn.de

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Overview

Freeze-Out ?

- The early Universe was in local thermal equilibrium.

$$\Gamma \gtrsim H \qquad or \qquad t_{int} \lesssim t_H \qquad (coupled)$$

- During expansion, the rate of expansion of the Universe (H) became larger than rate of interaction (Γ) $\Rightarrow$ particles fell out of the thermal equilibrium.

$$\Gamma \lesssim H \qquad or \qquad t_{int} \gtrsim t_H \qquad (decoupled)$$

Freeze-Out

Local Thermal Equilibrium

- Is $\Gamma \gg H$ satisfied for SM processes for $T \gtrsim 100\ GeV$?

$$\Gamma \equiv n\sigma v, \quad v \sim 1, \quad n \sim T^3, \quad \sigma \sim \frac{\alpha^2}{T^2} \quad \rightarrow \Gamma \sim \alpha^2 T$$
$$H \sim \frac{\rho^{1/2}}{M_{Pl}}, \quad \rho \sim T^4 \quad \rightarrow H \sim \frac{T^2}{M_{Pl}}$$

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl}}{T} \sim \frac{10^{16} GeV}{T} \qquad \text{with} \quad \alpha \sim 0.01$$

$\Gamma \gg H$ is satisfied for $100 GeV \lesssim T \lesssim 10^{16} GeV$

Local Thermal Equilibrium

- Distribution function (the number of particles per unit volume in phase space):

$$f(E) = \frac{1}{e^{(E-\mu)/T} \pm 1} \qquad (+ \text{ for fermions, } - \text{ for bosons})$$

- Relativistic particles dominate density and pressure of the primordial plasma.

  When $T \ll m$, particles become non-relativistic, then $f(E) \rightarrow e^{-m/T}$

- Total energy density (ρ_r)is approximated by:

$$\rho_r \propto \sum_i \int E_i(p)\, f_i(p) d^3p \qquad \Rightarrow \rho_r = \frac{\pi^2}{30} g_\star(T)\, T^4$$

  At early times, all particles are relativistic $\rightarrow g_\star = 106.75$

$g_\star(T)$ vs. T

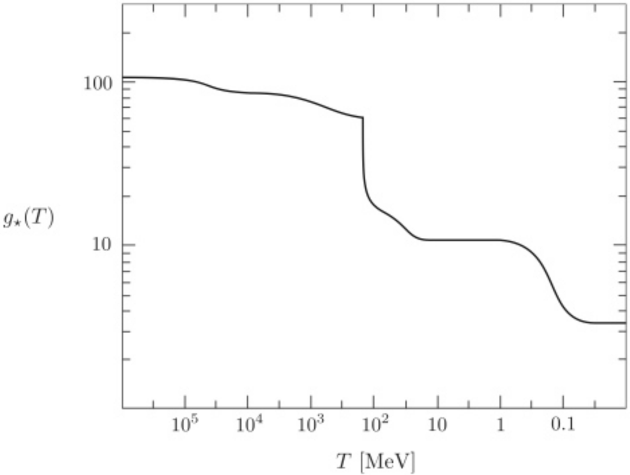

Figure: Evolution of the number of relativistic degrees of freedom as a function of temperature in the Standard Model.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology

Freeze-Out

- Deviation from equilibrium led to the *freeze-out* of massive particles.

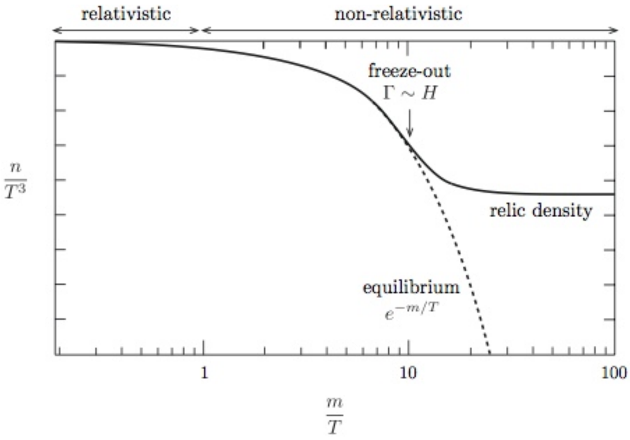

Figure: A schematic illustration of particle freeze-out.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology

Decoupling

- Below $100\,GeV$, $W^{\pm}$ and Z receive masses, $M_W \sim M_Z$.
- Cross section associated with processes mediated by weak interaction:

$$\sigma \sim \frac{\alpha^2}{M_W^4} T^2$$

$$\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl} T^3}{M_W^4} \sim \left(\frac{T}{1MeV}\right)^3$$

Particles that interact with the primordial plasma only through the weak interaction decouple at $T_{dec} \sim 1MeV$

A brief History of the Universe

Key events are:

- Baryogenesis
- Electroweak transition
- Dark matter freeze-out
- QCD transition
- Neutrino decoupling
- $e^+ - e^-$ annihilation
- Big Bang nucleosynthesis
- Recombination

Key events

Event	Time	Temperature
Inflation	$10^{-34}\,s(?)$	–
Baryogenesis	?	?
EW phase transition	$20\,ps$	$100\,GeV$
Dark matter freeze-out	?	?
QCD phase transition	$20\,\mu s$	$150\,MeV$
Neutrino decoupling	$1\,s$	$1\,MeV$
$e^+ - e^-$ annihilation	$6\,s$	$500\,keV$
Big Bang nucleosynthesis	$3\,min$	$100\,keV$
Matter-radiation equality	$60\,kyr$	$0.75\,eV$
Recombination	$260 - 380\,kyr$	$0.26 - 0.33\,eV$
Photon decoupling	$380\,kyr$	$0.23 - 0.38\,eV$
Reionization	$100 - 400\,Myr$	$2.6 - 7.0\,meV$
Dark energy-matter equality	$9\,Gyr$	$0.33\,meV$
Present	$13.8\,Gyr$	$0.24\,meV$

Number density, Energy density, Pressure

- *Phase space* is useful for description of the system.
- Density of states in momentum space: $L^3/h^3 = V/h^3$.
- Density of states in phase space with g internal degrees of freedom is $g/h^3 = g/(2\pi)^3$.
- Information about the distribution of particles amongst momentum eigenstates is in *phase space distribution function* $f(x, \vec{\mathbf{p}})$ (because of homogeneity and isotropy $f(x, \vec{\mathbf{p}}) = f(p)$).
- Particle density in phase space is $\frac{g}{(2\pi)^3} f(p)$.

Number density, Energy density, Pressure

For a weakly interacting gas of particles

- Number density:

$$n = \frac{g}{(2\pi)^3} \int f(p)\, d^3p$$

- Energy density:

$$\rho = \frac{g}{(2\pi)^3} \int E(p)\, f(p)\, d^3p$$

- Pressure:

$$p = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E} f(p)\, d^3p$$

with $E^2(p) = p^2 + m^2$

Local Thermal Equilibrium

- If particles exchange energy and momentum efficiently, system is in kinetic equilibrium.

  Fermi-Dirac and Bose-Einstein distributions: $f(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}$

- At low T, ($T < E - \mu$), these reduce to Maxwell-Boltzmann distribution:

$$f(p) \approx e^{-(E(p)-\mu)/T}$$

- If a species is in chemical equilibrium, for a process like $i + j \leftrightarrow k + l$, μ_i is related to μ_j which implies: $\mu_i + \mu_j = \mu_k + \mu_l$

  For $X + \bar{X} \leftrightarrow \gamma + \gamma$: $\mu_X + \mu_{\bar{X}} = \mu_\gamma + \mu_\gamma \rightarrow \mu_X = -\mu_{\bar{X}}$

Thermal equilibrium is achieved for species which are both in kinetic and chemical equilibrium.

Number and Energy Densities

- At early times, μ of all particles were so small ($\mu = 0$).
- Number density:

$$n = \frac{g}{(2\pi)^3} \int f(p)\, d^3p = \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2}{exp(\sqrt{p^2 + m^2}/T) \pm 1}$$

- Energy density:

$$\rho = \frac{g}{(2\pi)^3} \int E(p)\, f(p)\, d^3p = \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2 \sqrt{p^2 + m^2}}{exp(\sqrt{p^2 + m^2}/T) \pm 1}$$

Number and Energy Densities

- Defining $x \equiv m/T$ and $\xi \equiv p/T$:

$$n = \frac{g}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{exp(\sqrt{\xi^2 + x^2}) \pm 1}$$

$$\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty d\xi \frac{\xi^2 \sqrt{\xi^2 + x^2}}{exp(\sqrt{\xi^2 + x^2}) \pm 1}$$

Relativistic Limit

- In the limit $x \to 0$ or $m/T \to 0$ $(m \ll T)$:

$$n = \frac{g}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{e^\xi \pm 1} = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & \text{bosons} \\ \frac{3}{4} \frac{\zeta(3)}{\pi^2} g T^3 & \text{fermions} \end{cases}$$

$$\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty d\xi \frac{\xi^3}{e^\xi \pm 1} = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{bosons} \\ \frac{7}{8} \frac{\pi^2}{30} g T^4 & \text{fermions} \end{cases}$$

$$P = \rho/3$$

- For relic photons using the temperature of CMB ($T_0 = 2.73K$)

$$n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} {T_0}^3 \approx 410 \text{ photons } cm^{-3}$$

$$\rho_{\gamma,0} = \frac{\pi^2}{15} {T_0}^4 \approx 4.6 \times 10^{-34} g\ cm^{-3} \qquad \Rightarrow \Omega_\gamma h^2 \approx 2.5 \times 10^{-5}$$

Non-relativistic Limit

- In the limit $x \gg 1$ or $m/T \gg 1$ ($m \gg T$):

$$n = \frac{g}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{e^{\sqrt{\xi^2+x^2}}} = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

Massive particles are rare at low temperature.

At the lowest order $E(p) \approx m$ and energy density is equal to mass density:

$$\rho \approx mn$$

A non-relativistic gas of particles acts like a pressureless dust:

$$P = nT \ll \rho = mn$$

Effective Number of Relativistic Species

- Total radiation density: $\rho_r = \sum_i \rho_i \frac{\pi^2}{30} g_\star(T) T^4$

  where $g_\star(T)$ is effective number of relativistic d.o.f at temperature T

- Two types of contributions:

  - Relativistic species are in thermal equilibrium with the photons, $T_i = T \gg m_i$

    $$g_\star^{th}(T) = \sum_{i=bosons} g_i + \frac{7}{8} \sum_{i=fermions} g_i$$

    Away from mass threshold, thermal contribution is independent of T.

  - Relativistic species are not in thermal equilibrium with the photons, $T_i \neq T \gg m_i$

    $$g_\star^{dec}(T) = \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^4$$

$g_\star(T)$

- At $T \gtrsim 100\,GeV$, all particles of SM are relativistic.
  Internal d.o.f:
  $g_b = 28$ photons (2), $W^\pm$ and Z^0 $(3 \cdot 3)$, gluons $(8 \cdot 2)$, Higgs (1)
  $g_f = 90$ quarks $(6 \cdot 12)$, charged leptons $(3 \cdot 4)$, neutrinos $(3 \cdot 2)$

  Therefore: $$g_\star = g_b + \frac{7}{8} g_f = 106.75$$

  1. Top quarks: $g_\star = 106.75 - \frac{7}{8} \times 12 = 96.25$
  2. Higgs, $W^\pm$ and Z^0: $g_\star = 96.25 - (1 + 3 \times 3) = 86.25$
  3. Bottom quarks: $g_\star = 86.25 - \frac{7}{8} \times 12 = 75.75$
  4. Charm quarks and tau leptons: $g_\star = 75.75 - \frac{7}{8} \times (12 + 4) = 61.75$
  5. Matter undergoes the QCD transition and quarks combine into baryons and mesons.
  6. For $\pi^\pm, \pi^0, e^\pm, \mu^\pm, \nu' s, \gamma$: $g_\star = 2 + 3 + \frac{7}{8} \times (4 + 4 + 6) = 17.25$
  7. In order to understand electron-positron annihilation we need entropy.

$g_\star(T)$ vs. T

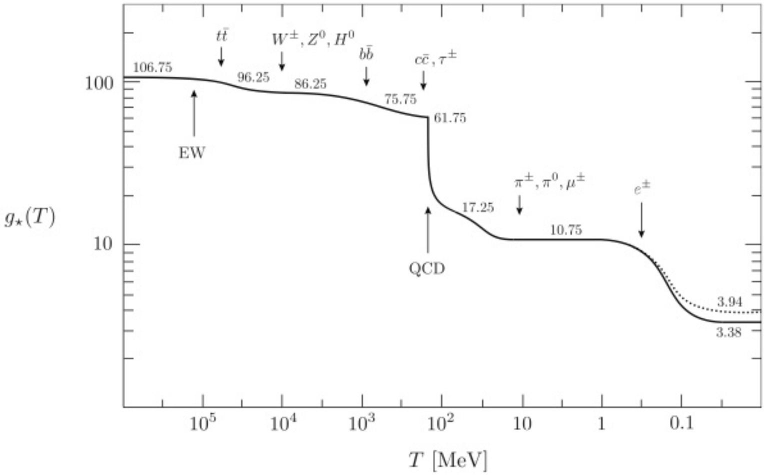

Figure: Evolution of the number of relativistic degrees of freedom as a function of temperature in the Standard Model.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology

Entropy

- To describe the evolution of the Universe, we need a conserved quantity.
- Total entropy increases or stays constant.
- Entropy is conserved in equilibrium.
- To a good approximation, expansion of the Universe is adiabatic, therefore the total entropy stays constant beyond equilibrium.

Effective Number of degrees of freedom in Entropy

- Considering 2nd law of thermodynamics: $TdS = dU + PdV$. Using $U = \rho V$, then:
  $dS = d\left(\frac{\rho+P}{T}V\right)$
  $\Rightarrow$ entropy is conserved in equilibrium, $\frac{dS}{dt} = 0$.
  $(\dot{\rho} + 3H(\rho + P) = 0$ and $\frac{\partial P}{\partial T} = \frac{\rho+P}{T})$
- Define entropy density: $s \equiv S/V \rightarrow s = \frac{\rho+P}{T}$
  The total energy density for a collision of different particle species:
  $s = \sum_i \frac{\rho_i+P_i}{T_i} = \frac{2\pi^2}{45} g_{\star S}(T) T^3$

Effective Number of degrees of freedom in Entropy is
$$g_{\star S}(T) = g_{\star S}^{th}(T) + g_{\star S}^{dec}(T)$$

- For species in equilibrium: $g_{\star S}^{th}(T) = g_{\star}^{th}(T)$
- For decoupled species:

$$g_{\star S}^{dec}(T) = \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^3 \neq g_{\star}^{dec}(T)$$

- Conservation of entropy has 2 consequences:
  1. $s \propto a^{-3} \rightarrow$ the number of particles in a comoving volume: $N_i \equiv n_i/s$
  2. $g_{\star S}(T) T^3 a^3 = constant$ or $T \propto g_{\star S}^{-1/3}(T) a^{-1}$
     The factor $g_{\star S}^{-1/3}(T)$ accounts for the fact whenever a particle species becomes non-relativistic, its entropy is transferred to the other relativistic particles present in the thermal plasma.

Neutrino Decoupling

- We already knew $\frac{\Gamma}{H} \sim \left(\frac{T}{1MeV}\right)^3$
- After decoupling, ν's move freely along geodesics and preserve relativistic Fermi-Dirac distribution.
- Momentum $p \propto 1/a$, thus we define $q \equiv ap$:

$$n_\nu \propto \frac{1}{a^3} \int \frac{d^3q}{e^{(q/aT_\nu)} + 1}$$

  consistent if $T_\nu \propto 1/a$
- $T_\gamma \sim T_\nu$, however particle annihilations will cause a deviation from $T_\gamma \propto 1/a$

$e^+ - e^-$ annihilation

- Shortly after neutrinos decouple, T drops below m_e and $e^+ - e^-$ annihilation occurs:

$$e^+ + e^- \leftrightarrow \gamma + \gamma$$

- If we neglect ν's and other species:

$$g_{\star S}^{th}(T) = \begin{cases} 2 + \frac{7}{8} \times 4 = \frac{11}{2} & T \gtrsim m_e \\ 2 & T < m_e \end{cases}$$

- Since $g_{\star S}^{th}(aT_\gamma)^3 = constant$, then aT_γ increases after $e^+ - e^-$ annihilation by a factor $(11/4)^{1/3}$. Then $T_\nu = (4/11)^{1/3} T_\gamma$.

- For $T \ll m_e$:

$$N_{eff} = 3.046$$

$$g_\star = 2 + \frac{7}{8} 2 N_{eff} \left(\frac{4}{11}\right)^{4/3} = 3.36$$

$$g_{\star S} = 2 + \frac{7}{8} 2 N_{eff} \left(\frac{4}{11}\right) = 3.94$$

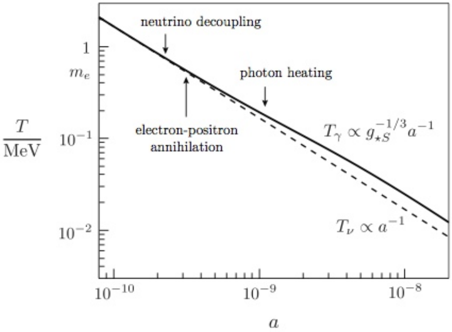

Figure: Thermal history through $e^+ - e^-$ annihilation

Image Source: Daniel D. Baumann, Lecture notes on Cosmology

Boltzmann equation

- In the absence of interactions, the number density of particle species evolves as:

$$\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = \frac{dn_i}{dt} + 3\frac{\dot{a}}{a}n_i = 0$$

Number of particles in a fixed volume is conserved.

- To include interactions of particles, a collision term is added to the other side of equation:

$$\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = C_i[\{n_j\}]$$

Boltzmann equation

Boltzmann equation

- We can limit ourselves to single-particle and two-particle scatterings / annihilations.
  Consider process $1+2 \leftrightarrow 3+4$. Suppose we track the number density n_1, then:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -An_1 n_2 + Bn_3 n_4$$

- $A = \langle \sigma v \rangle$: Thermally averaged cross section
  Since the collision term has to vanish in equilibrium: $B = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} A$
  where n_i^{eq} are the equilibrium number densities. Then:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle \sigma v \rangle \left[n_1 n_2 + \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} n_3 n_4 \right]$$

Boltzmann equation

- In terms of number of particles in a comoving volume:

$$\frac{d \ln N_1}{d \ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq} \frac{N_3 N_4}{N_1 N_2}\right]$$

where $\Gamma_1 \equiv n_2 \langle \sigma v \rangle$

$\frac{\Gamma_1}{H}$ describes the interaction efficiency

1. For $\Gamma_1 \gg H$, the system quickly relaxes to a steady state.
2. When $\Gamma_1 < H$, r.h.s. gets suppressed and comoving density of particles approaches a constant relic density.

Dark Matter Relics

- We assume that the dark matter is a weakly interacting massive particle (WIMP).
- Solving Boltzmann eq. and determine the epoch of freeze-out and its relic abundance.
- Assumptions are:
  1. Annihilation of a heavy dark matter particle and anti-particle produce two light particles: $X + \bar{X} \leftrightarrow l + \bar{l}$
  2. Light particles are tightly coupled to the cosmic plasma, thus $n_l = n_l^{eq}$
  3. There is no asymmetry between X and $\bar{X}$ $(n_X = n_{\bar{X}})$

$$\frac{dN_X}{dt} = -s\langle \sigma v \rangle \left[N_X^2 - (N_X^{eq})^2 \right]$$

$$N_X \equiv n_X/s, N_X^{eq} \equiv n_X^{eq}/s$$

Dark Matter Relics

- Define $x \equiv M_X/T$, then write Boltzmann eq. in terms of x:

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{M_X}{T}\right) = -\frac{1}{T}\frac{dT}{dt}x \simeq Hx$$

By assuming the radiation domination $h = H(M_X)/x^2$, then:

$$\frac{dN_X}{dx} = -\frac{\lambda}{x^2}\left[N_X^2 - (N_X^{eq})^2\right]$$

where $\lambda \equiv \frac{2\pi^2}{45} g_{\star S} \frac{M_X^2 \langle \sigma v \rangle}{H(M_X)}$

It is called Riccati equation.

There is no analytic solution for it.

Freeze-out density of DM

- At high temperature ($x < 1$): $N_X \approx N_X^{eq} \simeq 1$
- At low temperature ($x \gg 1$): $N_X^{eq} \sim e^{-x}$

  Thus X particles will become so rare and they cannot find each other fast enough to maintain equilibrium abundance.

- Freeze-out happens at $x_f \sim 20$, where solution starts to deviate from equilibrium.
- Final relic abundance: $N_X^\infty \equiv N_X(x = \infty)$ (freeze-out density of DM)
- After freeze-out $N_X \gg N_X^{eq}$, thus we can drop N_X^{eq}:

  $$\frac{dN_X}{dx} \simeq -\frac{\lambda N_X^2}{x^2} \quad \rightarrow \quad \int_{x_f}^{\infty}(\dots) \quad \rightarrow \quad \frac{1}{N_X^\infty} - \frac{1}{N_X^f} = \frac{\lambda}{x_f}$$

  where $N_X^f \equiv N_X(x_f)$. Typically, $N_X^f \gg N_X^\infty$, then: $N_X^\infty \simeq \frac{x_f}{\lambda}$

  This predicts the freeze-out abundance N_X^∞ decreases as the interaction rate λ increases.

Dark Matter Relics

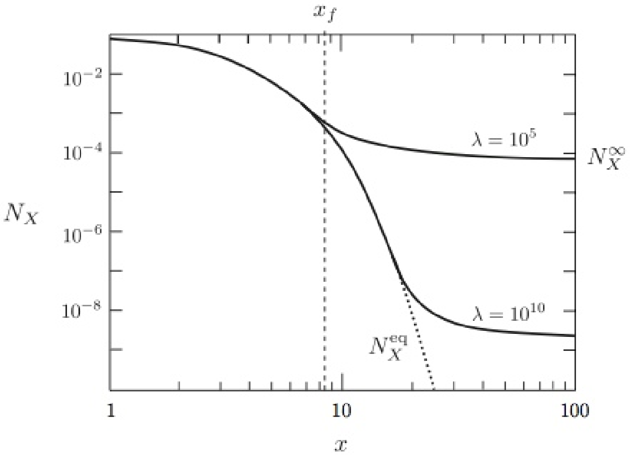

Figure: Abundance of dark matter particles as the temperature drops below the mass.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology

WIMP "Miracle"

Relating the freeze-out abundance of DM relics to the DM density today:

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_{c,0}}$$

$$= \frac{M_X n_{X,0}}{3M_{PL}^2 H_0^2} = \frac{M_X N_{X,0} s_0}{3M_{PL}^2 H_0^2} = M_X N_X^\infty \frac{s_0}{3M_{PL}^2 H_0^2}$$

$$= \frac{H(M_X)}{M_X^2} \frac{x_f}{\langle \sigma v \rangle} \frac{g_{\star S}(T_0)}{g_{\star S}(M_X)} \frac{T_0^3}{3M_{PL}^2 H_0^2}$$

$$= \frac{\pi}{9} \frac{x_f}{\langle \sigma v \rangle} \sqrt{\frac{g_\star(M_X)}{10}} \frac{g_{\star S}(T_0)}{g_{\star S}(M_X)} \frac{T_0^3}{3M_{PL}^2 H_0^2}$$

WIMP "Miracle"

$$\Rightarrow \Omega_X h^2 \sim 0.1 \frac{x_f}{10} \sqrt{\frac{10}{g_\star(M_X)}} \frac{10^{-8} GeV^{-2}}{\langle \sigma v \rangle}$$

This reproduces the observed DM density if:

$$\sqrt{\langle \sigma v \rangle} \sim 10^{-4} GeV^{-1} \sim 0.1 \sqrt{G_F}$$

The fact that a thermal relic with a cross section characteristic of the weak interaction gives the right dark matter abundance is called the WIMP miracle.

Summary

- The early Universe was in thermal equilibrium and freeze-out happened when $\Gamma \lesssim H$.
- After freeze-out particles fell out of equilibrium and decoupled.
- Deviations from equilibrium explain why there is something rather than nothing in the Universe.
- Boltzmann equation is useful to explain non-equilibrium processes.
- Calculate the relic abundance of Dark Matter.
- Cross section of the order of weak interaction gives the right Dark Matter abundance.

References

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Daniel D. Baumann, Lecture notes on Cosmology. http://www.damtp.cam.ac.uk/user/db275/Cosmology/ (version: 29.09.2015).