Freeze-Out of Particles

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Overview

1. The Basic Principles
2. Equilibrium Thermodynamics
3. Boltzmann Equation and Beyond Equilibrium
4. Summary
The early Universe was in local thermal equilibrium.

$$\Gamma \gtrsim H \quad \text{or} \quad t_{\text{int}} \lesssim t_H \quad \text{(coupled)}$$

During expansion, the rate of expansion of the Universe ($H$) became larger than rate of interaction ($\Gamma$) $\Rightarrow$ particles fell out of the thermal equilibrium.

$$\Gamma \lesssim H \quad \text{or} \quad t_{\text{int}} \gtrsim t_H \quad \text{(decoupled)}$$
Local Thermal Equilibrium

- Is $\Gamma \gg H$ satisfied for SM processes for $T \gtrsim 100$ GeV?

\[
\Gamma \equiv n \sigma v, \quad v \sim 1, \quad n \sim T^3, \quad \sigma \sim \frac{\alpha^2}{T^2} \quad \rightarrow \Gamma \sim \alpha^2 T
\]

\[
H \sim \frac{\rho^{1/2}}{M_{Pl}}, \quad \rho \sim T^4 \quad \rightarrow H \sim \frac{T^2}{M_{Pl}}
\]

\[
\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl}}{T} \sim \frac{10^{16} \text{GeV}}{T} \quad \text{with} \quad \alpha \sim 0.01
\]

$\Gamma \gg H$ is satisfied for $100 \text{GeV} \lesssim T \lesssim 10^{16} \text{GeV}$
Local Thermal Equilibrium

- Distribution function (the number of particles per unit volume in phase space):
  \[ f(E) = \frac{1}{e^{(E-\mu)/T} \pm 1} \quad (+ \text{ for fermions, } - \text{ for bosons}) \]

- Relativistic particles dominate density and pressure of the primordial plasma.
  
  When \( T \ll m \), particles become non-relativistic, then \( f(E) \rightarrow e^{-m/T} \)

- Total energy density (\( \rho_r \)) is approximated by:
  \[ \rho_r \propto \sum_i \int E_i(p) f_i(p) d^3p \quad \Rightarrow \rho_r = \frac{\pi^2}{30} g_* (T) T^4 \]

  At early times, all particles are relativistic \( \rightarrow g_* = 106.75 \)
Figure: Evolution of the number of relativistic degrees of freedom as a function of temperature in the Standard Model.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology
Deviation from equilibrium led to the freeze-out of massive particles.

Figure: A schematic illustration of particle freeze-out.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology
Below 100\,GeV, $W^\pm$ and $Z$ receive masses, $M_W \sim M_Z$.

Cross section associated with processes mediated by weak interaction:

$$
\sigma \sim \frac{\alpha^2}{M_W^4} T^2
$$

$$
\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{Pl} T^3}{M_W^4} \sim \left( \frac{T}{1 \text{MeV}} \right)^3
$$

Particles that interact with the primordial plasma only through the weak interaction decouple at $T_{\text{dec}} \sim 1 \text{MeV}$
Key events are:

- Baryogenesis
- Electroweak transition
- Dark matter freeze-out
- QCD transition
- Neutrino decoupling
- $e^+ - e^-$ annihilation
- Big Bang nucleosynthesis
- Recombination
### Key events

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- *Phase space* is useful for description of the system.

- Density of states in momentum space: \( \frac{L^3}{h^3} = \frac{V}{h^3} \).

- Density of states in phase space with \( g \) internal degrees of freedom is \( \frac{g}{h^3} = \frac{g}{(2\pi)^3} \).

- Information about the distribution of particles amongst momentum eigenstates is in *phase space distribution function* \( f(x, p) \) (because of homogeneity and isotropy \( f(x, p) = f(p) \)).

- Particle density in phase space is \( \frac{g}{(2\pi)^3} f(p) \).
Number density, Energy density, Pressure

For a weakly interacting gas of particles

- Number density:

\[
    n = \frac{g}{(2\pi)^3} \int f(p) \, d^3p
\]

- Energy density:

\[
    \rho = \frac{g}{(2\pi)^3} \int E(p) \, f(p) \, d^3p
\]

- Pressure:

\[
    p = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E} \, f(p) \, d^3p
\]

with \( E^2(p) = p^2 + m^2 \)
Local Thermal Equilibrium

- If particles exchange energy and momentum efficiently, system is in kinetic equilibrium.

Fermi-Dirac and Bose-Einstein distributions: \[ f(p) = \frac{1}{e^{(E(p) - \mu)/T} \pm 1} \]

- At low \( T \), \( T < E - \mu \), these reduce to Maxwell-Boltzmann distribution:

\[ f(p) \approx e^{-(E(p) - \mu)/T} \]

- If a species is in chemical equilibrium, for a process like \( i + j \leftrightarrow k + l \), \( \mu_i \) is related to \( \mu_j \) which implies: \( \mu_i + \mu_j = \mu_k + \mu_l \)

For \( X + \bar{X} \leftrightarrow \gamma + \gamma \):

\[ \mu_X + \mu_{\bar{X}} = \mu_{\gamma} + \mu_{\gamma} \rightarrow \mu_X = -\mu_{\bar{X}} \]

Thermal equilibrium is achieved for species which are both in kinetic and chemical equilibrium.
At early times, $\mu$ of all particles were so small ($\mu = 0$).

Number density:

$$n = \frac{g}{(2\pi)^3} \int f(p) \, d^3p = \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2}{\exp(\sqrt{p^2 + m^2/T})} \pm 1$$

Energy density:

$$\rho = \frac{g}{(2\pi)^3} \int E(p) \, f(p) \, d^3p = \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2 \sqrt{p^2 + m^2}}{\exp(\sqrt{p^2 + m^2/T})} \pm 1$$
Defining $x \equiv m/T$ and $\xi \equiv p/T$:

$$n = \frac{g}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{\exp(\sqrt{\xi^2 + x^2}) \pm 1}$$

$$\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty d\xi \frac{\xi^2 \sqrt{\xi^2 + x^2}}{\exp(\sqrt{\xi^2 + x^2}) \pm 1}$$
Relativistic Limit

- In the limit $x \to 0$ or $m / T \to 0 (m \ll T)$:

\[
n = \frac{g}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{e^{\xi} \pm 1} = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & \text{bosons} \\ \frac{3}{4} \frac{\zeta(3)}{\pi^2} g T^3 & \text{fermions} \end{cases}
\]

\[
\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty d\xi \frac{\xi^3}{e^{\xi} \pm 1} = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{bosons} \\ \frac{7}{8} \frac{\pi^2}{30} g T^4 & \text{fermions} \end{cases}
\]

\[P = \rho / 3\]

- For relic photons using the temperature of CMB ($T_0 = 2.73K$)

\[
n_{\gamma,0} = \frac{2\zeta(3)}{\pi^2} T_0^3 \approx 410 \text{ photons cm}^{-3}
\]

\[
\rho_{\gamma,0} = \frac{\pi^2}{15} T_0^4 \approx 4.6 \times 10^{-34} g \text{ cm}^{-3} \quad \Rightarrow \Omega_\gamma h^2 \approx 2.5 \times 10^{-5}
\]
Non-relativistic Limit

- In the limit $x \gg 1$ or $m/T \gg 1$ ($m \gg T$):

\[
n = \frac{g}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{e^{\sqrt{\xi^2 + x^2}}} = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-m/T}
\]

Massive particles are rare at low temperature.

At the lowest order $E(p) \approx m$ and energy density is equal to mass density:

\[
\rho \approx mn
\]

A non-relativistic gas of particles acts like a pressureless dust:

\[
P = nT \ll \rho = mn
\]
Effective Number of Relativistic Species

- Total radiation density: \( \rho_r = \sum_i \rho_i \frac{\pi^2}{30} g_\star(T) T^4 \)

  where \( g_\star(T) \) is effective number of relativistic d.o.f at temperature \( T \)

- Two types of contributions:
  
  - Relativistic species are in thermal equilibrium with the photons, \( T_i = T \gg m_i \)
    
    \[
    g_{\star}^{\text{th}}(T) = \sum_{i=\text{bosons}} g_i + \frac{7}{8} \sum_{i=\text{fermions}} g_i
    \]

    Away from mass threshold, thermal contribution is independent of \( T \).

  - Relativistic species are not in thermal equilibrium with the photons, \( T_i \neq T \gg m_i \)
    
    \[
    g_{\star}^{\text{dec}}(T) = \sum_{i=\text{bosons}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left( \frac{T_i}{T} \right)^4
    \]
At $T \gtrsim 100$ GeV, all particles of SM are relativistic.

Internal d.o.f:

$g_b = 28$ photons (2), $W^\pm$ and $Z^0$ (3 · 3), gluons (8 · 2), Higgs (1)

$g_f = 90$ quarks (6 · 12), charged leptons (3 · 4), neutrinos (3 · 2)

Therefore:

$g_* = g_b + \frac{7}{8} g_f = 106.75$

1. Top quarks:

$g_* = 106.75 - \frac{7}{8} \times 12 = 96.25$

2. Higgs, $W^\pm$ and $Z^0$:

$g_* = 96.25 - (1 + 3 \times 3) = 86.25$

3. Bottom quarks:

$g_* = 86.25 - \frac{7}{8} \times 12 = 75.75$

4. Charm quarks and tau leptons:

$g_* = 75.75 - \frac{7}{8} \times (12 + 4) = 61.75$

5. Matter undergoes the QCD transition and quarks combine into baryons and mesons.

6. For $\pi^\pm, \pi^0, e^\pm, \mu^\pm, \nu's, \gamma$:

$g_* = 2 + 3 + \frac{7}{8} \times (4 + 4 + 6) = 17.25$

7. In order to understand electron-positron annihilation we need entropy.
Figure: Evolution of the number of relativistic degrees of freedom as a function of temperature in the Standard Model.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology
To describe the evolution of the Universe, we need a conserved quantity.

Total entropy increases or stays constant.

Entropy is conserved in equilibrium.

To a good approximation, expansion of the Universe is adiabatic, therefore the total entropy stays constant beyond equilibrium.
Considering 2nd law of thermodynamics: \( TdS = dU + PdV \). Using \( U = \rho V \), then:
\[ dS = d \left( \frac{\rho + P}{T} V \right) \]
\( \Rightarrow \) entropy is conserved in equilibrium, \( \frac{dS}{dt} = 0 \).
\( (\dot{\rho} + 3H(\rho + P) = 0 \) and \( \frac{\partial P}{\partial T} = \frac{\rho + P}{T} ) \)

Define entropy density: \( s \equiv \frac{S}{V} \rightarrow s = \frac{\rho + P}{T} \)
The total energy density for a collision of different particle species:
\[ s = \sum_i \frac{\rho_i + P_i}{T_i} = \frac{2\pi^2}{45} g_\ast s(T) T^3 \]

Effective Number of degrees of freedom in Entropy is
\[ g_\ast s(T) = g_\ast s^{th}(T) + g_\ast s^{dec}(T) \]
For species in equilibrium: \[ g_{*S}^{th}(T) = g_{*}^{th}(T) \]

For decoupled species:

\[
g_{*S}^{dec}(T) = \sum_{i=\text{bosons}} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left( \frac{T_i}{T} \right)^3 \neq g_{*}^{dec}(T) \]

Conservation of entropy has 2 consequences:

1. \( s \propto a^{-3} \rightarrow \) the number of particles in a comoving volume: \( N_i \equiv n_i/s \)

2. \( g_{*S}(T)T^3a^3 = \text{constant} \) or \( T \propto g_{*S}^{-1/3}(T)a^{-1} \)

The factor \( g_{*S}^{-1/3}(T) \) accounts for the fact whenever a particle species becomes non-relativistic, its entropy is transferred to the other relativistic particles present in the thermal plasma.
Neutrino Decoupling

- We already knew $\frac{\Gamma}{H} \sim \left( \frac{T}{1\text{MeV}} \right)^3$.
- After decoupling, $\nu$'s move freely along geodesics and preserve relativistic Fermi-Dirac distribution.
- Momentum $p \propto 1/a$, thus we define $q \equiv ap$:

$$n_\nu \propto \frac{1}{a^3} \int \frac{d^3q}{e(q/aT_\nu) + 1}$$

consistent if $T_\nu \propto 1/a$

- $T_\gamma \sim T_\nu$, however particle annihilations will cause a deviation from $T_\gamma \propto 1/a$.
Shortly after neutrinos decouple, $T$ drops below $m_e$ and $e^+ - e^-$ annihilation occurs:

$$e^+ + e^- \leftrightarrow \gamma + \gamma$$

If we neglect $\nu$'s and other species:

$$g_{\times S}^{th}(T) = \begin{cases} 
2 + \frac{7}{8} \times 4 = \frac{11}{2} & T \gtrsim m_e \\
2 & T < m_e
\end{cases}$$

Since $g_{\times S}^{th}(aT_{\gamma})^3 = constant$, then $aT_{\gamma}$ increases after $e^+ - e^-$ annihilation by a factor $(11/4)^{1/3}$. Then $T_{\nu} = (4/11)^{1/3} T_{\gamma}$. 
For $T \ll m_e$: $N_{\text{eff}} = 3.046$

$$g_* = 2 + \frac{7}{8} 2 N_{\text{eff}} \left(\frac{4}{11}\right)^{4/3} = 3.36$$

$$g_\star s = 2 + \frac{7}{8} 2 N_{\text{eff}} \left(\frac{4}{11}\right) = 3.94$$

**Figure:** Thermal history through $e^+ - e^-$ annihilation

Image Source: Daniel D. Baumann, Lecture notes on Cosmology
In the absence of interactions, the number density of particle species evolves as:

\[
\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = \frac{dn_i}{dt} + 3 \frac{\dot{a}}{a} n_i = 0
\]

Number of particles in a fixed volume is conserved.

To include interactions of particles, a collision term is added to the other side of equation:

\[
\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = C_i[\{n_j\}]
\]

Boltzmann equation
Boltzmann equation

- We can limit ourselves to single-particle and two-particle scatterings / annihilations.
  Consider process $1 + 2 \leftrightarrow 3 + 4$. Suppose we track the number density $n_1$, then:

\[
\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = -A n_1 n_2 + B n_3 n_4
\]

- $A = \langle \sigma v \rangle$: Thermally averaged cross section
  Since the collision term has to vanish in equilibrium: $B = \left( \frac{n_1 n_2}{n_3 n_4} \right)_\text{eq} A$
  where $n_i^{eq}$ are the equilibrium number densities. Then:

\[
\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = -\langle \sigma v \rangle \left[ n_1 n_2 + \left( \frac{n_1 n_2}{n_3 n_4} \right)_\text{eq} n_3 n_4 \right]
\]
The Boltzmann equation in terms of number of particles in a comoving volume is given by:

\[
\frac{d \ln N_1}{d \ln a} = -\frac{\Gamma_1}{H} \left[ 1 - \left( \frac{N_1 N_2}{N_3 N_4} \right)_{eq} \frac{N_3 N_4}{N_1 N_2} \right]
\]

where \( \Gamma_1 \equiv n_2 \langle \sigma v \rangle \)

\( \frac{\Gamma_1}{H} \) describes the interaction efficiency.

1. For \( \Gamma_1 \gg H \), the system quickly relaxes to a steady state.

2. When \( \Gamma_1 < H \), r.h.s. gets suppressed and comoving density of particles approaches a constant relic density.
We assume that the dark matter is a weakly interacting massive particle (WIMP).

Solving Boltzmann eq. and determine the epoch of freeze-out and its relic abundance.

Assumptions are:

1. Annihilation of a heavy dark matter particle and anti-particle produce two light particles: \( \chi + \bar{\chi} \leftrightarrow l + \bar{l} \)

2. Light particles are tightly coupled to the cosmic plasma, thus \( n_l = n_l^{eq} \)

3. There is no asymmetry between \( \chi \) and \( \bar{\chi} \) \( (n_\chi = n_{\bar{\chi}}) \)

\[
\frac{dN_\chi}{dt} = -s\langle \sigma v \rangle \left[ N_\chi^2 - (N_\chi^{eq})^2 \right]
\]

\( N_\chi \equiv n_\chi / s, \ N_\chi^{eq} \equiv n_{\chi^{eq}} / s \)
Define $x \equiv M_X / T$, then write Boltzmann eq. in terms of $x$:

$$\frac{dx}{dt} = \frac{d}{dt} \left( \frac{M_X}{T} \right) = -\frac{1}{T} \frac{dT}{dt} x \simeq H x$$

By assuming the radiation domination $h = H(M_X)/x^2$, then:

$$\frac{dN_X}{dx} = -\frac{\lambda}{x^2} \left[ N_X^2 - (N_X^{eq})^2 \right]$$

where $\lambda \equiv \frac{2\pi^2}{45} g_* s \frac{M_X^2 \langle \sigma v \rangle}{H(M_X)}$

It is called Riccati equation.

There is no analytic solution for it.
Freeze-out density of DM

- At high temperature \((x < 1)\): \(N_X \approx N_X^{eq} \approx 1\)
- At low temperature \((x \gg 1)\): \(N_X^{eq} \sim e^{-x}\)

Thus \(X\) particles will become so rare and they cannot find each other fast enough to maintain equilibrium abundance.

- Freeze-out happens at \(x_f \sim 20\), where solution starts to deviate from equilibrium.

- Final relic abundance: \(N_X^\infty \equiv N_X(x = \infty)\) (freeze-out density of DM)

- After freeze-out \(N_X \gg N_X^{eq}\), thus we can drop \(N_X^{eq}\):

\[
\frac{dN_X}{dx} \sim -\frac{\lambda N_X^2}{x^2} \quad \rightarrow \quad \int_{x_f}^{\infty} (...) \quad \rightarrow \quad \frac{1}{N_X^\infty} - \frac{1}{N_X^f} = \frac{\lambda}{x_f}
\]

where \(N_X^f \equiv N_X(x_f)\). Typically, \(N_X^f \gg N_X^\infty\), then: \(N_X^\infty \sim \frac{x_f}{\lambda}\)

This predicts the freeze-out abundance \(N_X^\infty\) decreases as the interaction rate \(\lambda\) increases.
Figure: Abundance of dark matter particles as the temperature drops below the mass.

Image Source: Daniel D. Baumann, Lecture notes on Cosmology
Relating the freeze-out abundance of DM relics to the DM density today:

\[ \Omega_X \equiv \frac{\rho_{X,0}}{\rho_{c,0}} \]

\[ = \frac{M_X n_{X,0}}{3M_{PL}^2 H_0^2} = \frac{M_X N_{X,0} s_0}{3M_{PL}^2 H_0^2} = M_X N_\infty \frac{s_0}{3M_{PL}^2 H_0^2} \]

\[ = \frac{H(M_X)}{M_X^2} \frac{x_f}{\langle \sigma v \rangle} \frac{g_*(T_0)}{g_*(M_X)} \frac{T_0^3}{3M_{PL}^2 H_0^2} \]

\[ = \frac{\pi}{9} \frac{x_f}{\langle \sigma v \rangle} \sqrt{\frac{g_*(M_X)}{10}} \frac{g_*(T_0)}{g_*(M_X)} \frac{T_0^3}{3M_{PL}^2 H_0^2} \]
\[
\Rightarrow \Omega_X h^2 \sim 0.1 \frac{x_f}{10} \sqrt{\frac{10}{g^*(M_X)}} \frac{10^{-8} \text{GeV}^{-2}}{\langle \sigma v \rangle}
\]

This reproduces the observed DM density if:

\[
\sqrt{\langle \sigma v \rangle} \sim 10^{-4} \text{GeV}^{-1} \sim 0.1 \sqrt{G_F}
\]

The fact that a thermal relic with a cross section characteristic of the weak interaction gives the right dark matter abundance is called the WIMP miracle.
The early Universe was in thermal equilibrium and freeze-out happened when $\Gamma \lesssim H$.

After freeze-out particles fell out of equilibrium and decoupled.

Deviations from equilibrium explain why there is something rather than nothing in the Universe.

Boltzmann equation is useful to explain non-equilibrium processes.

Calculate the relic abundance of Dark Matter.

Cross section of the order of weak interaction gives the right Dark Matter abundance.
References
