Abundance of Thermal Relics in Non-standard Cosmological Scenarios

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Refs:
- PRD73 123502 (2006)
- PRD76 103524 (2007)
- work in progress

KIAS seminar
1. Motivation

- Observations of
  - cosmic microwave background
  - structure of the universe
  - etc.

- Non-baryonic dark matter: \( \Omega_{DM} h^2 = 0.1143 \pm 0.0034 \)

- Weakly interacting massive particles (WIMPs) \( \chi \) are good candidates for cold dark matter (CDM)

- The predicted thermal relic abundance naturally explains the observed dark matter abundance: \( \Omega_{\chi,\text{standard}} h^2 \sim 0.1 \)

- Neutralino (LSP); 1\textsuperscript{st} KK mode of the B boson (LKP); etc.
Investigation of early universe using DM abundance

- The abundance of thermal relics (e.g. DM) is determined by the Boltzmann equation:
  \[ \dot{n}_\chi + 3H n_\chi = -\langle \sigma_{\text{eff}} v \rangle (n_\chi^2 - n_{\chi,\text{eq}}^2) \]
  (and the reheat temperature: \(T_R\))
  Numerical calculation needed in evaluating the relic density in many cases

- The (effective) cross section \(\sigma_{\text{eff}}\) can be determined from collider and DM detection experiments

Analytic methods should be developed in various scenarios

We can test the standard CDM scenario and investigate conditions of very early universe: \(T_R, H, \cdots\)
Outline

- Analytic treatment applicable to low-reheat-temperature scenarios
- Dark matter = thermal WIMPs
  - constraints on the reheating temperature and on modifications of the Hubble parameter
- Analytic treatment that connects the hot and cold relic solutions

1. Motivation
2. Standard calculation of WIMP relic abundance (review)
3. Low-temperature scenario
4. Constraints on the very early universe from WIMP dark matter
5. Abundance of semi-relativistic relics
6. Summary
2. Standard calculation of the WIMP relic abundance

[Scherrer, Turner, PRD33(1986); Griest, Seckel, PRD43(1991); ...]

- Conventional assumptions:
  - $\chi = \bar{\chi}$, single production of $\chi$ is forbidden
  - Thermal equilibrium was maintained:
    \[ T_R(\text{Reheat temp}) \geq T_F(\text{Freezeout temp}) \]
  - For adiabatic expansion the Boltzmann eq. is
    \[
    \frac{dY_\chi}{dx} = -\frac{\langle \sigma v \rangle s}{H x} (Y_\chi^2 - Y_{\chi,eq}^2),
    \]
    \[ Y_{\chi,eq} = \frac{n_{\chi,eq}}{s}, \quad x = \frac{m_\chi}{T} \]
- $\chi$ decoupled when they were non-relativistic in RD epoch:
  \[
  \langle \sigma v \rangle = a + 6b/x + \mathcal{O}(1/x^2), \quad n_{\chi,eq} = g_\chi \left( \frac{m_\chi T}{2\pi} \right)^{3/2} e^{-m_\chi/T}
  \]

\[ \Omega_{\chi,\text{standard}} h^2 \simeq 0.1 \times \left( \frac{a + 3b/x_F}{10^{-9} \text{ GeV}^{-2}} \right)^{-1} \left( \frac{x_F}{22} \right) \left( \frac{g_*}{90} \right)^{-1/2} \simeq \Omega_{\text{DM}} h^2 \]
3. Low-temperature scenario

- $T_R$: Reheat temperature

  The initial abundance is assumed to be negligible: $Y_\chi(x_0) = 0$, $x_0 = \frac{m_\chi}{T_R}$

- Zeroth order approximation:

  $T_R < T_F$ \quad $\chi$ annihilation is negligible:

  $\frac{dY_0}{dx} = 0.028 \ g_\chi^2 g_*^{-3/2} m_\chi M_{\text{Pl}} e^{-2x} x \left( a + \frac{6b}{x} \right)$

  The solution is proportional to the cross section:

  At late times,

  $Y_0(x \gg x_0) \sim 0.014 \ g_\chi^2 g_*^{-3/2} m_\chi M_{\text{Pl}} e^{-2x_0} x_0 \left( a + \frac{6b}{x_0} \right)$

  This solution should be smoothly connected to the standard result
First order approximation

- Add a correction term describing annihilation to $Y_0$: $Y_1 = Y_0 + \delta$ ($\delta < 0$)

- As long as $|\delta| \ll Y_0$, the evolution equation for $\delta$ is

$$\frac{d\delta}{dx} = -1.3 \sqrt{g_* m_X M_{PL}} \left( a + \frac{6b}{x} \right) \frac{Y_0(x)^2}{x^2}$$

The solution is proportional to $\sigma^3$

At late times,

$$\delta(x \gg x_0) \simeq -2.5 \times 10^{-4} g_X^4 g_*^{-5/2} m^3 M_{PL}^3 e^{-4x_0} x_0 \left( a + \frac{3b}{x_0} \right) \left( a + \frac{6b}{x_0} \right)^2$$

- $|\delta|$ soon dominates over $Y_0$ for not very small cross section

$Y_1$ fails to track the exact solution
Re-summed ansatz

- It is noticed that $Y_0 \propto \sigma > 0$, $\delta \propto \sigma^3 < 0$
  For large cross section, $Y_\chi(x \to \infty)$ should be $\propto 1/\langle \sigma v \rangle$

- This observation suggests the re-summed ansatz:

  $$Y(x) = Y_0 + \delta = Y_0 \left(1 + \frac{\delta}{Y_0}\right) \approx \frac{Y_0}{1 - \delta/Y_0} \equiv Y_{1,r}$$

- For $|\delta| \gg Y_0$,
  $$Y_{1,r}(x) \approx -\frac{Y_0^2}{\delta} \propto \frac{1}{\sigma}$$
  At late times, $Y_{1,r}(x \to \infty) = \frac{1.3}{\sqrt{g_*} m_\chi M_{Pl}} \frac{a + 3 b/x_0}{(a + 3 b/x_0)}$

- In the case where $\chi$ production is negligible but the initial abundance is sizable, $Y_{1,r}$ is exact
4. Constraints on the very early universe from WIMP DM

- Out-of-equilibrium case: \( \sigma \rightarrow \Omega h^2 \); \( T_0 = m_\chi / x_0 \rightarrow \Omega h^2 \)
- Equilibrium case: \( \sigma \rightarrow \Omega h^2 \); \( \Omega_\chi h^2 \) is independent of \( T_R \)

- Thermal relic abundance in the RD universe:

\[ 0.8 < \Omega_{DM} h^2 < 0.12 \]

Requirement that \( \Omega_\chi h^2 \approx 0.1 \)

Lower bound on the reheat temperature: \( T_R > m_\chi / 23 \)
When WIMPs were in full thermal equilibrium, in terms of the modification parameter $A(x) = H_{st}(x)/H(x)$, the relic abundance is

$$\Omega X h^2 = 0.1 \left( \frac{I(x_F)}{8.5 \times 10^{-10} \text{ GeV}^{-2}} \right)^{-1}$$

$$I(x_F) = \int_{x_F}^{\infty} dx \frac{\sqrt{g_*} \langle \sigma v \rangle A(x)}{x^2}, \quad x_F = \ln \left[ \sqrt{\frac{45}{\pi^5} \xi m_X M_{Pl} g_*} \frac{\langle \sigma v \rangle A(x)}{\sqrt{x g_*}} \right]_{x=x_F}$$

If $A(x) = 1$, $x_F = x_{F, st}$ and we recover the standard formula.

This formula is capable of predicting the final relic density correctly.

Various cosmological models predict a non-standard early expansion [e.g. Scherrer et al., PRD(1985); Salati, PLB(2003); Fernengo et al., PRD(2003); Chung et al., PRD (1999); ...]

Predicted WIMP relic abundances are also changed.
Constrains on modifications of the Hubble parameter

- In terms of $z \equiv T/m_\chi = 1/x$
  we need to know $A(z)$ only for $z_{\text{BBN}} = 10^{-5} - 10^{-4} \leq z \leq z_F \sim 1/20 \ll O(1)$

This suggests a parametrization of $A(z)$ in powers of $(z - z_{F,\text{st}})$:

$$A(z) = A(z_{F,\text{st}}) + (z - z_{F,\text{st}})A'(z_{F,\text{st}}) + \frac{1}{2}(z - z_{F,\text{st}})^2 A''(z_{F,\text{st}})$$

subject to the BBN limit: $0.8 \leq k \equiv A(z \rightarrow z_{\text{BBN}}) \leq 1.2$

- Once we know $\sigma$, we can constrain $A(z)$:

$\Omega_\chi h^2$ depends on all $H(T_{\text{BBN}} < T < T_F)$ Larger allowed region for $H(T_F)$
5. Abundance of semi-relativistic relics

- Precise evaluation of the abundance of particles that freeze out when they are semi-relativistic \((x_F \sim 3)\) is complicated

Goal: simple analytic treatment that describes the transition from non-relativistic to relativistic relics

- Assume the Maxwell-Boltzmann distribution:

\[
Y_{\chi,eq} \equiv \frac{n_{\chi,eq}}{s} = 0.115 \frac{g_{\chi}}{g_\ast s} x^2 K_2(x) \quad (K_n(x): \text{modified Bessel function})
\]

Thermal average of cross section \(\sigma\):

\[
\langle \sigma v \rangle = \frac{1}{8m_\chi^4 T K_2^2(m_\chi/T)} \int_{4m_\chi^2}^{\infty} ds \, \sigma(s - 4m_\chi^2) \sqrt{s} \, K_1(\sqrt{s}/T)
\]
Ansatz for approximate cross sections

- Consider neutrinos as stable relic:
- Annihilation cross section:

\[ \sigma_v^{\text{Dirac}} \nu = \frac{G_F^2 s}{16\pi} \]

\[ \sigma_v^{\text{Majorana}} \nu = \frac{G_F^2 s v^2}{16\pi} \]

- Ansatz for the thermally-averaged annihilation cross section:

\[ \langle \sigma v \rangle_{\text{app}}^{\text{Dirac}} = \frac{G_F^2 m_X^2}{16\pi} \left( \frac{12}{x^2} + \frac{5 + 4x}{1 + x} \right) \]

\[ \langle \sigma v \rangle_{\text{app}}^{\text{Majorana}} = \frac{G_F^2 m_X^2}{16\pi} \left( \frac{12}{x^2} + \frac{3 + 6x}{(1 + x)^2} \right) \]

The approx. cross sections reproduce the exact results with accuracy of a few %
Approximate abundance of semi-relativistic relics

- Define the freeze-out temperature by
  \[ \Gamma(x_F) = H(x_F) \]
  (different from the standard definition of \( x_F \))

- Assume the relic abundance does not change after decoupling
  Final abundance:
  \[ Y_{\chi, \infty} = Y_{\chi, eq}(x_F) \]

- Comparison between the numerical and approx solutions

![Graph comparing numerical and approximate solutions for different mass values of \( m_\chi \).]
Applications of semi-relativistic relics

- As DM candidates
  
  Hypothetical semi-relativistic relics should decouple before BBN
  
  \[ m_\chi \sim T_F > T_{\text{BBN}} \sim \text{MeV} \Rightarrow \Omega_\chi h^2 > 10^3 \]

  The relic abundance is too high!

- As source of large entropy production
  
  Out-of-equilibrium decay of relic particles produces entropy

  Ratio of the final to initial entropy:
  
  \[ \frac{S_f}{S_i} = \frac{g^*_i}{g^*_f} \frac{m_\chi Y_{\chi, i} \tau_{\chi, i}^{1/2}}{M_{\text{Pl}}^{1/2}} \propto \Omega_\chi h^2 \]

  Semi-relativistic relics can produce significant entropy!
Example: sterile neutrino

- Consider a sterile neutrino mixed with an active neutrino (mixing angle: $\theta$)

- Decay rate of the sterile neutrino:
  \[ \Gamma_\chi = \frac{G_F^2 m_\chi^5}{192\pi^3} \sin^2 \theta \]
  large enough not to spoil BBN

- By introducing a new particle, U, large pair annihilation can be induced:
  \[ \sigma v = \frac{sv^2 g_X^2 g_f^2}{12\pi M_U^4} \]

- Entropy production $S_f / S_i$ by the decay of semi-relativistic sterile neutrinos
5. Summary

- Using the DM relic density we can probe very early universe at around $T \sim m_\chi/20 \sim \mathcal{O}(10) \text{ GeV}$ (well before BBN $T_{\text{BBN}} \sim \mathcal{O}(1) \text{ MeV}$)

- We find an approximate analytic formula for the WIMP abundance that is valid for all $T_R \leq T_F$

- Lower bound on the reheat temperature: $T_R > m_\chi/23$

- The sensitivity of $\Omega_{\chi,\text{thermal}} h^2$ on $H(T_F)$ is weak

- We find an approximate analytic formula for the abundance of semi-relativistic relics

- Semi-relativistic relics are useful for producing a large amount of entropy
Backup slides
Hot relics

- Hot relics (decouple for $x_F < 3$):
  
  $Y_{\chi,eq}(x)$ almost constant

  Final abundance is insensitive to the freeze out temperature:

  $$Y_{\chi,\infty} = Y_{\chi,eq}(x_F) = \frac{45}{2\pi^4} \frac{g_\chi}{g_{*s}(x_F)}$$
The re-summed ansatz $Y_{1,r}$ describes the full temperature dependence of the abundance when equilibrium is not reached. For larger cross section the deviation becomes sizable for $x - x_0 \sim 1$, but the deviation becomes smaller for $x \gg x_0$. 

\begin{itemize}
    \item The re-summed ansatz $Y_{1,r}$ describes the full temperature dependence of the abundance when equilibrium is not reached.
    \item For larger cross section the deviation becomes sizable for $x - x_0 \sim 1$, but the deviation becomes smaller for $x \gg x_0$.
\end{itemize}

$Y_\chi :$ Exact result, $Y_{1,r} :$ Re-summed ansatz, $b = 0$, $Y_\chi(x_0 = 22) = 0$
Semi-analytic solution

- $Y_{1,r}(x_0, x \to \infty) \propto \Omega_{1,r} h^2$ has a maximum (left)
- New semi-analytic solution can be constructed: $\Omega_{\text{semi}} h^2$ (right)

For $x_0 > x_{0,\text{max}}$, use $Y_{1,r}(x_0)$; for $x_0 < x_{0,\text{max}}$, use $Y_{1,r}(x_{0,\text{max}})$

The semi-analytic solution $\Omega_{\text{semi}} h^2$ reproduces the correct final relic density $\Omega_{\text{exact}} h^2$ to an accuracy of a few percent.