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4 Summary
Introduction: the need for Dark Matter

Several observations indicate existence of non-luminous Dark Matter (DM) (more exactly: missing force)
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Galactic rotation curves imply \( \Omega_{\text{DM}} h^2 \geq 0.05 \).

\( \Omega \): Mass density in units of critical density; \( \Omega = 1 \) means flat Universe.

\( h \): Scaled Hubble constant. Observation: \( h = 0.72 \pm 0.07 \)
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- Models of structure formation, X ray temperature of clusters of galaxies, . . .

- Cosmic Microwave Background anisotropies (WMAP etc.) imply $\Omega_{DM}h^2 = 0.112 \pm 0.006$  

PDG, 2012 edition
Need for non–baryonic DM

Total baryon density is determined by:

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Need for non–baryonic DM

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Consistent result: \( \Omega_{\text{bar}} h^2 \approx 0.02 \)

\[ \implies \text{Need non–baryonic DM!} \]
Need for exotic particles

Only possible non–baryonic particle DM in SM: Neutrinos!
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Make hot DM: do not describe structure formation correctly

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Make hot DM: do not describe structure formation correctly

$$\Rightarrow \Omega_{\nu}h^2 \leq 0.0062$$

$$\Rightarrow$$ Need exotic particles as DM!

Possible loophole: primordial black holes; not easy to make in sufficient quantity sufficiently early.
What we need

Since $h^2 \simeq 0.5$: Need $\sim 20\%$ of critical density in

- **Matter** (with negligible pressure, $w \simeq 0$)
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- and does not couple to elm radiation
Remarks

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- Precise “WMAP” determination of DM density hinges on assumption of “standard cosmology”, including assumption of nearly scale–invariant primordial spectrum of density perturbations: almost assumes inflation!

- Evidence for $\Omega_{DM} \gtrsim 0.2$ much more robust than that! (Does, however, assume standard law of gravitation.)
Particle DM: Possibilities

Theorist’s tasks:

- Introduce right kind of particle (stable, neutral, non–relativistic)
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⇒ Use theoretical “prejudice” as guideline: Only consider candidates that solve (at least) one additional problem!
Particle Candidates 1

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Gravitino $\tilde{G}$: Majorana spin–3/2 fermion
Particle Candidates 2

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Here: focus on WIMP (e.g. Neutralino) and Gravitino.
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Making DM (cont.’d)

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From decay of heavier particle ($\tilde{\chi}$, $\tilde{G}$)
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- Assume symmetric contribution annihilates away: only “particles” left (see: baryons)
- If same mechanism generates baryon asymmetry: “Naturally” explains $\Omega_{DM} \simeq 5\Omega_{baryon}$, if $m_\chi \simeq 5m_p$
- For WIMPs: Order of magnitude of $\Omega_{DM}$ is understood; $\Omega_{baryon}$ isn’t
Currently: Universe dominated by dark energy ($\sim 75\%$) and non–relativistic matter ($\sim 25\%$); $\Omega_{\text{rad}} \sim 10^{-4}$.

(Radiation $\equiv$ relativistic particles.)
Thermal history of the Universe

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- Early Universe was dominated by radiation! (Except in some extreme ‘quintessence’ or ‘brane cosmology’ models.)
Let $\chi$ be a generic DM particle, $n_\chi$ its number density (unit: GeV$^3$). Assume $\chi = \bar{\chi}$, i.e. $\chi\chi \leftrightarrow$ SM particles is possible, but single production of $\chi$ is forbidden by some symmetry.
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Evolution of $n_{\chi}$ determined by Boltzmann equation:

$$\frac{dn_{\chi}}{dt} + 3H n_{\chi} = -\langle \sigma_{\text{ann}} v \rangle (n_{\chi}^2 - n_{\chi,\text{eq}}^2)$$

$H = \dot{R}/R$: Hubble parameter
$\langle \ldots \rangle$: Thermal averaging
$\sigma_{\text{ann}} = \sigma(\chi\bar{\chi} \rightarrow \text{SM particles})$
$v$: relative velocity between $\chi$’s in their cms
$n_{\chi,\text{eq}}$: $\chi$ density in full equilibrium
\[
\frac{dn_{\chi}}{dt} + 3H n_{\chi} = -\langle \sigma_{\text{ann}} v \rangle \left( n_{\chi}^2 - n_{\chi, \text{eq}}^2 \right)
\]

2\textsuperscript{nd} Lhs term: Describes $\chi$ dilution by expansion of Universe:

\[
\frac{dR^{-3}}{dt} = -3R^{-4} \dot{R} = -3H R^{-3}
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**1\textsuperscript{st} rhs term:** describes \( \chi \) pair annihilation; assumes *shape* of \( n_\chi \) same as that of \( n_{\chi, \text{eq}} \): reactions \( \chi + f \leftrightarrow \chi + f \) are very fast (\( f \) : some SM particle).
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2\textsuperscript{nd} rhs term: describes \( \chi \) pair production; assumes CP conservation (\( \Rightarrow \) same matrix element).

Check: creation and annihilation balance iff \( n_\chi = n_{\chi, \text{eq}} \).
Rewriting the Boltzmann equation

In order to get rid of the $3H n_\chi$ term: introduce $Y_\chi \equiv \frac{n_\chi}{s}$ (s: entropy density)
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$$= \frac{1}{s} \left[ -3H n_\chi - \langle \sigma_{\text{ann}} v \rangle \left( n_\chi^2 - n_{\chi, \text{eq}}^2 \right) \right] + \frac{n_\chi}{s^2} 3H s$$
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$$= -s \langle \sigma_{\text{ann}} v \rangle \left( Y^2_\chi - Y^2_\chi, \text{eq.} \right)$$

$$s = \frac{2\pi^2}{45} g_* T^3 \quad (g_*: \text{no. of relativistic d.o.f.})$$
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If interactions are negligible: $Y_\chi \rightarrow \text{const.}, \text{i.e.} \chi$ density in co–moving volume is unchanged
Rewriting the Boltzmann equation (cont’d)

Write lhs entirely in terms of dimensionless quantities: introduce \( x = \frac{m \chi}{T} \).
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$$\Rightarrow \frac{d}{dt} (g_* T^3) = -3H (g_* T^3)$$
Rewriting the Boltzmann equation (cont’d)

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$\implies \frac{d}{dt} (g_* T^3) = -3H (g_* T^3)$

$\implies \dot{g}_* T^3 + 3g_* T^2 \dot{T} = -3H g_* T^3$
Rewriting the Boltzmann equation (cont’d)

Write lhs entirely in terms of dimensionless quantities: introduce $x = \frac{m_{\chi}}{T}$.

Had: $\dot{s} = -3Hs$

$$\implies \frac{d}{dt} \left(g_\ast T^3\right) = -3H \left(g_\ast T^3\right)$$

$$\implies \dot{g}_\ast T^3 + 3g_\ast T^2 \dot{T} = -3H g_\ast T^3$$

$$\implies \dot{T} = - \left( H + \frac{\dot{g}_\ast}{3g_\ast} \right) T$$
Rewriting the Boltzmann equation (cont’d)

Write lhs entirely in terms of dimensionless quantities: introduce \( x = \frac{m_X}{T} \).

Had: \( \dot{s} = -3Hs \)

\[
\Rightarrow \frac{d}{dt} (g_* T^3) = -3H (g_* T^3)
\]

\[
\Rightarrow g_* T^3 + 3g_* T^2 \dot{T} = -3H g_* T^3
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\[
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From now on: set \( \dot{g}_* = 0 \), since \( g_* \) changes only slowly with time. (Except during QCD phase transition.)
Rewriting the Boltzmann equation (cont’d)

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Had: \( \dot{s} = -3Hs \)

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\[ \implies \dot{x} = -\frac{m\chi}{T^2} \dot{T} = -\frac{x}{T} \dot{T} = xH \]
Boltzmann equation (cont’d)

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\frac{dY_x}{dx} = \frac{1}{\dot{x}} \frac{dY_x}{dt}
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Boltzmann equation (cont’d)

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\[
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Boltzmann equation (cont’d)

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\frac{dY_x}{dx} = \frac{1}{x} \frac{dY_x}{dt} = -\frac{1}{Hx} s \langle \sigma_{\text{ann}} v \rangle \left( Y_x^2 - Y_{x,\text{eq}} \right)
\]

Use \( s = \frac{2\pi^2}{45} g_* T^3 \)

\[
H = \sqrt{\frac{\rho_{\text{rad}}}{3M_P^2}} = \frac{\pi}{\sqrt{90}} \frac{\sqrt{g_*}}{M_P} T^2 \quad \text{for flat, rad.–dom. Universe}
\]
Boltzmann equation (cont’d)

\[
\frac{dY_x}{dx} = \frac{1}{\dot{x}} \frac{dY_x}{dt} = -\frac{1}{Hx} s \langle \sigma_{\text{ann}} v \rangle \left( Y^2_x - Y^2_x, \text{eq} \right)
\]

Use \( s = \frac{2\pi^2}{45} g_* T^3 \)

\[
H = \sqrt{\frac{\rho_{\text{rad}}}{3M_P^2}} = \frac{\pi}{\sqrt{90}} \frac{\sqrt{g_*} T^2}{M_P}
\]

for flat, rad.–dom. Universe

\[
\Rightarrow \frac{dY_x}{dx} = -\frac{4\pi \sqrt{g_*}}{\sqrt{90}} \frac{m_x M_P}{x^2} \langle \sigma_{\text{ann}} v \rangle \left( Y^2_x - Y^2_x, \text{eq} \right)
\]
Boltzmann equation (cont’d)

\[
\frac{dY_X}{dx} = \frac{1}{x} \frac{dY_X}{dt} = - \frac{1}{Hx} s \langle \sigma_{\text{ann}} v \rangle \left( Y_{\chi}^2 - Y_{\chi, \text{eq}}^2 \right)
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\]

\[
\Rightarrow \frac{dY_X}{dx} = - \frac{4\pi \sqrt{g^*} m_{\chi} M_P}{\sqrt{90}} x^2 \langle \sigma_{\text{ann}} v \rangle \left( Y_{\chi}^2 - Y_{\chi, \text{eq}}^2 \right)
\]

For \( T \gtrsim 200 \text{ MeV} \): \( 10 \lesssim \frac{4\pi \sqrt{g^*}}{\sqrt{90}} \lesssim 20 \) (SM, MSSM)
Condition for thermal equilibrium

\[ n_\chi \langle \sigma_{\text{ann}} v \rangle > H \text{ for some } T! \]
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- For non–renormalizable interactions: easiest to satisfy at maximal temperature, \( T \sim T_R \). (See: \( \tilde{G} \))

- For \( T_R < m_\chi \): Easiest to satisfy for \( T \sim T_R \) (see: WIMP at low \( T_R \)).
Example 1: WIMP

Decouple (freeze out) at temperature \( T \ll m_\chi \) (see below).
(N.B. Means \( \chi \) makes cold DM!)
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1) $n_\chi \simeq g_\chi \left( \frac{m_\chi T}{2\pi} \right)^{3/2} e^{-x}$

$$\langle \sigma_{\text{ann}} v \rangle \simeq \frac{x^{3/2}}{2\sqrt{\pi}} \int_0^\infty dv \, v^2 (\sigma_{\text{ann}} v) e^{-xv^2/4}$$
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2) Most of the time: can expand cross section in $\chi$ velocity:

$$\sigma_{\text{ann}} v = a + bv^2 + \ldots \implies \langle \sigma_{\text{ann}} v \rangle = a + 6\frac{b}{x} + \ldots$$
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Typically, \( a, b \lesssim \frac{\alpha^2}{m_\chi^2}, \quad \alpha^2 \sim 10^{-3} \), unless \( a \) is suppressed by some symmetry; e.g. for \( \tilde{\chi}\tilde{\chi} \rightarrow f\bar{f} : a \propto m_f^2 \).
Case 1: Low reheat temperature

Let $T_R$ be the highest temperature of the radiation–dominated universe (after inflation).
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\[ \implies Y_\chi (x \gg x_R) = \frac{45^2 g_\chi^2}{8\sqrt{90} g_*^{3/2} \pi^6} m_\chi M_P \cdot e^{-2x_R} \left[ \frac{a}{2} \left( x_R - \frac{1}{2} \right) + 3b \right]. \]
To get current $\Omega_{\chi} h^2$

Saw: $Y_{\chi} \rightarrow Y_{\chi,0} = \text{const.}$ for $x \gg x_R$. 
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$$\Rightarrow \Omega_\chi h^2 = \frac{\rho_\chi}{\rho_{\text{crit.}}} h^2 = \frac{n_\chi m_\chi}{3 H_0^2 M_P^2} \frac{H_0^2}{(100 \text{ km Mpc}^{-1} \text{ sec}^{-1})^2}$$
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Use 1 Mpc $= 3.09 \cdot 10^{19}$ km, 1 sec$^{-1} = 6.6 \cdot 10^{-25}$ GeV, 
$s_0 = 2.9 \cdot 10^3$ cm$^{-3} = 2.2 \cdot 10^{-38}$ GeV$^3$, and introduce dimensionless quantities $\hat{a} = a m_\chi^2$, $\hat{b} = b m_\chi^2$.
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Example: $\hat{a} = 0$, $\hat{b} = 10^{-4}$ $\Rightarrow$ need $T_R \simeq 0.04 m_{\chi}$
Case 2: Thermal WIMP

Assume $\chi$ was in full thermal equilibrium after inflation.
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For $T < T_F$: WIMP production negligible, only annihilation relevant in Boltzmann equation.
Thermal WIMP: solution of Boltzmann eq.

\[
\text{Had} \quad \frac{dY_\chi}{dx} = - \frac{4\pi \sqrt{g_*}}{\sqrt{90}} \frac{m_\chi \, M_P}{x^2} \langle \sigma_{\text{ann}} v \rangle \left( Y_\chi^2 - Y_\chi^2, \text{eq} \right)
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High temperature, \( T > T_F \): write \( Y_\chi = Y_{\chi, \text{eq}} + \Delta \), ignore \( \Delta^2 \) term.
Thermal WIMP: solution of Boltzmann eq.

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\[ \Rightarrow \frac{d\Delta}{dx} = -\frac{dY_\chi,_{\text{eq}}}{dx} + \frac{dY_\chi}{dx} \]

\[ \simeq -\frac{dY_\chi,_{\text{eq}}}{dx} - \frac{4\pi \sqrt{g_*}}{\sqrt{90}} \frac{m_\chi M_P}{x^2} \langle \sigma_{\text{ann}} v \rangle (2Y_\chi,_{\text{eq}} \Delta) \]
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Use \( \frac{dY_{\chi, \text{eq}}}{dx} = - \frac{3}{2} \frac{Y_{\chi, \text{eq}}}{x} - Y_{\chi, \text{eq}} \simeq -Y_{\chi, \text{eq}} \quad (x \gg 1) \):
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\approx -\frac{dY_{\chi, \text{eq}}}{dx} - \frac{4\pi\sqrt{g_*}}{\sqrt{90}} \frac{m_\chi M_P}{x^2} \langle \sigma_{\text{ann}} v \rangle (2Y_{\chi, \text{eq}} \Delta)
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Use \( \frac{dY_{\chi, \text{eq}}}{dx} = -\frac{3}{2} \frac{Y_{\chi, \text{eq}}}{x} - Y_{\chi, \text{eq}} \approx -Y_{\chi, \text{eq}} \) \((x \gg 1)\):

To keep \( \frac{d\Delta}{dx} = 0 \): need

\[
\Delta \approx \frac{x^2}{2.64m_\chi M_P \sqrt{g_*} \langle \sigma_{\text{ann}} v \rangle}
\]
Low−$T$ solution

$Y_{\chi, \text{eq}} \rightarrow 0 \implies$ can ignore production term in Boltzmann eq.
Low–$T$ solution

$Y_{\chi, \text{eq}} \to 0 \implies$ can ignore production term in Boltzmann eq.

$$\frac{d\Delta}{dx} \simeq -1.32 m_{\chi} M_P \sqrt{g_*} x^{-2} \langle \sigma_{\text{ann}} v \rangle \Delta^2$$
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\[ \implies \frac{1}{\Delta(x_F)} - \frac{1}{\Delta(\infty)} = -1.32 m_\chi M_P \sqrt{g_*} \int_{x_F}^{\infty} dx x^{-2} \langle \sigma_{\text{ann}} v \rangle \]

\[ \equiv -1.32 m_\chi M_P \sqrt{g_*} J(x_F) \]
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Assume $\Delta(\infty) \ll \Delta(x_F)$

\[\implies Y_{\chi,0} = \frac{1}{1.32\sqrt{g_*} m_\chi M_P J(x_F)} \]
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Typically, $x_F \simeq 22$; depends only logarithmically on $\sigma_{\text{ann}}$. 

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Typically, $x_F \simeq 22$; depends only logarithmically on $\sigma_{\text{ann}}$.

Non-relativistic expansion: $J(x_F) = \frac{a}{x_F} + \frac{3b}{x_F^2} \ldots$
\[ \Omega \chi h^2 \approx \frac{8.7 \cdot 10^{-11}}{\sqrt{g^* J(x_F)}} \text{ GeV}^{-2} \]

- Solution validated numerically.
\[ \Omega_\chi h^2 \simeq \frac{8.7 \cdot 10^{-11}}{\sqrt{g_\star J(x_F)}} \text{ GeV}^{-2} \]

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- Smooth transition to previous case (\( T_R < T_F \)): MD, Iminniyaz, Kakizaki, hep-ph/0603165
Co–annihilation

Is important for SUSY scenarios with small mass splitting between LSP and NLSP: $\delta m \equiv m_{\tilde{\chi}'} - m_{\tilde{\chi}} \ll m_{\tilde{\chi}}$
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$\tilde{\chi}, \tilde{\chi}'$ retain relative equilibrium well after sparticles decouple from SM particles: $n_{\tilde{\chi}'} = n_{\tilde{\chi}} e^{-\delta m/T}$
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Previous treatment still applies, with replacement:

\[ \sigma_{\text{ann}} \rightarrow \sigma_{\text{eff}} \sim \sigma_{\text{ann}} + f_B \sigma(\tilde{\chi}\tilde{\chi}' \rightarrow \text{SM}) + f_B^2 \sigma(\tilde{\chi}'\tilde{\chi}' \rightarrow \text{SM}) \]

\( f_B : \) relative Boltzmann factor = \( \left(1 + \frac{\delta m}{m_{\tilde{\chi}}} \right)^{3/2} e^{-\delta m/T} \)
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\( \sigma(\tilde{\chi}\tilde{\chi}'), \sigma(\tilde{\chi}'\tilde{\chi}') \gg \sigma(\tilde{\chi}\tilde{\chi}) \) possible!
Case 3: Freeze–in

Assume very weak, renormalizable interaction (Feebly Interacting Massive Particle, FIMP): never achieved thermal equilibrium
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Final relic density proportional to cross section, independent of FIMP mass
Thermal WIMPs, FIMPs: Assumptions

- $\chi$ is effectively stable, $\tau_\chi \gg \tau_U$: partly testable at colliders
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- No entropy production after $\chi$ decoupled: Not testable at colliders
- $H$ at time of $\chi$ decoupling is known: partly testable at colliders
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\( \implies \) Most important \( \tilde{G} \) production mechanism for \( m \tilde{G} \gtrsim \) MeV: associated production with other sparticle!

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\sigma \tilde{G} \approx \frac{1}{24\pi (m \tilde{G} M_P)^2} \left( 26 g_s^2 M_g^2 + \ldots \right)
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Thermal Gravitino Dark Matter

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\[ \sigma_{\tilde{G}} \simeq \frac{1}{24\pi(m_{\tilde{G}}M_P)^2} \left( 26g_s^2M_{\tilde{g}}^2 + \ldots \right) \]

\( \tilde{G} \) annihilation can be ignored; write Boltzmann eq. for \( \tilde{Y}_{\tilde{G}} \equiv n_{\tilde{G}}/n_\gamma \):

\[ \frac{d\tilde{Y}_{\tilde{G}}}{dT} = - \frac{n_\gamma \sigma_{\tilde{G}}}{4TH(T)} \]
Solution of Boltzmann eq.:

\[ \tilde{Y}_{\tilde{G},0} = \frac{4n_\gamma(T_R)\sigma_\tilde{G}}{4H(T_R)} \propto T_R \] (assuming \( \tilde{Y}_{\tilde{G}}(T_R) = 0 \))
Gravitino DM (cont.’d)

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Inclusion of thermal corrections: e.g. Pradler & Steffen, hep-ph/0612291
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In general, have to add \( \Omega_{NLSP} \frac{m_{\tilde{G}}}{m_{NLSP}} \) from (late) decays of NLSPs. (BBN!)
DM Production from Inflaton Decay

Only consider *perturbative* decays here.
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Inflatons are non–relativistic when they decay.
Energy conserved during $\phi$ decay

$$\Rightarrow n_\phi m_\phi = \rho_{\text{rad}}(T_R) = \frac{\pi^2}{30} g_* T_R^4$$
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DM Production from Inflaton Decay (cont.’d)

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If $\chi$ production and annihilation at $T < T_R$ is negligible, universe evolves adiabatically:

$$\implies \Omega_\chi h^2 = 2.1 \cdot 10^8 \frac{m_\chi}{m_\phi} \frac{T_R}{1 \text{ GeV}} B(\phi \rightarrow \chi)$$
Possibilities for $B(\phi \rightarrow \chi)$

- If $\chi = \text{LSP}$: expect $B(\phi \rightarrow \chi) \approx 1$: Excludes charged LSP for $m_\phi > 2m_\chi$, $T_R \gtrsim 1$ MeV!
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- $\phi \rightarrow f \bar{f} \chi \chi$ (4–body):
  $$B(\phi \rightarrow \chi) \sim \frac{\alpha_\chi^2}{96\pi^3} \left(1 - \frac{4m_\chi^2}{m_\phi^2}\right)^2 \left(1 - \frac{2m_\chi}{m_\phi}\right)^{5/2}$$

(Assumes $\sigma(\chi \chi \leftrightarrow f \bar{f}) \sim \frac{\alpha_\chi^2}{m_\chi^2}$, $\phi \rightarrow f \bar{f}$ dominates.)
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- Can be most important production mechanism for superheavy Dark Matter ($m_\chi \sim 10^{12}$ GeV) in chaotic inflation ($m_\phi \sim 10^{13}$ GeV); for LSP if $T_R \lesssim 0.03m_\chi$; ...
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- Only the thermal WIMP scenario can be tested using collider data and results from WIMP search experiments. Other scenarios can only be tested with additional input to constrain cosmology ($T_R$, ...).