Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1

SS2011

Exercises Quantum Field Theory I

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Some general remarks

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If you have any questions, remarks, need for further discussion etc. concerning the exercises or the lectures, we highly advice you to consult us during our office hour on `Wednesday` from `2-4 pm`. Feedback is also highly welcome at any time.

The lecture times are

- `Monday 4-6 pm` and `Thursday 12-13` at HS1 PI.

You can find informations about the lecture and the exercises on the webpage
`http://www.th.physik.uni-bonn.de/klemm/QFT1SS2011/index.php`

1 Classical Electrodynamics and Field Theory

In this exercise session we want to study some basics in field theory. We focus on the simplest setup, which you should be familiar with from classical electrodynamics.

1.1 Warm-Ups

1. Recall the definition of the field strength tensor $F_{\mu\nu}$ from electrodynamics. What is its relation to the potential A_μ?

2. How does gauge invariance manifest itself in this setup?

3. Consider a theory which is described by a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$, depending on the field ϕ and its derivative. Recall the expression for the equations of motion and furthermore recall the statement of Noether's theorem. To which symmetry is the energy-momentum tensor associated and what is its general formula from Noether's theorem?

1.2 Exercises

1. Consider the following action for electrodynamics without sources
$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}, \tag{1.1}$$
and derive Maxwell equations from that.

2. Calculate the energy-momentum tensor. Check whether it is symmetric!

3. In order to obtain a symmetric energy-momentum tensor, we can add a term of the form $\partial_\alpha K^{\alpha\mu\nu}$ (recall why?). Demanding that $\partial_\alpha K^{\alpha\mu\nu}$ is divergenceless, what does this imply for its symmetries? Use
$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu, \tag{1.2}$$
to introduce the completed energy-momentum tensor
$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\alpha K^{\alpha\mu\nu}, \tag{1.3}$$
and derive the expressions for the electromagnetic energy and momentum densities.

4. How to incorporate a source term? What would change in 1.-3. in this case?

2 The Lorentz group I

In this exercise we would like to recall some basic facts about the Lorentz group, which you hopefully encountered earlier in your studies. We work with the convention, that the metric $g_{\mu\nu}$ has the following form $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and four-vectors are usually denoted by $x = (x^0, \vec{x})$. We sometimes write $\langle x, y\rangle = g_{\mu\nu}x^\mu y^\nu$. Furthermore the canonical basis of $\mathbb{R}^4$ is denoted by e_μ with 1 in the μ-th component and 0 otherwise.

1. Remind yourself, what are space-, light- and time-like vectors!

2. What is the defining property of a Lorentz transformation?

3. How can one interpret spatial rotations as a Lorentz transformation?

4. Show explicitly, that the two transformations $T = \mathrm{diag}(-1,1,1,1)$ and $S = \mathrm{diag}(1,-1,-1,-1)$ are elements of the Lorentz group. What is the interpretation of their action?

5. Following the definition, show that $|\Lambda^0_0| \geq 1$ and $\det\Lambda = \pm 1$. What is the meaning of $\Lambda^0_0 < 0$?

6. Show that the Lorentztransformations make up a group, called the Lorentz group $L = O(1,3)$ which has four disconnected branches (as a manifold)
$$\begin{array}{ll} \mathcal{L}_+^\uparrow : \Lambda^0_0 \geq 1, \ \ \det\Lambda = 1, & \mathcal{L}_-^\uparrow : \Lambda^0_0 \geq 1, \ \ \det\Lambda = -1, \\ \mathcal{L}_-^\downarrow : \Lambda^0_0 \leq -1, \ \ \det\Lambda = -1, & \mathcal{L}_+^\downarrow : \Lambda^0_0 \leq -1, \ \ \det\Lambda = 1. \end{array} \tag{2.4}$$
Which component is connected to $\mathbf{1}$? For the case that $\Lambda^0_0 \geq 1$, Λ is called orthochronous. If $\det\Lambda = 1$, then Λ is called proper. The proper Lorentz transformations $L_+ := L_+^\uparrow \cup \mathcal{L}_+^\downarrow$ are called $SO(1,3)$. How can one obtain the other branches of the Lorentz group from the orthochronous, proper branch?

7. Consider the set of skew-symmetric matrices $\hat{L}$, i.e. $\langle x, Ay\rangle = -\langle Ax, y\rangle$ w.r.t. the Minkowski metric. Define furthermore

$$\omega_{\mu\nu} x = e_\mu \langle e_\nu, x\rangle - e_\nu \langle e_\mu, x\rangle, \quad \mu < \nu \tag{2.5}$$

Show that $\hat{L}$ is a six dimensional vector space with basis $\omega_{\mu\nu}$ and $[A, B] = AB - BA \in \hat{L}$ for $A, B \in \hat{L}$.

8. Calculate $[\omega_{\mu\nu}, \omega_{\kappa\lambda}]$!

9. In addition, show that

$$\Lambda(\tau) = \exp(\tau A) \in L_+, \tag{2.6}$$

Hint: You might want to use that $\det \exp(A) = \exp(\mathrm{Tr} A)$.

This promotes the vector space $\hat{L}$ to a Lie algebra of the Lie group $L_+$and $\omega_{\mu\nu}$ are called the generators of L_+ (why only L_+?).

3 Homework

3.1 The complex scalar field

Let $\phi : \mathbb{R}^4 \to \mathbb{C}$ be a field, that obeys the Klein-Gordon equation. The action is given by

$$S = \int d^4x \, (\partial_\mu \phi^* \partial^\mu \phi^\mu - m^2 \phi^* \phi). \tag{3.7}$$

This theory is best analyzed by considering $\phi(x)$ and $\phi^*(x)$ as the basic dynamic variables.

1. Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is given by

$$H = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi\right). \tag{3.8}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

2. Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

3. Show that the theory is invariant under $\phi(x) \mapsto e^{i\alpha}\phi(x)$ and therefore exhibits a global $U(1)$ symmetry.

4. Rewrite the conserved charge

$$Q = \int d^3x \, \frac{i}{2} \left(\phi^* \pi^* - \pi \phi\right), \tag{3.9}$$

in terms of creation and annihilation operators and evaluate the charge of the particles of each type.

5. Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of the part above, and the other three given by

$$Q^i = \int d^3x \frac{i}{2} \left(\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b\right), \tag{3.10}$$

where σ^i are the Pauli matrices. Show that these three charges have the commutation relations of angular momentum (SU(2))! This leads to the concept of anti-particles.