Physikalisches Institut
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Theoretische Physik

Exercise 10

SS2011

Exercises Quantum Field Theory I

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1 Yukawa theory - spin zero interaction bosons

Consider the interacting theory, called Yukawa theory, of a single massive Dirac fermion Ψ and a single real massive boson ϕ with Hamiltonian H given by

$$H = H_{\text{Dirac}} + H_{\text{KG}} + \int dx^3 g \bar{\Psi}\Psi\phi. \tag{1.1}$$

Here H_{Dirac}, H_{KG} denote the Hamiltonians of the free Dirac fermion with mass m_ψ respectively the free Klein-Gordon field with mass m_ϕ and g denotes the dimensionless coupling constant, that is a real number.

(i) Recall the Feynman rules of the free theory (Use normal and dashed lines for fermions and bosons, respectively). What is the difference between external fermions and anti-fermions (introduce one arrow for particle number flow and one arrow for momentum)? Identify the interaction Hamiltonian $H_I(t)$?

(ii) Consider the S-matrix of the Yukawa theory for two-fermion to two-fermion scattering

$$iT_{2\to 2} = \langle \mathbf{p}'\mathbf{k}'|iT|\mathbf{p}\mathbf{k}\rangle\,, \tag{1.2}$$

where $\mathbf{p}, \mathbf{k}$ respectively $\mathbf{p}', \mathbf{k}'$ denote the momenta in the initial respectively final state. What is the leading contribution to $iT_{2\to 2}$ in t-channel? Evaluate it and extract the matrix element $i\mathcal{M}$.

(iii) From your experience with this calculation, introduce reasonable Feynman rules (be careful about fermion ordering).

(iv) Work out the diagrams for the u-channel as well and compare to the t-channel result.

In the following we want to extract the effective potential $V(\mathbf{x})$ between to distinguishable fermions generated by the exchange of the field ϕ. For this purpose we compare with the Born approximation from non-relativistic quantum mechanics

$$\langle p'|iT|p\rangle = -i\tilde{V}(\mathbf{q})\delta^{(3)}(E_{\mathbf{p}'} - E_{\mathbf{p}})\,, \quad \mathbf{q} = \mathbf{p}' - \mathbf{p}\,, \tag{1.3}$$

where $\tilde{V}(\mathbf{q})$ is the Fourier transform of the potential $V(\mathbf{x})$.

(v) Restrict to the non-relativistic limit by expanding the momentum four-vectors p, p', k, k' to leading order in $\vec{p}$ ($\vec{p}^2 \ll m^2$). Evaluate also $(p-p')^2$, give the spinors $u^s(p)$ and evaluate the spinor product $\bar{u}^{s'}(p')u^s(p)$. Read off the matrix element $i\mathcal{M}$ from (ii) in this limit (Focus on the t-channel contribution only).

(vi) Compare with the Born approximation (1.3) to extract $\tilde{V}(\mathbf{q})$. Perform a Fourier transform to obtain $V(\mathbf{x})$. What is the range of this Yukawa potential? Assume a range of $1fm$ to calculate the mass m_ϕ. In the physical context, $1fm$ is the range of the strong nuclear interactions, i.e. the radius of the atomic core, and m_ϕ the pion mass.

(vii) Show that $4\pi/g^2 V(\mathbf{x})$ is the Greens function of $\Delta - m_\phi^2$.

(viii) Redo the same calculations for fermion-anti-fermion scattering. Is the potential still attractive?

2 Electrodynamics - spin one interaction bosons

Quantum electrodynamics can be viewed as a generalization of the Yukawa theory obtained by replacing the spin zero field ϕ by a spin one field A_μ, the gauge potential of electromagnetism. The interaction Hamiltonian is then given by

$$H_{int} = \int dx^3 e\bar{\psi}\gamma^\mu\psi A_\mu \,, \tag{2.4}$$

where e denotes again a dimensionless coupling constant, that is physically just the electric charge of the electron. $\mathcal{L}_{\mathrm{KG}}$ similarly has to be replaced by the Maxwell Lagrangian

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,. \tag{2.5}$$

(i) Guess the corresponding Feynman rules, i.e. the rules for the vertex, the photon propagator and external photon lines (use wavy lines for photons). Use the mode expansion for A_μ in Lorentz gauge using polarization vectors ϵ_μ to justify your guess. What is the physical condition on the initial and final state polarization?

(ii) From your experience with the Yukawa theory, what is the leading order contribution to the S-matrix of two-fermion scattering? Read off the matrix element.

(iii) Go to the non-relativistic limit and compare to the Born approximation (1.3) to first obtain $\tilde{V}(\mathbf{q})$ and then the Fourier transform, $V(\mathbf{x})$ (Use that for $p = p' = 0$, $\bar{u}(p')\gamma^i u(p)$ is neglegible compared to $\bar{u}(p')\gamma^0 u(p)$).

(iv) Does $V(\mathbf{x})$ have a Greens function interpretation and why is this expected?

(v) What changes for fermion-anti-fermion scattering? Compare to the Yukawa theory.

3 The linear sigma model - the pion

In contemporary physics quantum field theory is mostly used as an effective description of physics than as a microscopic theory. Such an effective quantum field theory is given by the linear sigma model that appropriately describes pions at low energies. Pions in turn yield an effective decription of strong nuclear interactions. The linear sigma model consists of N real scalars ϕ^i interacting via ϕ^4-interactions in an $O(N)$-invariant way,

$$H = \int dx^3 \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\phi^i)^2 + V(\phi^2)\right) \,, \qquad V(\phi^2) = \frac{1}{2}m^2(\phi^i)^2 + \frac{\lambda}{4}((\phi^i)^2)^2 \,. \tag{3.6}$$

Here Π^i denote the canonically conjugate momenta.

(i) Show that the Lagrangian is invariant under rotations $O(N)$.

(ii) Analyze the linear sigma model for $m^2 > 0$. Note that for $\lambda = 0$ the model reduces to N non-interacting identical Klein-Gordon fields ϕ^i. Define the theory by perturbation theory around $\lambda = 0$. Derive the propagator as

$$\langle T\phi^i(x)\phi^j(y)\rangle = \delta^{ij} D_F(x-y)\,, \tag{3.7}$$

where $D_F(x-y)$ denotes the propagator of a single Klein-Gordon field of mass m. What is the Feynman rule for the single vertex?

(iii) Compute the differential cross-section at leading order in λ for the processes

$$\phi^1\phi^2 \to \phi^1\phi^2\,, \quad \phi^1\phi^1 \to \phi^2\phi^2\,, \quad \phi^1\phi^1 \to \phi^1\phi^1 \tag{3.8}$$

as a function of the center-of-mass energy E_{CMS}.

(iv) Consider now the case $-\mu^2 := m^2 < 0$. Draw a three-dimensional graph for $V(\phi)$ and convince yourself that V has a local maximum at 0 and a local minimum at $(\phi^i)^2 = v$. Expand the fields around this new vacuum by writing

$$\phi^i(x) = \pi^i(x) \ (i = 1, \ldots, N-1) \quad \phi^N(x) = v + \sigma(x)\,, \tag{3.9}$$

where we used the rotational symmetry $O(N)$ to achieve this form. Show that the linear sigma model reduces in these field variables to a theory of a massive field σ and $N-1$ massless pion fields $\pi^i(x)$, that interact via a cubic and quartic potential, which becomes small for $\lambda \to 0$. What is the geometry of the vacuum manifold and how can you geometrically interpret the fields π^i in this context[1]. Introduce Feynman rules by using double lines for the field σ and normal lines for the fields π^i.

(v) Compute the scattering amplitude for $\pi^i(p_1)\pi^j(p_2) \to \pi^j(p_3)\pi^k(p_4)$ to leading order in λ. Show that there are four contributing diagrams and that they cancel at threshold $\mathbf{p}_i = 0$ (You can consider the process $\pi^1\pi^1 \to \pi^2\pi^2$ first before trying the general case). Show that in the case $N = 2$ also the terms of order $O(p^2)$ cancel.

(vi) Add to V a symmetry-breaking term

$$\Delta V = -a\phi^N\,, \tag{3.10}$$

for small and constant a. Which symmetries are broken and unbroken? Find the new value of v minimizing V and expand the theory around this new vacuum. What is the field content and their properties? Show that the pion mass $m_\pi^2 \cong a$ and that the above scattering amplitude at threshold $\mathbf{p}_i = 0$ does not vanish and is proportional to a.

[1] Thus, the quantum field theory of the $\pi^i(x)$ alone is the non-linear sigma model with target S^{N-1}, see Peskin, Schroeder p. 454ff for details.