Physikalisches Institut
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Theoretische Physik

Exercise 11

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Exercises Quantum Field Theory I

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1 Fierz Identities

Products of Dirac bilinears obey interchange relations that are also called Fierz rearrangements. Introduce the 16 independent antisymmetric combinations of γ-matrices

$$\Gamma^a = \{1, \gamma^\mu, \frac{1}{2}[\gamma^\mu, \gamma^\nu], \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}, \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}\}\,. \tag{1.1}$$

These form a basis of the vector space of 4×4-matrics with scalar product given by the trace of two matrices, that can be used to construct even an orthonormal basis. This implies Fierz-relations for every product of bilinears of the form

$$(\bar{u}_1\Gamma^a u_2)(\bar{u}_3\Gamma^b u_4) \tag{1.2}$$

for four 4-component Dirac spinors u_i, that we will prove in the following.

(i) Begin by normalizing the 16 matrices Γ^a to the convention

$$\mathrm{Tr}(\Gamma^a\Gamma^b) = 4\delta^{ab}\,. \tag{1.3}$$

Give all 16 normalized matrices explicitly.

(ii) Write the general Fierz identity as

$$(\bar{u}_1\Gamma^a u_2)(\bar{u}_3\Gamma^b u_4) = \sum_{c,d} C^{ab}_{cd}(\bar{u}_1\Gamma^c u_4)(\bar{u}_3\Gamma^d u_2) \tag{1.4}$$

with unkown coefficients. Use the completeness of the Γ^a to show that

$$C^{ab}_{cd} = \frac{1}{16}\mathrm{Tr}(\Gamma^c\Gamma^a\Gamma^d\Gamma^b)\,. \tag{1.5}$$

Hint: Derive the completeness relation in the form $\sum_a \frac{1}{4}\Gamma^a_{ij}\Gamma^a_{kl} = \delta_{il}\delta_{jk}$.

(iii) Work out the explicit Fierz transformations for the products $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.

The process of using Fierz relations is also called "fierzen".

2 Compton Scattering

In QED, compton scattering is understood as the process of scattering one incoming fermion with momentum p and an incoming photon with momentum k and polarization ϵ^μ to a final state with fermion and photon of momentum p' respectively k' with polarization ϵ^ν.

(i) Draw the two contributing Feynman diagrams at leading order. What is the corresponding matrix element $i\mathcal{M}$? Evaluate $(p+k)^2 - m^2$ and $(p-k')^2 - m^2$ as well as use some Dirac algebra to obtain

$$i\mathcal{M} = -ie^2 \epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\Big[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\Big]u(p)\,. \tag{2.6}$$

(ii) As a next step we square this amplitude and sum (or average) over the electron and photon polarization states in the incoming and outgoing states. For this purpose use the fact that $\sum_{\text{polarizations}} \epsilon^*_\mu \epsilon_\nu \to -g_{\mu\nu}$ in any contraction with physical amplitudes[1]. Show that the final result reads

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4}\Big[\frac{N_1}{(2p\cdot k)^2} + \frac{N_2}{(2p\cdot k)(2p\cdot k')} + \frac{N_3}{(2p\cdot k')(2p\cdot k)} + \frac{N_4}{(2p\cdot k')^2}\Big]\,, \tag{2.7}$$

such that $N_4 = N_1$ after replacing k with $-k'$ as well as $N_2 = N_3$ by reversing the order of γ-matrices in traces (prove this latter fact using $-\gamma^T = C\gamma C$ with $C = \gamma^0\gamma^2$.)

(iii) Compute N_1 and N_2 explicitly yielding

$$N_1 = N_4|_{k'\to -k} = 16\big(4m^4 - 2m^2(p\cdot p') + 4m^2(p\cdot k) - 2m^2(p'\cdot k) + 2(p\cdot k)(p'\cdot k)\big) \tag{2.8}$$

and a similar expression for $N_2 = N_3$. Introduce the Mandelstam variables $s = (p+k)^2$, $t = (p'-p)^2$, $u = (k'-p)^2$ to rewrite this as

$$\begin{aligned} N_1 &= 16\big(2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\big)\,, && (2.9)\\ N_4 &= 16\big(2m^4 + m^2(u-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\big) && (2.10)\\ N_2 &= N_3 = -8\big(4m^4 + m^2(s-m^2) + m^2(u-m^2)\big)\,. && (2.11)\end{aligned}$$

(iv) Use your results to finally obtain

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\Big[\frac{(p\cdot k')}{(p\cdot k)} + \frac{(p\cdot k)}{(p\cdot k')} + 2m^2\Big(\frac{1}{(p\cdot k)} - \frac{1}{(p\cdot k')}\Big) + m^4\Big(\frac{1}{(p\cdot k)} - \frac{1}{(p\cdot k')}\Big)^2\Big]\,. \tag{2.12}$$

(v) Go to the lab frame in which the electron is initially at rest, i.e.

$$k = (\omega, \omega e_3)\,, \quad p = (m, \mathbf{0})\,, \quad k' = (\omega', \omega' sin(\theta), 0, \omega'\cos(\theta))\,, \quad p' = (E', \mathbf{p}') \tag{2.13}$$

for ω, ω' the frequencies before and after scattering, θ the angle w.r.t. to the z-axis and e_3 the unit vector in the z-direction. Evaluate all kinematical quantities, in particular

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos(\theta))}\,, \tag{2.14}$$

as well as the two-body phase space integral

$$\int d\Pi_2 = \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(k'+p'-k-p)\,. \tag{2.15}$$

(Make a variable transformation to $d\Omega$ and $d\omega'$ to reduce all integrals to one remaining integral $\int d\cos(\theta)$.)

[1]A formal prove of this identity makes use of the Ward identity $k^\mu \mathcal{M}_\mu(k) = 0$.

(vi) Use $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 1$ (why?) in equation (1.2) on sheet 9 to evaluate the differential cross-section as

$$\frac{d\sigma}{d\cos(\theta)} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2(\theta)\right], \tag{2.16}$$

what is known as the spin-averaged Klein-Nishina formula.

(vii) Evaluate the differential cross-section in the limit $\omega \to 0$ and determine the total cross-section. The result is the familiar Thomson cross-section for scattering of classical electromagnetic waves by free electrons.