Physikalisches Institut
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Theoretische Physik

Exercise 2

SS2011

# Exercises Quantum Field Theory I

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In this exercise, we want to recall some basic notions from complex analysis and especially on integration in the complex plane. This becomes important in the evaluation of propagators, which are a central object in QFT.

## 1 Some basics on integration in the complex plane

This exercise is a short recap of some facts about complex integration, that we need in the next exercise.

1. Integrate the function $f(z) = z^n, n \in \mathbb{Z}$ along the two closed paths in the complex plane paramterized by
$$\gamma(t) = re^{\pm it},\ 0 \leq t \leq 2\pi \tag{1.1}$$
for $n = -1$ and $n \neq -1$. Do the results depend on the parameter $r$? Do they depend on the orientation of the path (i.e. the $\pm$ sign)?

2. Relate this result to the existence or non-existence of a primitive along the complete path(s).

3. The Laurent series is a generalization of the Taylor series that includes poles/ singularities,
$$f(z) = \sum_{\nu=-\infty}^{\infty} a_\nu (z - z_0)^\nu. \tag{1.2}$$
The coefficient of the single poles is called the residues of $f$ in $z_0$,
$$a_{-1} = \mathrm{res}_{z_0} f. \tag{1.3}$$
From the above investigations, motivate the residue theorem
Let $\gamma$ be a closed path in $U \subset \mathbb{C}, S$ a discrete set, $f$ holomorphic in $U - S$, and $Sp(\gamma) \cap S = \emptyset$. Then
$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{a \in S} n(\gamma, a) \mathrm{res}_a f, \tag{1.4}$$
where $n(\gamma, a)$ is the winding number of the path $\gamma$ with respect to $a$.

## 2 Properties of the Propagator $\Delta(x-y)$

Expand the field $\phi(x)$ in annihilation and creation operators, calculate $\Delta(x-y) = -i\langle 0|\phi(x)\phi(y)|0\rangle$ and show that it has the following properties:

1. $\Delta(x-y)$ is a Lorentz-invariant function,
2. $\Delta(x-y) = -\Delta(y-x)$,
3. $\Delta(x-y)$ obeys the following boundary conditions:
$$\delta(0, \vec{x}-\vec{y}) = 0,\ \frac{\partial}{\partial x^0}\,\Delta(x^0-y^0, \vec{x}-\vec{y})\big|_{x^0=y^0} = -\delta^{(3)}(\vec{x}-\vec{y}). \tag{2.5}$$
4. $\Delta(x-y)$ obeys the homogeneous Klein-Gordon equation
$$(\Box + m^2)\Delta(x-y) = 0. \tag{2.6}$$
5. $\Delta(x-y)$ vanishes for spacelike arguments:
$$\Delta(x-y) = 0, \text{ if } (x-y)^2 < 0. \tag{2.7}$$

## Homework

## 3 Poincaré algebra

Suppose we have a theory described by a Lagrangian $\mathcal{L}(\phi_\mu, \partial\phi_\mu)$. The total 4-momentum of this theory $P_\mu$ and the total angular momentum $M_{\mu\nu}$ are given by

$$P_\mu = \int d^3x\, T_{\mu 0}, \quad M_{\mu\nu} = \int d^3x\, \tilde{M}_{0\mu\nu}, \tag{3.8}$$

where $\tilde{M}_{\lambda\mu\nu}$ is the angular momentum tensor.

1. Using canonical commutation relations, check that the operators $P_\mu$, $M_{\mu\nu}$ obey the Poincare algebra
$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\lambda] &= i(g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu), \\ [M_{\mu\nu}, M_{\sigma\tau}] &= i(g_{\nu\sigma}M_{\mu\tau} + g_{\mu\tau}M_{\nu\sigma} - g_{\nu\tau}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\tau}) \end{aligned} \tag{3.9}$$
2. The Pauli-Lubanski four-vector is defined in terms of $P_\mu, M_{\mu\nu}$ as $W^\lambda = \frac{1}{2}\epsilon^{\lambda\sigma\mu\nu}M_{\mu\nu}P_\sigma$. Prove the following commutation relations
$$[W^\lambda, M^{\mu\nu}] = i\left(W^\mu g^{\nu\lambda} - W^\nu g^{\mu\lambda}\right),\ [W^\lambda, W^\sigma] = i\epsilon^{\lambda\sigma\mu\nu}W_\mu P_\nu. \tag{3.10}$$
3. Show that $P_\mu P^\mu$ and $W_\mu W^\mu$ are Casimir operators of the Poincare group.