Physikalisches Institut
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Theoretische Physik

Exercise 3

SS2011

Exercises Quantum Field Theory I

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1 Warm-Ups

We want to recall some notions when passing to the field formalism, that we encountered and used in previous exercises.

Transition to field formalism

Let $q_n(t), n = 1, 2, \dots$ be a complete set of canonical coordinates of a given dynamical quantum system. Let $f_n(\vec{x}), n = 1, 2, \dots$ be an orthonormal, complete basis of a Hilbert space and define the quantum field

$$\phi(x) = \phi(\vec{x}, t) = \sum_{n=1}^{\infty} q_n(t) f_n(\vec{x}). \tag{1.1}$$

1. From the Euler-Lagrange equation for the discrete variable $q_n(t)$, deduce the Euler-Lagrange equation for the field $\phi(x)$, regarded as a continuous canonical quantum coordinate.
2. What is the momentum conjugate to $\phi(x)$?
3. Starting from the canonical commutation relations for the discrete variables, deduce those for the field variables.

2 Lorentz algebra and trace relations

In this exercise, we prove some identities, that will turn out to be useful in our ongoing discussion of quantum field theory.
Using just the algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{a} = a_\mu\gamma^\mu$, and $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, prove the following results:

1. $(\gamma^5)^2 = 1$,
2. $\not{a}_1\not{a}_2 = 2a_1 \cdot a_2 - \not{a}_2\not{a}_1 = a_1 \cdot a_2 + 2S^{\mu\nu}a_{1\mu}a_{2\nu}$,
3. $\mathrm{Tr}\not{a}_1\not{a}_2 = 4a_1 \cdot a_2$,
4. $\mathrm{Tr}\gamma^5 = 0$,
5. $\mathrm{Tr}\not{a}_1\not{a}_2 \cdots \not{a}_r = 0$ if r is odd,
6. $\mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = 4\{(a_1 \cdot a_2)(a_3 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3) - (a_1 a_3)(a_2 \cdot a_4)\}$
7. $\mathrm{Tr}(\gamma^5\not{a}_1\not{a}_2)$

8. $\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$,
9. $\gamma_\mu \not{a}_1 \not{a}_2 \gamma^\mu = 4a_1 \cdot a_2$,
10. $\gamma_\mu \not{a}_1 \not{a}_2 \not{a}_3 \gamma^\mu = -2\not{a}_3 \not{a}_2 \not{a}_1$,
11. $\gamma_\mu \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4 \gamma^\mu = 2\not{a}_4 \not{a}_1 \not{a}_2 \not{a}_3 + 2\not{a}_3 \not{a}_2 \not{a}_1 \not{a}_4$
12. $\mathrm{Tr}(\gamma^5 \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4) = 4i\epsilon_{\lambda\mu\nu\rho} a_1^\lambda a_2^\mu a_3^\nu a_4^\rho$

Homework

The first exercise should provide a better understanding of the process of quantization, starting from a classical setup. The second exercise illustrates the relation $SO(3) \simeq SU(2)/\mathbb{Z}_2$.

3 The quantum string

The Hamiltonian of the classical discrete chain is given by

$$H(p,q) = \sum_{j=1}^{N} \left(\frac{p_j^2}{2m} + \frac{k}{2}(q_j - q_{j+1})^2 \right), \tag{3.2}$$

where $p_j = mq_j$. The system can be quantized by replacing p_j and q_j by Hermitian operators satisfying the commutation relations

$$[p_j, q_k] = -i\delta_{jk}. \tag{3.3}$$

1. Impose periodic boundary conditions $q_{N+1} = q_1$, $p_{N+1} = p_1$ and consider the expansion in the normal modes

$$\begin{aligned} q_j &= \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} Q_n e^{2\pi inj/N}, \quad p_j = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2} P_n e^{2\pi inj/N}, \\ P_n^\dagger &= P_{-n}, Q_n^\dagger = Q_{-n}. \end{aligned} \tag{3.4}$$

   What are the commutation relations beween P_n and Q_n?

2. Rewrite the Hamiltonian, using the variables P_n, Q_n.
3. Find the spectrum of the Hamiltonian.
4. Perform the continuum limit in the classical Hamiltonian, introducing the field variables $p(\vec{x}, t), q(\vec{x}, t)$ instead of the discrete variables p_j, q_j.
5. How are the variables $p(x,t), q(x,t)$ quanitzed?

4 $SU(2)$ and $SO(3)$

The rotational group $SO(3)$ is given by the set of 3×3 matrices R with real entries, such that $R^T R = \mathbf{1}$ and $\det R = 1$.

1. Show that these matrices form a group.

2. Write down explicit matrix representations for rotations about an angle θ about the x-, y- and z-axis and denote them by $R_j\theta)$, $j = x, y, z$. Calculate the generator $J_i, i = x, y, z$ given by
$$J_j = \frac{1}{i} \left. \frac{dR_j(\theta)}{d\theta} \right|_{\theta=0} . \tag{4.5}$$
Calculate the commutator between the generators $[J_i, J_j] = J_i J_j - J_j J_i$.
Up to a factor of $\hbar$ you should recover the commutation relations of angular momentum. Therefore angular momentum operators are the generators of rotations. Generators describe an infinitesimal rotation about the corresponding axis. Given the generator, which is an element of the Lie algebra, one obtains the group element by the following procedure
$$R_j(\theta) = \lim_{N\to\infty} (1 + iJ_j\,\theta/N)^N = e^{iJ_j\theta}. \tag{4.6}$$
Check that this is true for the three generators obtained in the previous calculation.

3. The group $SU(2)$ is given by the 2×2 matrices U with complex entries, such that $U^\dagger U = 1, \det U = 1$. Using the defintion on a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{C})$, what does it imply for the entries? What is the geometric interpretation?

4. Interpret this as a transformation in a 2-complex dimensional space with basic spinor $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ with $\xi \to U\xi$, $\xi^\dagger \to \xi^\dagger U^\dagger$. How does the transformation act on $\xi\xi^\dagger$? Show that ξ and $\xi' = \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix}$ transform in the same way under $SU(2)$.
The spinors ξ' and ξ^* are related by a matrix ζ, which you should determine!

5. Calculate the trace of $-H = \xi\xi'^\dagger$. What are its transformation properties under $SU(2)$? Let $\vec{r} = (x, y, z)^T$ and h a traceless 2×2 matrix with the same transformation properties as H under $SU(2)$. Show that h is given by
$$h = \vec{\sigma} \cdot \vec{r}, \tag{4.7}$$
with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and σ_i are the Pauli matrices given as
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$
Show that $\det h' = \det h$ where h' is the transformed h under SU(2).
Conclude that a $SU(2)$ transformation on ξ corresponds to a $SO(3)$ transformation on $\vec{r}$.
Express x, y, z in terms of ξ_1 and ξ_2.

6. Consider a rotation under $SU(2)$. How do the components of $\vec{r}$ transform under $SU(2)$? Setting $a = e^{i\alpha/2}, b = 0$ (why?), what is the result for the rotation of x', y', z'? What is therefore the relation between $SU(2)$ and $SO(3)$? Rewrite the group elements in terms of generators. For the two other rotations about the axis, set $a = \cos(\beta/2), b = \sin(\beta/2)$ and $a = \cos(\gamma/2), b = i\sin(\gamma/2)$.
From this we conclude, that the relation between $SU(2)$ and $SO(3)$ is given by
$$SU(2) : U = e^{i\sigma\cdot\theta/2} \leftrightarrow SO(3) : R = e^{iJ\cdot\theta}. \tag{4.9}$$
7. Calculate the commutator of the generators of $SU(2)$. We finish with the observation, that U and $-U$ in $SU(2)$ correspond to R in $SO(3)$ and therefore $SO(3) \simeq SU(2)/\mathbb{Z}_2$.