Physikalisches Institut
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Theoretische Physik

Exercise 4

SS2011

Exercises Quantum Field Theory I

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1 Warm-Ups

In this exercise we want to study additional properties of the Poincaré group and its algebra. As a warm-up, recall some of the facts from the lectures and the exercises.

1. What is the definition of the Poincaré group?

2. What is the commutator of the generators $[M^{\mu\nu}, M^{\rho\sigma}]$ with $(M^{\mu\nu})^{\rho}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma})$? What is the relation between the group element Λ and the generators of the Lie algebra $M^{\mu\nu}$?

3. Let $J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}$ and $K^i = M^{0i}$. Calculate the commutation relations between these generators, i.e. show that

$$[J^i, J^j] = i\epsilon^{ijk}J^k, \quad [J^i, K^j] = i\epsilon^{ijk}K^k, \quad [K^i, K^j] = -i\epsilon^{ijk}J^k. \tag{1.1}$$

4. Define

$$T^i_{L/R} = \frac{1}{2}\left(J^i \pm iK^i\right) \tag{1.2}$$

and show that

$$[T^i_L, T^j_L] = i\epsilon^{ijk}T^k_L, \quad [T^i_R, T^j_R] = i\epsilon^{ijk}T^k_R, \quad [T^i_L, T^j_R] = 0. \tag{1.3}$$

From this we can conclude, that we can classify the Lorentz algebra using two non-negative integers (j_L, j_R).

Homework

2 Weyl spinors

Define $\vec{\alpha}$ and $\vec{\beta}$ via $\omega_{ij} = \epsilon_{ijk}\alpha_k$, $\beta_i = \omega_{0i}$. We denote furthermore by $D(\Lambda)$ the representation of the Lorentz-group element Λ.

1. Show that

$$\begin{aligned} D(\Lambda) &= \exp(-i[\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}]) \\ &= \exp(-i[\vec{\alpha} - i\vec{\beta}]\cdot\vec{T}_L)\exp(-i[\alpha + i\vec{\beta}]\cdot\vec{T}_R), \end{aligned} \tag{2.4}$$

where we only know the commutation relations of $\vec{T}_L$ and $\vec{T}_R$. For an explicit representation, these have to be chosen.

2. Specialize to a particular representation, where $\vec{T}_R$ and $\vec{T}_L$ are the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

The simplest representations of the Lorentz group are $(1/2, 0)$ and $(0, 1/2)$. An object, that transforms in the representation $(1/2, 0)$ is called a left-chiral Weyl spinor. For a right-handed Weyl spinor the definition is analogous. How many entries does a Weyl Spinor have? Write down the transformation laws for the left- and right-handed Weyl spinors.

3. Next we want to rewrite the transformation laws for the Weyl spinors. We start with

$$D(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right), \tag{2.6}$$

and we introduce the Pauli matrices σ^μ and $\bar{\sigma}^\mu$ by

$$\sigma^\mu = (\mathbf{1}, \sigma^i), \quad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i), \tag{2.7}$$

and furthermore by

$$\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \; \bar{\sigma}^{\mu\nu} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu). \tag{2.8}$$

Denote the left/right-handed chiral Weyl spinor by $\psi_{L/R}$ respectively. Denote the corresponding transformation matrices by D_L and D_R. Show, that the Weyl spinors transform as

$$\psi_L \mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi_L, \; \psi_R \mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\psi_R \tag{2.9}$$

4. Prove the following identities: $D_L^{-1} = D_R^\dagger$, $\sigma_2 D_L \sigma_2 = D_R^*$, $\sigma_2 = (D_L)^T\sigma_2 D_L$. From the last identity, what do you conclude concerning the role of σ_2 in the spinor space?

5. Show that $\sigma_2\psi_L^*$ transforms in the $(0, 1/2)$ representation and $\sigma_2\psi_R^*$ transforms in the $(1/2, 0)$ representation.

6. Let $\psi_L, \psi_R, \phi_L, \phi_R$ by Weyl spinors. Show that the following expressions are invariant under Lorentz transformations

$$\begin{aligned} i(\phi_L)^T\sigma_2\psi_L, \\ i(\phi_R)^T\sigma_2\psi_R, \\ (\phi_R)^\dagger\psi_L, \\ (\phi_L)^\dagger\psi_R. \end{aligned} \tag{2.10}$$

7. Choose $\phi_L = \psi_L$ and compute $i(\psi_L)^T\sigma_2\psi_L$.

8. Show that the parity operator acts as follows on the generators of the Lorentz algebra

$$J^i \mapsto J^i, K^i \mapsto -K^i \tag{2.11}$$

9. Show that under parity transformations a representation (m,n) of the Lorentz algebra goes to (n,m). Therefore, if $m \neq n$, the parity transformation maps an element of the vector space of the representation to an element, that is not part of the vector space.

10. Show that the dimension of the representation (m,n) is $(2m+1)(2n+1)$.

11. Show that the 4 dim. Minkowski space is the vector space of the $(1/2,1/2)$ representation.

3 Dirac spinors

Since the vector spaces of the left- and right-chiral Weyl spinors are not mapped to themselves under parity, we consider the following (reducible) representation of the Lorentz algebra $(1/2,0) \oplus (0,1/2)$. In other words: we take a left-chiral Weyl spinor ψ_L and a right-chiral Weyl spinor ϕ_R and take them as the components of a new 4-component spinor, called the Dirac spinor

$$\Psi = \begin{pmatrix} \psi_L \\ \phi_R \end{pmatrix}. \tag{3.12}$$

Note that this is only possible for the chiral representation of the Clifford algebra.

1. Show that the Dirac spinor transforms under a Lorentz transformation as

$$\psi \mapsto \Psi' = D\Psi = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right)\Psi, \tag{3.13}$$

with $\gamma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$ and γ^μ in the Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{3.14}$$

Here D denotes a representation of the proper Lorentz group, which contains the identity and can therefore be expressed by the exponential function.

2. Prove the following equations

$$[\gamma^\mu,\gamma^{\nu\sigma}] = (M^{\nu\sigma})^\mu_\rho\gamma^\rho,\ D^{-1}\gamma^\mu D = \Lambda^\mu{}_\nu\gamma^\nu. \tag{3.15}$$

3. Show that in the chiral representation $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ can be written as

$$\gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \tag{3.16}$$

and prove that $[\gamma^5, D] = 0$.

4. Show that the following operators are a complete set of projection operators

$$P_L = \frac{1}{2}(\mathbf{1}-\gamma^5),\ P_R = \frac{1}{2}(\mathbf{1}+\gamma^5). \tag{3.17}$$

What is their action on a Dirac spinor in the chiral representation?

5. Show that
$$D^\dagger = \gamma^0 D^{-1} \gamma^0, \tag{3.18}$$
and from this it follows
$$\bar{\Psi} \mapsto \bar{\Psi} D^{-1}, \tag{3.19}$$
where $\bar{\Psi} = \Psi^\dagger \gamma^0$.

6. Consider the parity operator D_P, i.e. ($\Lambda_P = \mathrm{diag}(1, -1, -1, -1)$. Show that one representation of the parity operator is $D_P = \gamma^0$. Examine its action on a Dirac spinor in a chiral representation.

7. Check the covariance and the behavior under parity of the five bilinear covariants
$$\begin{array}{rl} \text{scalar} & \bar{\Psi}\Psi, \\ \text{vector} & \bar{\Psi}\gamma^\mu\Psi, \\ \text{tensor} & \bar{\Psi}\gamma^{\mu\nu}\Psi, \\ \text{pseudo-scalar} & \bar{\Psi}\gamma^5\Psi, \\ \text{pseudo-vector} & \bar{\Psi}\gamma^5\gamma^\mu\Psi. \end{array} \tag{3.20}$$