Physikalisches Institut
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Theoretische Physik

Exercise 5

SS2011

Exercises Quantum Field Theory I

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1 Warm-up: Lie algebra and Lie groups

In the warm-up exercise we want to fix the relation between a Lie group and a Lie algebra. Please discuss in detail the definitions in class. A Lie group G is on the one hand a group and on the other hand it is also a differentiable manifold such that the group action is also a differentiable manifold. We encountered as an example $SU(2)$. To which geometrical object does it correspond?
A Lie algebra $\mathfrak{g}$ is a vector space V with a bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$, such that the following properties hold

- $[\cdot,\cdot]$ is antisymmetric, i.e. $[x,y]=-[y,x]$,
- $[\cdot,\cdot]$ satisfies the Jacobi identity, i.e.

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0, \tag{1.1}$$

It can be shown, that the Lie algebra is given as the tangent space of the Lie group G at the identity, such that $\mathfrak{g}\simeq T_{\mathbf{1}}G$ is the tangent space at the identity. This is also the reason, why we always consider the component connected to the group identity. Show that the commutator satisfies the properties of a Lie-algebra.
The connection between the Lie group and the Lie slgebra is given by the exponential map, as we have encountered in various examples

$$\exp:\mathfrak{g}\longrightarrow G. \tag{1.2}$$

Therefore we can obtain the generators $\tilde{g}$ of the Lie algebra by taking the derivative of the group element

$$\tilde{g}=\frac{d}{dt}\bigg|_{t=0}\exp(t\,\tilde{g}). \tag{1.3}$$

Consider again the orthogonal group $SO(N)$ and determine the form of the generators!

2 Little group

In this exercise we discuss the little group under which particle states transform. The *little Group* is given by the following construction. We consider the action of a group (e.g. Lorentz group) on a set X, i.e.

$$\begin{aligned} G\times X &\longrightarrow X\\ (g,x)&\mapsto gx. \end{aligned} \tag{2.4}$$

The little group G_x of an element $x \in X$ is the set of transformations which leaves x invariant, i.e.

$$G_x = \{g \in G | \, gx = x\} \tag{2.5}$$

1. Show that G_x is indeed a subgroup of G.
2. In quantum field theory we are interested in the little group of the momentum p of a particle as a subgroup of the proper orthochronous Lorentz group, i.e.

   $$\Lambda^\mu{}_\nu p^\nu = p^\mu, \quad \Lambda \in SO(1,3) \tag{2.6}$$

   What is the corresponding condition for the Lie algebra $\mathfrak{so}(1,3)$?
3. Recall once more, that a basis for the Lie-algebra is given by the following set of matrices

   $$(M^{\mu\nu})^\rho_\sigma = i(\eta^{\mu\nu}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma). \tag{2.7}$$

   Now we look at a massive particle. Its momentum can be rotated to the form

   $$p = (m, 0, 0, 0)^T. \tag{2.8}$$

   Which generators leave p invariant? What is the little group of a massive state.
4. The momentum of a massless particle can be chosen $p = (p, 0, 0, p)^T$. Find three generators which leave p invariant. Describe the corresponding group action. The group they generate is isomorphic to the so-called Euclidean group $E(2)$.
5. Show that two of these three generators correspond to non-compact directions by explicitly computing the group elements. Find the maximal compact subgroup of $E(2)$. Since non-trivial irreducible representations of non-compact groups are infinite dimensional, we restrict the little group of massless particles to the maximal compact subgroup by projecting the states onto their representations.

We saw in an earlier exercise, that $p_\mu p^\mu$ is a Casimir operator, that is left invariant under Lorentz transformations. Therefore, we have six distinct classes of the representation

- $p^2 = m^2 > 0$, $p^0 > 0$,
- $p^2 = m^2 > 0$, $p^0 < 0$,
- $p^2 = 0$, $p^0 > 0$,
- $p^2 = 0$, $p^0 < 0$,
- $p = 0$,
- $p^2 < 0$.

Together with our knowledge from the above exercise, one can show that we only need to know the representations of the rotation group in order to study the representations of the Lorentz group for a time-like particle.
Note, that spin corresponds to the Pauli-Lubanski pseudovector W_μ and its Casimir $W_\mu W^\mu$. As the Poincaré group has rank 2 these are the only two Casimir operators and hence a massive state can be labled by its mass and its spin. However, on a massless particle $W_\mu W^\mu$ and $P_\mu P^\mu$ both act trivial and as they are orthogonal we conclude that they must be proportional to each other. Therefore a massless particle is characterized by one number, called the *helicity*.

3 Boosts

In this exercise we want to study the free particle solutions of the Dirac equation. The other solutions are then obtained by boosting the free solution.

1. Argue physically, why the Dirac field $\psi(x)$ is subject to the following solution
$$\psi(x) = u(p)e^{-ipx}, \text{ with } p^2 = m^2. \tag{3.9}$$
Which additional condition has to be satisfied by $u(p)$.

2. By going to the restframe, i.e. $p = p_0 = (m, 0, 0, 0)^T$, show that $u(p)$ has to satisfy the following condition
$$(m\gamma^0 - m)u(p) = 0. \tag{3.10}$$
We choose the γ-matrices to be given by the chiral representation, i.e.
$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{3.11}$$
The normalization condition is given by $u^\dagger(p_0)u(p_0) = 2m$. Show that for a boost in the z-direction one gets
$$\begin{aligned} U(\omega) &= \cosh(\omega/2) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} - \sinh(\omega/2) \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \\ &= \frac{1}{\sqrt{m}} \begin{pmatrix} \sqrt{E - p^3} & 0 & 0 & 0 \\ 0 & \sqrt{E + p^3} & 0 & 0 \\ 0 & 0 & \sqrt{E + p^3} & 0 \\ 0 & 0 & 0 & \sqrt{E - p^3} \end{pmatrix}. \end{aligned} \tag{3.12}$$

3. Show that the boosted spinor can be written as
$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} \tag{3.13}$$

4 Discrete symmetries of the Dirac theory

In this exercise we want to discuss the discrete symmetries of the Dirac theory, namely charge conjugation C, time reversal T and parity P. It turns out, that a relativistic theory doesn't have to be invariant under each of these symmetries. However, the symmetry CPT turns out to be a perfect symmetry of nature. For this exercise you may seek help and details from Peskin-Schröder.

1. Argue, that under parity the Dirac field transforms as
$$P\psi(\vec{x}, t)P = \eta_a \gamma^0 \psi(-\vec{x}, t), \tag{4.14}$$
where η_a is just a phase.

2. Show that under time reversal T the fermion field $\psi(t,\vec{x})$ transforms as

$$T\psi(\vec{x},t)T = \gamma^1\gamma^3\psi(-t,\vec{x}). \qquad (4.15)$$

For this implement that T flips not only the time but also the spin.

3. Charge conjugation C maps a fermion with given spin to an anti-fermion with the same spin. Show that

$$C\psi(x)C = -i\gamma^2\psi^* = -i(\bar{\psi}\gamma^0\gamma^2)^T \qquad (4.16)$$

4. Show that the following table is correct

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1