Physikalisches Institut
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Theoretische Physik

Exercise 6

SS2011

Exercises Quantum Field Theory I

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1 Quantization of the Dirac Field

Use the Fourier decomposition for the free Dirac field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(p)} \sum_s \left(b(p,s)u(p,s)e^{-ipx} + d(p,s)v(p,s)e^{ipx} \right) \tag{1.1}$$

and quantize the field, assuming

$$[b(p,s), b^\dagger(p',s')] = [d(p,s), d^\dagger(p',s')] = \delta_{s\,s'}(2\pi)^3 2\omega(p)\delta^{(3)}(p-p'). \tag{1.2}$$

Like the Klein-Gordon field, all other commutators of the creation and annihilation operators are zero.

1. Show that we arrive at the usual equal-time commutator
$$[\psi(x), \pi(x')]\big|_{t=t'} = i\delta^{(3)}(x-x'). \tag{1.3}$$
Use the relations
$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m, \qquad \sum_s v(p,s)\bar{v}(p,s) = \not{p} - m. \tag{1.4}$$

2. Check that
$$D_R(x-x') = \Theta(t-t')\langle 0|[\phi(x), \phi(x')]|0\rangle. \tag{1.5}$$
is a Greens function of the free Klein-Gordon equation ($\phi(x)$ being a scalar field), i.e. show that
$$(\partial^2 + m)D_R(x-x') = (-i)\delta^{(4)}(x-x') \tag{1.6}$$

3. Use the relation
$$[\phi(x), \phi(x')] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2w(p)} \left(e^{-ip(x-x')} - e^{ip(x-x')} \right) \tag{1.7}$$
to derive
$$[\psi(x), \bar{\psi}(x')] = (i\not{\partial} + m)[\phi(x), \phi(x)]. \tag{1.8}$$

4. We define
$$S_R(x-x') = \Theta(t-t')\langle 0|[\psi(x), \bar{\psi}(x')]|0\rangle. \tag{1.9}$$
Show first that
$$S_R(x-x') = (i\not{\partial}_x + m)D_R(x-x') \tag{1.10}$$
and then that
$$(i\not{\partial} - m)S_R(x-x') = i\delta^{(4)}(x-x'). \tag{1.11}$$
This shows that S_R is a Greens function of the Dirac equation. It can be shown that S_R does not violate causality, which you should show.

5. Express the Hamiltonian

$$H = \int d^3x \bar{\psi}(x)(-i\partial\!\!\!/ + m)\psi(x) \tag{1.12}$$

through the creation and annihilation operators

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(p)} \left[\omega(p) b^\dagger(p,s) b(p,s) - \omega(p) d^\dagger(p,s) d(p,s)\right]. \tag{1.13}$$

Since the Hamiltonian is not bounded form below, the quantization via commutators is not acceptable.

6. To take care of this problem, one has to use anticommutators instead of commutators. Repeat the above calculations for this setup and show in particular

$$S_R(x - x') = \Theta(t - t)\langle 0|\{\psi(x), \bar{\psi}(x)\}|0\rangle. \tag{1.14}$$

2 Majorana Spinors

We write a four component Dirac spinor in the chiral representation as a compostion of two Weyl spinors

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{2.15}$$

A Majorana spinor is a Dirac spinor Ψ with the following constraint

$$\Psi^c := C\bar{\Psi}^T = \psi, \tag{2.16}$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation operator.

1. Show that $(\Psi^c)^c = \Psi$.

2. What does the Majorana condition imply for ψ_L and ψ_R and what is the physical meaning of this condition.

3. The Lagrangian $\mathcal{L}$ for a Dirac spinor has the form

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/)\Psi - m\bar{\Psi}\Psi, \tag{2.17}$$

where the second term is called the Dirac mass term. Rewrite $\mathcal{L}$ in ψ_L and ψ_R.

4. Remember the projectors $P_{L/R} = \frac{1}{2}(\mathbf{1} \mp \gamma^5)$. As you know $P_{L/R}$ project Ψ onto the left/right handed part, respectively. We denote $\Psi_{L/R} := P_{L/R}\Psi$. Show that

$$\begin{gathered} (\Psi_{L/R})^c = (\Psi^c)_{R/L} \\ \overline{(\Psi_{L/R})^c}(\Psi_{R/L})^c = \overline{\Psi_{R/L}}\Psi_{L/R}. \end{gathered} \tag{2.18}$$