Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 7

SS2011

Exercises Quantum Field Theory I

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Some Information on the mid-term exam

The midterm exam will take place on June 6th, 2011 in Hörsaal I of the PI from 16 to 18 o'clock. With the next exercise sheet we also distribute a checklist of notions, that we assume to be known.

1 Warm-Up

1. Recall the definition of the Dirac and the Klein-Gordon propagator and comment on their differences.
2. Which deeper lesson can you learn from this?
3. What is the Majorana condition?

2 A first glance at the path integral and Feynman graphs

In this exercise we want to give a first introduction to the Feynman path integral and how this leads to Feynman graphs. We will not cover the details but refer to the literature for these.
We introduce the partition function Z of a field $X : M \to \mathbb{R}$, where we consider a toy model such that M is a point. This is also referred to as 0 dimensional quantum mechanics. The corresponding action $S[X]$ is given by the following expression

$$Z := \int dX\, e^{-S[X]}. \tag{2.1}$$

We are interested in correlation functions of some funtion $f(x)$, that are given by

$$\langle f(X) \rangle := \int dX\, f(X)\, e^{-S[X]}. \tag{2.2}$$

In addition, we also define the normalized correlation functions by

$$\frac{\int dX\, f(X)\, e^{-S[X]}}{\int dX\, e^{-S[X]}}. \tag{2.3}$$

1. Consider a variation of the action of the form

$$S \mapsto S' = S + \sum_i a_i f_i(X). \tag{2.4}$$

How can you obtain the correlation function $\langle f_i(X) \rangle$?

2. As an example consider the following action

$$S[X] = \frac{\alpha}{2}X^2 + i\epsilon X^3. \tag{2.5}$$

The partition function now depends on two parameters $Z = Z(\alpha, \epsilon)$. Evaluate the partition function $Z(\alpha, 0)$.

3. We term with X^3 can be considered as a perturbation. For the case that $\epsilon \ll 1$, we can expand the partition function in powers of ϵ. Perform this expansion.

We come back to to our toy model but first we want to introduce the notion of Feynman diagrams/graphs.

4. Consider the function

$$f(\alpha, J) = \int dX\, e^{-\frac{\alpha X^2}{2} + JX}, \tag{2.6}$$

and perform the integration by completing the square. Calculate

$$\left.\frac{\partial^r f(\alpha, J)}{\partial J^r}\right|_{J=0}. \tag{2.7}$$

5. Observe the following: Each derivative w.r.t. J leads to a factor J/α, which can be absorbed by another derivative w.r.t. J. What would happen if it were not absorbed when setting $J = 0$? Therefore in computing the integral of X^r, one has to choose all pairs of X and contract them, which is called Wick contraction. Therefore one obtains for $\frac{d^r f(\alpha,J)}{dJ^r}$

$$\frac{d^r f(\alpha, J)}{dJ^r} = \frac{1}{\alpha^{r/2}} \times (\#\text{ways of contracting}). \tag{2.8}$$

Possible contractions are represented by lines which are called propagators, which is weighted by a factor $\frac{1}{\alpha}$.

6. Return to the model described above and consider the first non-trivial correction to $Z(\alpha, 0)$.

$$\frac{(-i\epsilon)}{2!} \int dX\, X^3 \times X^3 \times e^{-\frac{\alpha}{2}X^2}. \tag{2.9}$$

Now do the following: For each X^k draw a vertex with k edges emanating from it (in this case two vertices with three edges each). Now write down all possible contractions of two edges and show that there are just two toplogical distinct graphs which come with the corresponding factors $\frac{1}{2}(-i\epsilon)^2 \left(\frac{1}{\alpha}\right)^3 \times 3!$ and $\frac{1}{2}(-i\epsilon)^2 \left(\frac{1}{\alpha}\right)^3 \times 3^2$. So the number of possible contractions is $3! + 3^2 = 15$. Why is this the number we expected?
In general the combinatorial factor associated to each connected graph is given by

$$(-3! i\epsilon)^V \alpha^{-E} / |\mathrm{Aut}(G)|, \tag{2.10}$$

where V is the number of vertices of the graph, E is the number of the edges and $|\mathrm{Aut}(G)|$ denotes the order of the automorphism group of the graph.

7. For the multivariable case we have X_i with $i = 1, \dots, N$ and the action is given by

$$S(X_i, M, C) = \frac{1}{2} X^i M_{ij} X^j + C_{ijk} X^i X^j X^k, \tag{2.11}$$

where M is a positive definite and invertible matrix. Evaluate the partition function to show that

$$Z(M, C = 0) = \frac{(2\pi)^{N/2}}{\sqrt{\det M}}. \tag{2.12}$$

The term $C_{ijk} X^i X^j X^k$ leads to a vertex with three lines metting at a point and a factor of $-C_{ijk}$. Again one can expand the partition function for small C in powers of C. The propagator connecting X^i and X^j carries a factor $(M^{-1})_{ij}$.

Majorana fermions II

In this exercise we use again $\sigma^\mu = (1, \sigma)$ and $\bar{\sigma}^\mu = (1, -\sigma)$. In the chiral representation, the Dirac matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

The left- and right-handed components of the Dirac spinor $\Psi = (\psi_L, \psi_R)$ are given by

$$\Psi_{R,L} = \frac{1}{2}(1 \pm \gamma^5)\Psi, \quad \Psi_R = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}, \quad \Psi_L = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}. \tag{2.14}$$

Under infinitesimal Lorentz transformations,

$$\psi_{L/R} \to (1 - \frac{i}{2}\theta \cdot \sigma \mp \frac{1}{2}\beta \cdot \sigma)\psi_{L/R}, \tag{2.15}$$

where θ and β are the parameters of an infinitesimal rotation and a boost.

1. Consider the two-component spinor field $\psi_{L/R}$. Using the identity $\sigma^2 \sigma^* = -\sigma \sigma^2$, show that $\sigma^2 \psi^*_{L/R}$ transforms like $\psi_{R/L}$ under Lorentz transformation.

2. Show that the equation

$$i\bar{\sigma} \cdot \partial \psi - im\sigma^2 \psi^* = 0 \tag{2.16}$$

   is relativistically invariant.

3. Show that the components of ψ obey the Klein-Gordon equation. (Here: $\psi \equiv \psi_{L/R}$.)

4. Determine the equation of motion of the following action

$$S = \int d^4x \left(\psi^\dagger i\bar{\sigma} \cdot \partial \psi + \frac{im}{2}(\psi^T \sigma^2 \psi - \psi^\dagger \sigma^2 \psi^*) \right). \tag{2.17}$$

   Classically, the two component field ψ is an anti-commuting Grassmann variable.

5. Quantize the theory, promoting $\psi(x)$ to a quantum field satisfying the canonical anti-commutation relations

$$\{\psi_a(x), \psi_b^\dagger(y)\} = \delta_{ab}\delta^{(3)}(x-y). \tag{2.18}$$

Construct a Hermitian Hamiltonian, and find a representation of the canoncial commutation relations that diagonalizes the Hamiltonian in terms of a set of creation and annihilation operators.

6. Writing the Dirac field as $\Psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$, recall that the lower component ψ_R transforms in a way equivalent by a unitary transformation to a complex conjugate of the representation ψ_L. In this way, we can rewrite the 4-component Dirac field in terms of two 2-component spinors $\psi_L = \psi_1$ and $\psi_R = i\sigma^2\psi_2^*$, which you should know from the previous exercises.

7. Show that the above action has a global symmetry. Compute the divergences of the currents

$$J^\mu = \psi^\dagger\bar{\sigma}^\mu\psi, \quad J^\mu = \psi_1^\dagger\bar{\sigma}^\mu\psi_1 - \psi_2^\dagger\bar{\sigma}^\mu\psi_2, \tag{2.19}$$

for the above theory and relate your results to the symmetries of these theories. Construct a theory of N free massive 2-component fermion fields with $O(n)$ symmetry.