Physikalisches Institut
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Theoretische Physik

Exercise 8

SS2011

Exercises Quantum Field Theory I

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1 Perturbation theory from the interaction picture

In this exercise, we want to review some of the aspects of the Feynman graph expansion by using the time evolution operator for ϕ^4 theory. The corresponding Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1.1}$$

where the term with ϕ^4 describes an interaction and hence a perturbation of the free theory.

1. How does the field $\phi(\vec{x}, t)$ look like for a fixed time $t = t_0$?

2. In the Heisenberg picture the time evolution is given by

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}. \tag{1.2}$$

   What is the definition of the field ϕ_I in the interaction picture? What is its mode expansion?

3. Show that $\phi(t, \vec{x})$ can be written as

$$\phi(t, \vec{x}) = U^\dagger(t, t_0)\phi_I(t_0, \vec{x})U(t, t_0), \tag{1.3}$$

   with the time evolution operator in the interaction picture

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{1.4}$$

4. Show that $U(t, t_0)$ is the unique solution to the following differential equation with a suitable initial condition (which one?)

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I(t)U(t, t_0), \quad H_I(t) = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} = \int d^3x\, \frac{\lambda^4}{4!}\phi_I^4, \tag{1.5}$$

   with $H_{\text{int}} = \int d^3x\, \frac{\lambda^4}{4!}\phi_I^4$.

5. Show that $U(t, t_0)$ can be written as

$$U(t, t_0) = T\left\{\exp -i\int_{t_0}^{t} dt'\, H_I(t')\right\}. \tag{1.6}$$

   Why is time-ordering essential?
   Conclude the corresponding generalization for a time t'' instead of our reference time t_0.
   Show that for $t_1 \geq t_2 \geq t_3$ we have

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3), \quad U(t_1, t_3)\left[U(t_2, t_3)\right]^\dagger = U(t_1, t_2). \tag{1.7}$$

   Is $U(t_1, t_2)$ unitary?

6. Finally show that we have for a vaccum $|\Omega\rangle$ of the interacting theory, i.e. of the full Hamiltonian $H = H_0 + H_{\text{int}}$, the following expression for the correlator

$$\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|T(\phi_I(x)\phi_I(y)\exp(-i\int_{-T}^{T} dt\, H_I(t)))|0\rangle}{\langle 0|T(\exp(-i\int_{-T}^{T} dt\, H_I(t)))|0\rangle} \tag{1.8}$$

   Expanding the exponential yields to perturbation theory. In which case is this a meaningful expansion?

7. The evaluation of the time ordered product of fields can be performed, using Wick's theorem and by expanding the exponential in (1.8). Show that

$$\begin{aligned}\langle 0|T(\phi(x)\phi(y)(-i\int dt \int d^3z\, \frac{\lambda\phi^4}{4!})|0\rangle &= 3\frac{-i\lambda}{4!} D_F(x-y)\int d^4z\, D_F(z-z)D_F(z-z) \\ &+ 12\frac{-i\lambda}{4!}\int d^4z D_F(x-z)D_F(y-z)D_F(z-z).\end{aligned} \tag{1.9}$$

   Draw the corresponding Feynman diagrams and derive the corresponding Feynman rules!

2 Checklist

For the midterm exam, we assume that the following topics are know. Please note, that this not a complete list.

- Basics in Field theory, e.g. Lagrangian and Hamiltonian description,
- Noether's theorem and conserved quantities
- The Klein-Gordon field and the Dirac field, commutation relations
- Klein-Gordon and Dirac propagators
- Canonical quantization, annihilation and creation operators,
- Lorentz group and its representations
- Gamma matrices and its properties, Weyl-, Dirac- and Majorana Spinors
- Dirac equation and solutions
- Dirac field bilinears
- Discrete symmetries of the Dirac theory, CPT theorem
- Idea of Feynman graphs and perturbation theory