Physikalisches Institut
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Theoretische Physik

Exercise 9

SS2011

Exercises Quantum Field Theory I

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1 S-matrix and cross-sections from Feynman diagrams - general considerations

In the last lectures the concept of the S-matrix S was introduced. This allows to relate the concepts of QFT, in particular the Feynman graph expansion of perturbation theory, to physically measurable quantities like the cross-section σ of a particular scattering event or to the decay rate Γ of a particle. Splitting the S-matrix into a trivial, i.e. non-interacting, part 1 and an interacting part T, the T-matrix, as $S = 1 + T$ one can further separate kinematics and dynamics as

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_b - \sum_i p_i) \cdot i\mathcal{M}(k_A, k_b \to \{p_i\}_i)\,. \tag{1.1}$$

Here the delta-function just expresses the kinematical conservation of energy-momentum whereas the invariant matrix element $\mathcal{M}$ encodes the non-trivial dynamics described by the underlying QFT. The latter is directly associated to a particular set of Feynman diagrams. Then, as was shown in the lectures the infinitesimal cross-section $d\sigma$ of two-particle to n-particle scattering takes the form of Fermi's golden rule,

$$d\sigma = \frac{1}{2E_A E_B |v_A - v_b|} \left(\int d\Pi_n \right) \cdot |\mathcal{M}(p_A, p_b \to \{p_i\}_i)|^2, \tag{1.2}$$

where the relativistic n-body phase space[1] is denoted by

$$\int d\Pi_n = \left(\prod_i \int \frac{dp_i^3}{2\pi}^3 \frac{1}{2E_i} \right) (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_i p_i)\,. \tag{1.3}$$

In order to explicitly calculate S in an interacting QFT, a relation between the states of the free QFT, denoted $|\mathbf{p}_1 \mathbf{p}_2 \cdots \rangle_0$, and perturbation theory, encoded in the interaction picture Hamiltonian $H_I(t)$, was established as

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{k}_A \mathbf{k}_B \rangle = \left(\lim_{T' \to \infty(1 - i\epsilon)} \left({}_0\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | T \left[\exp\left(-i \int_{T'}^{T'} dt H_I(t) \right) \right] | \mathbf{k}_A \mathbf{k}_B \rangle_0 \right) \right)_{\text{conn,ampt}}\,. \tag{1.4}$$

Here, on the right hand side only connected and amputated graphs contribute. In the following we want to understand this restriction on the contributing Feynman graphs for the example of two-particle scattering in the ϕ^4-theory.

[1] Cf. Peskin, Schroeder, page 105ff for details.

2 S-matrix and cross-sections from Feynman diagrams - ϕ^4-theory

Consider a two-particle to two-particle scattering process in the following.

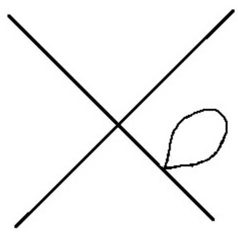
1. What is the Lagrangian of the scalar ϕ^4-theory? Identify H_{int} and H_I.

2. Give the S-matrix in this case and expand the exponential in (1.4). What is the contribution at the zeroth order from physical reasoning about scattering processes (Recall the split $S = 1 + T$)? Draw the corresponding Feynman diagrams to explain your result.

3. What is the contribution at first order: First, how is $|\mathbf{p}\rangle_0$ defined? Use this to evaluate $\phi_I(x)|\mathbf{p}\rangle_0$ and introduce a Feynman rule (external line) for the contraction of $\phi_I(x)$ with an external state $|\mathbf{p}\rangle_0$. Consequently, the contribution to the S-matrix is in general given by all contractions of field operators with each other *and* external states. Then, use Wick's theorem to write down all contractions of ϕ_I^4 and the external momentum states. Draw the Feynman diagrams to argue again from physical reasoning which terms do not contribute. Use your results to explain the restriction of (1.4) to only connected graphs. What is the only contributing term and Feynman diagram?

4. Extract the matrix element $i\mathcal{M}$ introduced in (1.1) at first order and evaluate the differential cross-section, that takes for the scattering of two identical particles the simplified form
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CMS}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CMS}}}\,, \tag{2.5}$$
where E_{CMS} denotes the center of mass energy. What is the total cross-section?

5. Proceed to second order by drawing all inequivalent Feynman diagrams. Which diagrams contribute to (1.4) in this case? What is the difficulty when evaluating the following diagram:



   What does this diagram describe physically? Explain how this leads to the concept of amputation in the diagramatic calculation of the S-matrix.

6. Conclude with a list of all Feynman rules for the ϕ^4-theory with external momentum states both in momentum and position space.