Advanced Quantum Theory

Dr. Hans Jockers und Urmi Ninad

`http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php`

Due Date: Oct. 16th, 2019

–Information–

We can be reached at: jockers@uni-bonn.de , urmi@th.physik.uni-bonn.de .

Lecture notes, exercise sheets, and other important information can be found on our course homepage at:
`http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php` .

The exercise sheets will be collected in the lecture on Wednesday. Exercise sheets can be submitted in groups of up to three people. For the admission to the exam it is required to get 50 % of the points on the exercise sheets.

The first and second exams will take place on 08.02.2020 from 09:00-12:00 and on 25.03.2020 from 09:00-12:00 respectively.

The distribution of tutors and the respective timing and venue of the tutorials can be found on the webpage.

Starting from the 18.10.2019 the drop-in will take place every Friday from 14:00-16:00 in the Hörsaal of the HISKP.

1.1 Review: Harmonic Oscillator & Perturbation Theory

We consider the Hamilton operator $\hat{H}_0$ together with the discrete and normalized energy eigenstates $|n\rangle$ with eigenvalues E_n^0, i.e.,

$$\hat{H}_0 \, |n\rangle = E_n^0 \, |n\rangle \ , \quad n = 0, 1, 2, \ldots \ .$$

Let us study the eigenvalues E_n of the perturbed Hamilton operator $\hat{H} = \hat{H}_0 + \lambda \hat{H}_1$ in perturbation theory, where λ is a small real number and $\hat{H}_1$ is a hermitian operator.

a) Assuming that the energy spectrum is not degenerate show that the corrections up to second order in perturbation theory are given by

$$E_n^1 = \langle n|\hat{H}_1|n\rangle \ , \qquad E_n^2 = \sum_{\substack{m \\ m\neq n}} \frac{|\, \langle m|\hat{H}_1|n\rangle \,|^2}{E_n^0 - E_m^0} \ ,$$

in terms of the perturbative expansion $E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \mathcal{O}(\lambda^3)$. *(3 Pts)*

Let us first consider a perturbation of the one-dimensional harmonic oscillator, which is described in terms of the operators

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \ , \qquad \hat{H}_1 = \omega(\hat{x}\hat{p} + \hat{p}\hat{x}) \ ,$$

with the lowering and raising operators $\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} + i\hat{p})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} - i\hat{p})$.

b) Express the perturbation $\hat{H}_1$ to the Hamiltonian of the harmonic oscillator in terms of the operators $\hat{a}$ und $\hat{a}^\dagger$, and compute the first two corrections E_n^1 and E_n^2 of the energy eigenvalues. *(2 Pts)*

c) Compute the energy spectrum of the *total* Hamilton operator $\hat{H}$ exactly. Compare your exact result with the perturbative result obtained in part b). *(3 Pts)*

Let us now consider the two-dimensional harmonic oscillator given in terms of the Hamiltonian

$$\hat{H}_0 = \frac{1}{2m} \left(\hat{p}_1^2 + \hat{p}_2^2 \right) + \frac{1}{2} m\omega^2 \left(\hat{x}_1^2 + \hat{x}_2^2 \right) \ ,$$

which in terms of raising and lowering operators is given by

$$\hat{H}_0 = \hbar\omega(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1) \ ,$$

where $[\hat{a}_k, \hat{a}_l^\dagger] = \delta_{kl}$, $[\hat{a}_k, \hat{a}_l] = [\hat{a}_k^\dagger, \hat{a}_l^\dagger] = 0$.

d) Determine the energy spectrum together with their degeneracies and list the energy eigenstates of the two-dimensional harmonic oscillator. *(2 Pts)*

e) We now consider the perturbation $\hat{H}_1 = m\,\delta^2\,\hat{x}_1\hat{x}_2$ in terms of a small (real) quantity δ, such that the perturbed Hamilton operator reads $\hat{H} = \hat{H}_0 + \hat{H}_1$. Express the perturbation in terms of raising and lowering operators. To first order in perturbation theory compute the perturbation to the ground state energies. Furthermore, to first order in perturbation theory calculate the split-up of the energy eigenvalues of the first excited energy eigenstates. *(3 Pts)*

1.2 Review: Bound States of the Ideal Hydrogen Atom

We consider the state $|\psi\rangle$ of the ideal hydrogen atom, which is given by

$$|\psi\rangle = \frac{1}{2}\left(\sqrt{2}\,|3,2,1\rangle + |3,1,1\rangle + |3,1,0\rangle\right) \ . \tag{1}$$

Here $|n,l,m\rangle$ are the normalized eigenstates of the operators $\hat{H}, |\hat{\vec{L}}|^2, \hat{L}_z$ with eigenvalues

$$\hat{H}\,|n,l,m\rangle = \frac{E_1}{n^2}\,|n,l,m\rangle\ ,\ |\hat{\vec{L}}|^2\,|n,l,m\rangle = \hbar^2 l(l+1)\,|n,l,m\rangle\ ,\ \hat{L}_z\,|n,l,m\rangle = \hbar m\,|n,l,m\rangle \ .$$

a) Show that the state $|\psi\rangle$ is normalized. Furthermore, compute the expectation value and the standard deviation of Hamilton operators $\hat{H}$ with respect to the state $|\psi\rangle$. *(1 Pt)*

b) Compute expectation value and the standard deviation of the operator $|\hat{\vec{L}}|^2$ with respect to the state $|\psi\rangle$. *(2 Pts)*

c) Compute expectation value of the operator $\hat{L}_y$ with respect to the state $|\psi\rangle$. *(2 Pts)*

d) Given the hydrogen atom in the state $|\psi\rangle$, we first perform a measurement with respect to the operator $|\hat{\vec{L}}|^2$ and *afterwards* we perform another measurement with respect to the operator $\hat{L}_z$. For the first measurement of $|\hat{\vec{L}}|^2$, list all possible results of the measurement together with their probabilities. Furthermore, for all possible results after the first measurement, determine the resulting quantum states together with the probability that the second measurement of $\hat{L}_z$ yields the measurement $\hbar$. *(2 Pts)*