Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

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–Exercises–

10.1 The Hydrogen Atom

Consider a hydrogen atom in which the electron is replaced by a particle of the same mass and the same charge but with spin 0 obeying the Klein-Gordon equation. The potential energy is given by

$$V(r) = -\frac{\alpha}{r} \ , \tag{1}$$

where $\alpha = \frac{\hbar^2}{m r_B}$ with r_B the Bohr radius.

a) Use the Klein-Gordon equation minimally coupled to the four-vector potential A^μ of the electromagnetic field, i.e., the four-momentum $\hat{P}^\mu$ is transformed as

$$\hat{P}^\mu \to \hat{P}^\mu - \frac{e}{c} A^\mu \ , \tag{2}$$

to derive the eigenvalue equation

$$\left(\Delta - \left(\frac{mc}{\hbar}\right)^2 + \frac{1}{\hbar^2 c^2}(E - V(r))^2 \right) \varphi(r,\theta,\phi) = 0 \ , \tag{3}$$

for the wavefunction $\varphi(r,\theta,\phi)$. Here Δ is the Laplacian in 3d and E is the energy eigenvalue of the electron. *(2 Pts)*

b) Use the separation of variables on the wavefunction $\varphi(r,\theta,\phi)$ of the electron to derive the differential equation

$$u''(r) = \left(-\frac{1}{\hbar^2 c^2} \left((E - V(r))^2 - (mc^2)^2 \right) + \frac{l(l+1)}{r^2} \right) u(r) \tag{4}$$

in terms of the ansatz $R(r) = \frac{u(r)}{r}$ for the radial part $R(r)$ of the wavefunction. *(2 Pts)*

c) Analyse the asymptotic behaviour of $u(r)$ for $r \to \infty$ and with the boundary condition $u(0) = 0$ for $r \to 0$ to show that

$$u(r) = r^\gamma e^{-\delta r} v(r) \ , \tag{5}$$

where $\gamma = \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - \left(\frac{\alpha}{\hbar c}\right)^2}$, $\delta = \frac{1}{\hbar c}\sqrt{(mc^2)^2 - E^2}$ and $v(r)$ is a polynomial in r of degree $k_{\max}$. *(4 Pts)*

d) Derive from the differential equation (3) a recursion relation for the coefficients c_k of the polynomial

$$v(r) = \sum_{k=0}^{k_{\max}} c_k \, r^k \ . \tag{6}$$

Argue that in order for $v(r)$ to be a polynomial of degree $k_{\max}$ there are discrete energy levels $E_{n,l}$ which can be written as

$$E_{n,l} = mc^2 \left(1 + \frac{(\alpha/\hbar c)^2}{(n-\epsilon_l)^2}\right)^{-1/2} \ , \tag{7}$$

for $n = l + 1 + k_{\max}$, $\epsilon_l = l + \frac{1}{2} - \left[(l+\frac{1}{2})^2 - \left(\frac{\alpha}{\hbar c}\right)^2\right]^{1/2}$. Here $n \in \mathbb{N}$ and $l = 0, 1, \ldots n-1$. *(4 Pts)*

e) Extract the non-relativistic energy spectrum by expanding (7) in powers of $1/c^2$.

Hint: Note that the leading energy contribution is the rest mass of the (spinless) electron, which diverges in the non-relativistic limit $c \to \infty$. *(3 Pts)*

10.2 Gamma Matrix Identities

The gamma matrices $\gamma^\mu = (\gamma^0, \gamma^i)$ are defined as

$$\gamma^0 = \begin{pmatrix} i\, \mathbb{1}_{2\times2} & 0 \\ 0 & -i\, \mathbb{1}_{2\times2} \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & i\,\sigma_i \\ -i\,\sigma_i & 0 \end{pmatrix} \ , \tag{8}$$

for $i = 1, 2, 3$.

a) Show that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. *(1 Pt)*

b) Show that

$$\frac{1}{4}\left[[\gamma^\rho, \gamma^\nu], \gamma^\mu\right] = \eta^{\mu\nu}\gamma^\rho - \eta^{\mu\rho}\gamma^\nu \ . \tag{9}$$

Hint: Use the Clifford algebra of the gamma matrices. *(2 Pts)*

c) Prove the gamma matrix identities for $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$,

(i) $(\gamma^5)^2 = \mathbb{1}_{2\times2}$,

(ii) $\{\gamma^\mu, \gamma^5\} = 0$.

(2 Pts)