Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

Due Date: Jan. 8th, 2020

–Exercises–

11.1 Properties of two vectors $\widetilde{Y}_l^{j,m}$

In the lecture the following ansatz was made for the eigenfunctions $\psi_{j,m,\omega}(r,\theta,\phi)$ of the Dirac Hamiltonian in spherical coordinates,

$$\psi_{j,m,\omega} = \frac{1}{r}\begin{pmatrix} F(r)\ \widetilde{Y}_l^{j,m}(\theta,\phi) \\ iG(r)\ \widetilde{Y}_{l'}^{j,m}(\theta,\phi)\end{pmatrix} \quad ; \quad l = j + \frac{1}{2}\omega,\ l = j - \frac{1}{2}\omega\ , \tag{1}$$

where the two vectors $\widetilde{Y}_l^{j,m}(\theta,\phi)$ and $\widetilde{Y}_{l'}^{j,m}(\theta,\phi)$ are eigenfunctions of $\hat{\vec{J}}^2, \hat{J}_z$ and $\hat{\vec{L}}^2$ with the respective quantum numbers j, m and l(or l').

a) Compute the commutator $[\hat{\vec{J}}, \vec{\sigma}\cdot\frac{\vec{x}}{r}]$ and deduce that $\left(\vec{\sigma}\cdot\frac{\vec{x}}{r}\right)\widetilde{Y}_l^{j,m}$ and $\left(\vec{\sigma}\cdot\frac{\vec{x}}{r}\right)\widetilde{Y}_{l'}^{j,m}$ are eigenfunctions of $\hat{\vec{J}}^2$ and $\hat{J}_z$ and determine their eigenvalues. *(3 Pts)*

b) Show that the two vectors $\widetilde{Y}_l^{j,m}(\theta,\phi)$ and $\widetilde{Y}_{l'}^{j,m}(\theta,\phi)$ can be normalised such that they obey

$$\left(\vec{\sigma}\cdot\frac{\vec{x}}{r}\right)\widetilde{Y}_l^{j,m} = -\widetilde{Y}_{l'}^{j,m}\ , \quad \left(\vec{\sigma}\cdot\frac{\vec{x}}{r}\right)\widetilde{Y}_{l'}^{j,m} = -\widetilde{Y}_l^{j,m}\ . \tag{2}$$

<u>*Hint:*</u> Compute $\left(\vec{\sigma}\cdot\frac{\vec{x}}{r}\right)^2$ and consider the action of the parity operator on $\widetilde{Y}_l^{j,m}$ and $\widetilde{Y}_{l'}^{j,m}$ as a function of the quantum numbers l and l'. *(2 Pts)*

11.2 The Hydrogen Atom II

Consider a hydrogen atom in which the electron is a Dirac particle in the central potential

$$V(r) = -\frac{\alpha}{r}\ ,\ \alpha > 0\ . \tag{3}$$

The application of the Dirac equation on the wavefunction (1) yields a system of coupled differential equations for the radial functions $F(r)$ and $G(r)$

$$\left[-\frac{d}{dr} + \frac{\omega(j+\frac{1}{2})}{r}\right]G(r) = \frac{1}{\hbar c}\left(E - mc^2 + \frac{\alpha}{r}\right)F(r)\ , \tag{4}$$

$$\left[\frac{d}{dr} + \frac{\omega(j+\frac{1}{2})}{r}\right]F(r) = \frac{1}{\hbar c}\left(E + mc^2 + \frac{\alpha}{r}\right)G(r)\ . \tag{5}$$

a) Analyse the asymptotic behaviour of $F(r)$ and $G(r)$ for $r \to \infty$ and with the boundary condition $F(0) = G(0) = 0$ for $r \to 0$ to show that

$$F(r) = r^s e^{-\kappa r} v_F(r) \quad , \quad G(r) = r^s e^{-\kappa r} v_G(r) \ , \tag{6}$$

where $\kappa = \frac{1}{\hbar c}\left(\sqrt{(mc^2)^2 - E^2}\right)$, $s = \sqrt{(\omega(j+\frac{1}{2}))^2 - (\frac{\alpha}{\hbar c})^2}$ and $v_F(r), v_G(r)$ are polynomials in r of degree $k_{\max}$ atmost. *(4 Pts)*

b) Derive from the ansatz (6) a recursion relation for the coefficients f_k and g_k of the polynomials

$$v_F(r) = \sum_{k=0}^{k_{\max}} f_k \, r^k \quad , \quad v_G(r) = \sum_{k=0}^{k_{\max}} g_k \, r^k \ , \tag{7}$$

where $f_0 \neq 0, g_0 \neq 0$ and $\begin{pmatrix} f_{k_{\max}} \\ g_{k_{\max}} \end{pmatrix} \neq 0$. Show that there are discrete energy levels $E_{n,j}$ which can be written as

$$E_{n,j} = mc^2 \left(1 + \frac{(\alpha/\hbar c)^2}{(n - \epsilon_j)^2}\right)^{-1/2} \ , \tag{8}$$

for $n = j + \frac{1}{2} + k_{\max}$, $\epsilon_j = j + \frac{1}{2} - \left[(j+\frac{1}{2})^2 - (\frac{\alpha}{\hbar c})^2\right]^{1/2}$. Here $n \in \mathbb{N}$ and j takes all half integral values in the interval $(0, n)$, i.e., $j = \frac{1}{2}, \frac{3}{2}, \ldots n - \frac{1}{2}$. *(6 Pts)*

11.3 Gamma Matrix Identities II

The gamma matrices $\gamma^\mu = (\gamma^0, \gamma^i)$ are defined as

$$\gamma^0 = \begin{pmatrix} i \, \mathbb{1}_{2\times2} & 0 \\ 0 & -i \, \mathbb{1}_{2\times2} \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & i \, \sigma_i \\ -i \, \sigma_i & 0 \end{pmatrix} \ , \tag{9}$$

for $i = 1, 2, 3$ and σ_i the Pauli matrices.

a) Show first that

$$\left(\vec{\sigma} \cdot \hat{\vec{A}}\right) \left(\vec{\sigma} \cdot \hat{\vec{B}}\right) = \left(\hat{\vec{A}} \cdot \hat{\vec{B}}\right) + i\vec{\sigma} \cdot \left(\hat{\vec{A}} \times \hat{\vec{B}}\right) \ , \tag{10}$$

for operators $\hat{\vec{A}}$ and $\hat{\vec{B}}$. *(2 Pts)*

b) Use the result of a) to show that

$$\vec{\gamma} \cdot \hat{\vec{P}} = \left(\vec{\gamma} \cdot \frac{\vec{x}}{r}\right) \left(\hat{p}_r \cdot \mathbb{1}_{4\times4} + \frac{i}{4}(\hbar \, \mathbb{1}_{2\times2} + \vec{\sigma} \cdot \vec{L}) \otimes \mathbb{1}_{2\times2}\right) \ , \tag{11}$$

where $\hat{\vec{P}} = -i\hbar\vec{\nabla}$ is the momentum operator and $\vec{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$.

(3 Pts)