Physikalisches Institut
Universität Bonn
Theoretical Physics

Exercise Sheet 12
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Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

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–Exercises–

12.1 Dirac Equation in a Constant Magnetic Field

Consider a Dirac particle of charge e in a uniform and constant magnetic field $\vec{B}$ along the z-axis. We want to solve for the energy eigenvalues of this system.

a) Solve for the electromagnetic vector potential A^μ that gives rise to the constant magnetic field $\vec{B} = (0, 0, B)$ such that only one component of A^μ is non-zero. Distinct solutions to A^μ that give rise to the same magnetic field are known as *gauges*. *(1 Pt)*

b) In the following we will work with the gauge where only the x-component of A^μ non-zero. Solve the Dirac equation minimally coupled to the electromagnetic field

$$\left[\frac{i}{\hbar}\left(\hat{\not{P}} - \frac{e\not{A}}{c}\right) - \frac{mc}{\hbar}\right]\psi(t, \vec{x}) = 0 \ , \tag{1}$$

with the ansatz for the wavefunction $\psi(t, \vec{x})$ being an eigenfunction of the Hamiltonian, i.e.,

$$\psi(t, \vec{x}) = e^{\frac{-iEt}{\hbar}}\begin{pmatrix}\varphi_1(\vec{x}) \\ \varphi_2(\vec{x})\end{pmatrix} \ , \tag{2}$$

where φ_1 and φ_2 are 2-component spinors. *(2 Pts)*

c) Eliminate φ_2 from the system of equations resulting from (1) and (3) to obtain a second order differential equation in φ_1. *(1 Pt)*

d) Plug in the ansatz

$$\varphi_1(\vec{x}) = e^{\frac{i}{\hbar}(p_x x + p_z z)}\begin{pmatrix}\chi_1(y) \\ \chi_2(y)\end{pmatrix} \ , \tag{3}$$

into the differential equation resulting from c). Here $\vec{p} = (p_x, p_y, p_z)$ is the three-momentum. Introduce the dimensionless variable

$$\xi = \sqrt{\frac{|eB|}{\hbar}}\left(y + \frac{p_x}{eB}\right) \ , \tag{4}$$

to rewrite the differential equations in χ_1, χ_2 as

$$\left(\frac{d^2}{d\xi^2} - \xi^2 + a_{1/2}\right)\chi_{1/2} = 0 \ , \tag{5}$$

with $a_{1/2}$ being a function of energy. *(2 Pts)*

e) Equation (5) is a special form of the Hermite differential equation. To restore the conventional form of this differential equation introduce the variable $\zeta_i = \chi_i e^{\frac{\xi^2}{2}}$. Solve this differential equation and state the constraint on $a_{1/2}$ that ensures a polynomial solution.

Remark: Note that the requirement for polynomial solutions $\zeta_i(\xi)$ ensures the existence of normalisable solutions upon suitably integrating over the momentum eigenvalues p_x and p_z. *(3 Pts)*

f) Solve for the quantised energy eigenvalues, known as the relativistic Landau levels, given by

$$E = \sqrt{m^2c^4 + c^2p_z^2 + 2kmc^2\hbar\omega_c} \,. \tag{6}$$

Here $k \in \mathbb{N}_0$ and $\omega_c = \frac{|eB|}{m}$ is the cyclotron frequency. *(2 Pts)*

g) Compare the energy levels of part f) with the non-relativistic Landau levels of the electron given by

$$E = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m} - \frac{1}{2}\hbar\omega_c\sigma_z \,. \tag{7}$$

(1 Pt)

12.2 The Permutation Group

Suppose we have an ordered set of n elements, denoted $\{1, 2, \dots, n\}$, and a permutation σ that acts as

$$\{1, 2, \dots, n\} \mapsto \sigma\{1, 2, \dots, n\} \equiv \{\sigma(1), \sigma(2), \dots, \sigma(n)\}. \tag{8}$$

a) Show that the permutations of n elements form a group, denoted S_n, and compute its dimension.
Remark: A group G is a set with a binary operation $\cdot : G \times G \to G, (a, b) \mapsto a \cdot b$ such that:

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in G$ (associativity),
- there exists an element $e \in G$ with $e \cdot a = a \cdot e = a$ for any $a \in G$ (identity),
- and for all $a \in G$ there exists a' with $a \cdot a' = a' \cdot a = e$ (inverse).

(2 Pts)

b) For $1 \leq k \leq n$, let $a_1, a_2, \dots, a_k \in \{1, 2, \dots, n\}$ be pairwise different. A *cycle* $(a_1, \dots, a_k)$ is the cyclic permutation $a_1 \mapsto a_2$, $a_2 \mapsto a_3, \dots,$ $a_k \mapsto a_1$; it acts as the identity on all elements other than the a_i. We call k the length of the cycle.
Show that any permutation σ can be written as a product of disjoint cycles (two cycles are disjoint if they contain no common elements). *(3 Pts)*

c) A *transposition* is a cycle of length 2. For $n \geq 2$, show that every cycle (and thus every permutation) can be written as a product of transpositions. In particular, show that for $k \geq 2$, $(a_1, \dots, a_k) = (a_1, a_2)(a_2, a_3) \cdots (a_{k-1}, a_k)$. *(3 Pts)*