Physikalisches Institut
Universität Bonn
Theoretical Physics

Exercise Sheet 2
Oct. 16th, 2019
WS 2019/2020

Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

Due Date: Oct. 23rd, 2019

–Exercises–

2.1 Complete Set of States

Let $\{|x\rangle\}$ and $\{|p\rangle\}$ denote the complete set of position and momentum operator eigenstates in one dimension respectively, i.e., they satisfy

$$\hat{x}\,|x\rangle = x\,|x\rangle \quad , \quad \hat{p}\,|p\rangle = p\,|p\rangle \ . \tag{1}$$

They are normalised such that

$$\langle x|x'\rangle = \delta(x-x') \quad , \quad \langle p|p'\rangle = \delta(p-p') \ , \tag{2}$$

and they fulfil

$$\begin{aligned} \psi_p(x) &:= \langle x|p\rangle = (2\pi\hbar)^{-1/2} e^{ipx/\hbar} \ , \\ \phi_x(p) &:= \langle p|x\rangle = (2\pi\hbar)^{-1/2} e^{-ipx/\hbar} \ , \end{aligned} \tag{3}$$

where $\psi_p(x)$ and $\phi_x(p)$ are the position-space and momentum-space wave functions of $|p\rangle$ and $|x\rangle$ respectively.

a) Use the identity operator, $\mathbb{1} = \int dx\,|x\rangle\langle x| = \int dp\,|p\rangle\langle p|$, to show that the bracket of two normalised states $|\zeta\rangle$ and $|\chi\rangle$ is

$$\langle\zeta|\chi\rangle = \int dx\ \zeta^*(x)\chi(x) = \int dp\ \tilde{\zeta}^*(p)\tilde{\chi}(p) \ , \tag{4}$$

in terms of position-space wave functions $\zeta(x) = \langle x|\zeta\rangle$ and $\chi(x) = \langle x|\chi\rangle$ and in terms of momentum-space wave functions $\tilde{\zeta}(p) = \langle p|\zeta\rangle$ and $\tilde{\chi}(p) = \langle p|\chi\rangle$. *(3 Pts)*

b) Show that

$$\begin{aligned} \langle x|\hat{p}|\zeta\rangle &= -i\hbar\frac{\partial}{\partial x}\zeta(x) \quad \text{and} \\ \langle p|\hat{x}|\zeta\rangle &= i\hbar\frac{\partial}{\partial p}\tilde{\zeta}(p) \ . \end{aligned} \tag{5}$$

(4 Pts)

c) Verify the commutation relation $[\hat{x},\hat{p}] = i\hbar$ by computing $\langle x|[\hat{x},\hat{p}]|p\rangle$. *(3 Pts)*

2.2 Gaussian Wave Packets

We consider the wave function

$$\psi_g(\vec{x},t) = \frac{1}{(2\pi\hbar)^{3/2}} \int \frac{d^3k}{(2\pi)^{3/2}}\, g(\vec{k})\, \exp\left(i\vec{k}\cdot\vec{x} - \frac{i\hbar t|\vec{k}|^2}{2m} \right) , \tag{6}$$

for the wave packet

$$g(\vec{k}) \propto \exp\left(-\frac{\Delta_0^2}{2}(\vec{k}-\vec{k}_0)^2 - i\frac{\hbar\vec{k}\cdot\vec{k}_0 t_0}{m} + i\frac{\hbar t_0|\vec{k}|^2}{2m} \right) , \tag{7}$$

for a constant wave vector $\vec{k}_0$ at some time t_0. Prove that the probability density obeys

$$|\psi_g(\vec{x},t)|^2 \propto \Delta^{-3}\exp\left(-\frac{1}{\Delta^2}\left(\vec{x} - \frac{\hbar\vec{k}_0 t}{m} \right)^2 \right) , \tag{8}$$

where

$$\Delta = \left(\Delta_0^2 + \frac{\hbar^2(t-t_0)^2}{m^2\Delta_0^2} \right)^{1/2} . \tag{9}$$

(5 Pts)

2.3 The Green's Function

Show that the Green's function in three dimensions,

$$G_k(\vec{x}) = -\frac{e^{ik|\vec{x}|}}{4\pi|\vec{x}|} , \tag{10}$$

satisfies the inhomogeneous differential equation

$$(\Delta + k^2)G_k(\vec{x}) = \delta^{(3)}(\vec{x}) . \tag{11}$$

<u>*Hint:*</u> Show first that $(\Delta + k^2)G_k(\vec{x}) = 0$ for $\vec{x} \neq 0$. Then consider the integral

$$\int\limits_{|\vec{x}|\leq 1} d^3x\; (\Delta + k^2)G_k(\vec{x})$$

to deduce the final result. *(5 Pts)*