Physikalisches Institut
Universität Bonn
Theoretical Physics

Exercise Sheet 3
Oct. 23rd, 2019
WS 2019/2020

Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

Due Date: Oct. 30th, 2019

–Exercises–

3.1 Differential Cross Section and Rutherford Scattering

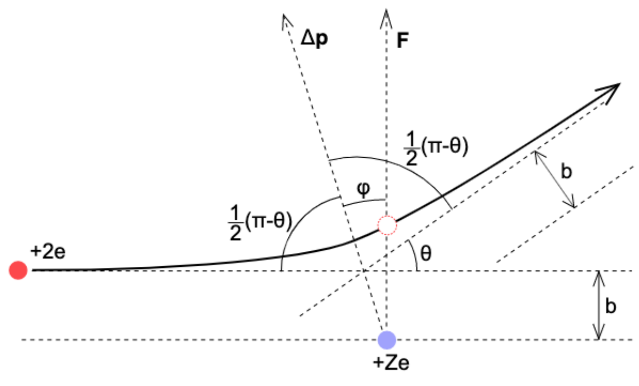
Let us compute the differential cross section of α particles scattering off a nucleus from a classical mechanics perspective. For this scattering process the repulsive Coulomb potential is given by

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \, , \tag{1}$$

where $Z_1(= 2)$ is the atomic number of the α particle, Z_2 is the atomic number of the target nucleus, and r denotes the separation between the α particle and the target nucleus.

a) An α particle approaching the nucleus head on reaches a distance of closest approach d before the repulsive Coulomb force makes it reverse its direction. Determine the distance of closest approach d as a function of the kinetic energy E_{kin} of the incoming α particle infinitely far away from the scattering center. *(1 Pt)*

For an α particle not approaching the nucleus head on, we define the distance between its line of incidence and the nucleus as the impact parameter b.



We want to prove the following relation between the distance of closest approach and the impact parameter:

$$\tan\left(\frac{\theta}{2}\right) = \frac{d}{2b} \, , \tag{2}$$

where θ is the scattering angle.

b) Let the initial and final momenta of the α particle be given by $\vec{p}_1$ and $\vec{p}_2$, and we define the momentum transfer $\vec{q} = \Delta\vec{p} = \vec{p}_1 - \vec{p}_2$. Under the assumption of elastic scattering show the relation

$$\frac{|\vec{p}_1 - \vec{p}_2|}{|\vec{p}_1|} \equiv \frac{|\vec{q}|}{|\vec{p}_1|} = 2\sin\left(\frac{\theta}{2}\right) . \tag{3}$$

(*2 Pts*)

c) Let the angle between the Coulomb force $\vec{F}$ on the α particle and the momentum transfer $\vec{q}$ be φ. Show that

$$b\,|\vec{p}_1| = \mu r^2 \dot{\varphi} , \tag{4}$$

where μ is the (reduced) mass of the α particle. (*2 Pts*)

d) Use Newton's second law of motion

$$|\vec{q}| = \int \left(\vec{F} \cdot \vec{e}_{\vec{q}}\right) \, dt , \tag{5}$$

together with the results of parts a), b) and c) to show the relation (2). (*3 Pts*)

We now want to calculate the differential cross section for the scattered α particle. Note that the infinitesimal area $d\sigma$ into which the α particle scatters is given by the area of the ring of radius b and width db.

e) Prove that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4E_{\text{kin}} \sin^2\left(\frac{\theta}{2}\right)}\right)^2 . \tag{6}$$

(*3 Pts*)

3.2 The Born Approximation for a Shielded Coulomb Potential

For spherically symmetric potentials, i.e., $V(\vec{x}) = V(|\vec{x}|)$, the first order Born approximation for the scattering amplitude is given by the formula

$$f_{\vec{k}}(\theta) = -\frac{m}{2\pi\hbar^2} \int d^3y \; V(|\vec{y}|) \; e^{-i\vec{q}\cdot\vec{y}} , \tag{7}$$

with the momentum transfer $\vec{q} \equiv \vec{k} - |\vec{k}|\vec{e}_{\vec{x}}$ and where $\vec{e}_{\vec{x}}$ is the direction of scattering. Due to the symmetry it depends on the angular variable θ only.

a) Show that (7) simplifies to the integral

$$f_{\vec{k}}(\theta) = -\frac{2m}{\hbar^2} \int_0^\infty dr \; r^2 \; V(r) \; \frac{\sin(qr)}{qr} , \tag{8}$$

where $q = 2k\sin(\frac{\theta}{2})$. (*2 Pts*)

b) If the potential corresponding to a nucleus scattering off a neutral atom is modelled as

$$V(r) = \frac{Z_1 Z_2 e^2}{r} e^{-\kappa r} \ , \tag{9}$$

with $\kappa > 0$, e the electric charge, Z_1 the atomic number of scattered nucleus and Z_2 atomic number of target atom, then prove that evaluating the first order Born approximation (8) yields

$$f_k(\theta) = -\frac{2\mu Z_1 Z_2 e^2}{q\hbar^2} \int\limits_0^\infty dr \ e^{-\kappa r} \sin(qr) = -\frac{2\mu Z_1 Z_2 e^2}{\hbar^2} \frac{1}{q^2 + \kappa^2} \ . \tag{10}$$

Here μ is the reduced mass of the scattered nucleus. *(2 Pts)*

c) For the $\kappa \to 0$ limit of the scattering amplitude (10), compare the differential cross section with the classical differential cross section of the classical Rutherford scattering analysed in the previous exercise. Discuss how the quantum mechanical probability interpretation of the differential scattering cross section for plane waves should be compared to the classical result of Rutherford, which assumes a deterministic scattering process. *(2 Pts)*

3.3 Total Cross Section

Find an upper bound for the total cross section σ_{tot}, if the scattering amplitude $f_{\vec{k}}(\vec{e}_{\vec{x}})$ is *independent* of the direction $\vec{e}_{\vec{x}}$. *(3 Pts)*