Physikalisches Institut
Universität Bonn
Theoretical Physics

Exercise Sheet 5
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WS 2019/2020

Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

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–Exercises–

5.1 Resonances

Suppose that in the scattering of a particle of mass m by an unknown potential, a resonance is observed at energy E_R for which the total cross section at the peak of the resonance is σ_{max}. Show how to use this data to give a value for the orbital angular momentum of the resonant state. *(2 Pts)*

5.2 The Square Well Potential

In this exercise we want to consider the spherically symmetric potential well in three dimensions,

$$V(r) = \begin{cases} -V_0 & r \leq a \\ 0 & r > a \ , \end{cases} \tag{1}$$

where $V_0 > 0$ and r is the radial coordinate.

a) Recall that for a spherically symmetric potential the wavefunction can be expanded as

$$\psi(r,\theta) = \sum_{l=0}^{\infty} R_l(r)\ P_l(\cos\theta)\ , \tag{2}$$

where $P_l(x)$ are Legendre polynomials. Solve the radial Schrödinger equation,

$$\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} + \left(k^2 - \frac{2m}{\hbar^2}V(r)\right)\right) R_l(r) = 0 \tag{3}$$

for the square well potential. *(3 Pts)*

b) Recall that the scattering amplitude can be expressed in term of phase shifts,

$$f_k(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\ (e^{2i\delta_l} - 1)P_l(\cos\theta)\ , \tag{4}$$

where each term is known as a partial wave,

$$f_l = \frac{1}{2ik}(2l+1)(e^{2i\delta_l} - 1)P_l(\cos\theta)\ . \tag{5}$$

Use the result of part a) to show that the s-wave phase shift δ_0 yields the partial wave

$$f_0 = \frac{1}{2ik}\left(\frac{e^{-iak}(2\kappa\cot(a\kappa)\sin(ak) - 2k\cos(ak))}{k + i\kappa\cot(a\kappa)}\right) . \tag{6}$$

Here $\kappa^2 := k^2 + \frac{2m}{\hbar^2}V_0$.

<u>*Hint*</u>: It might help to show first that the partial wave f_0 can be expressed as

$$f_0 = \frac{1}{k}\left(\frac{\tan\delta_0}{1 - i\tan\delta_0}\right) . \tag{7}$$

(*4 Pts*)

c) Show that at linear order in V_0 the first partial wave reads

$$f_0 = \frac{mV_0\left(ak - \frac{1}{2}\sin(2ak)\right)}{\hbar^2 k^3} . \tag{8}$$

<u>*Hint*</u>: Use the expansion

$$\alpha\sqrt{\beta x + 1}\cot\left(\alpha\sqrt{\beta x + 1}\right) = \alpha\cot(\alpha) + \frac{1}{2}\alpha\beta x\left(\cot(\alpha) - \alpha\left(\frac{1}{\sin\alpha}\right)^2\right) + \mathcal{O}\left(x^2\right) . \tag{9}$$

(*4 Pts*)

d) Using the first order Born approximation solve for the scattering amplitude $f_k^{\text{Born}}(\theta)$ and show that to the first order in energy (i.e. second order in k)

$$f_k^{\text{Born}}(\theta) = \frac{2a^3 mV_0}{3\hbar^2} - \frac{4a^5k^2 mV_0\sin^2\left(\frac{\theta}{2}\right)}{15\hbar^2} + \mathcal{O}(k^4) . \tag{10}$$

(*4 Pts*)

e) Compare the results of parts c) and d). (*3 Pts*)