Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

Due Date: Nov. 20th, 2019

–Exercises–

6.1 Levinson's Theorem

We want to deduce a relationship between bound states and phase shifts in the theory of elastic scattering.

a) Consider a free system enclosed in a (large) sphere S^2 with radius R, i.e. , the wavefunction vanishes outside the sphere S^2. We can model such a system with a potential

$$V(r) = \begin{cases} 0 & r < R \\ \infty & r \geq R \ . \end{cases} \tag{1}$$

Show that for large R, the number of states with angular momentum quantum number l in the energy range 0 to E is given by:

$$N_{R,l}^{\text{free}}(E) = \left\lfloor \frac{kR}{\pi} + \mathcal{O}(R^{-1}) \right\rfloor \ , \tag{2}$$

where $\lfloor \ldots \rfloor$ denotes the floor function. *(2 Pts)*

b) Now consider a potential $V(r)$, which vanishes for $r \to \infty$ at least as fast as r^{-2}. Consider now the theory with $V(r)$ on a sphere S^2 of (large) radius R. I.e., consider the potential

$$V(r) = \begin{cases} V(r) & r < R \\ \infty & r \geq R \ . \end{cases} \tag{3}$$

Show that for large R, the number of states with angular momentum quantum number l in the energy range 0 to E is given by:

$$N_{R,l}(E) = \left\lfloor \frac{kR}{\pi} + \frac{\delta_l(E)}{\pi} - \frac{\delta_l(0)}{\pi} + \mathcal{O}(R^{-1}) \right\rfloor \ . \tag{4}$$

(1 Pt)

c) Conclude that the number of bound states, i.e., states with $E < 0$, with angular momentum quantum number l for the potential $V(r)$ is given by

$$N_l = \frac{1}{\pi} \left(\delta_l(0) - \delta_l(\infty) \right) \ . \tag{5}$$

(2 Pts)

d) Consider now a Hamiltonian for some potential $V(r)$ that falls off as r^{-2} as $r \to \infty$, such that it has a continuum of particle states together with a number of discrete bound states N_l with angular momentum l and with energy $E < 0$. Suppose we add an interaction, which is given in terms of a local potential $\Delta V(r)$ (with $\Delta V(r)$ vanishing at least as fast as r^{-2} as $r \to \infty$), such that all discrete states become unstable and all continuum states remain in a continuum. Determine the change in the phase shifts $\delta_l(E)$ as the energy is scanned from $E = 0$ to $E = \infty$ and as we vary the interaction coupling λ in $V_\lambda(r) = V(r) + \lambda\, \Delta V(r)$ from $\lambda = 0$ to $\lambda = 1$. *(2 Pts)*

6.2 Coulomb Scattering

In this exercise we want to solve the Schrödinger equation for the Coulomb potential

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \ . \tag{6}$$

Here $Z_1 e$ is the charge of the scattered particle, $Z_2 e$ the charge of the scattering centre and r the distance between them. The Schrödinger equation for such a potential becomes

$$-\frac{\hbar^2}{2\mu}\Delta\psi + \frac{Z_1 Z_2 e^2}{r}\psi = \frac{\hbar^2 k^2}{2\mu}\psi \tag{7}$$

with μ being the reduced mass of the scattered particle.

a) Starting from the ansatz

$$\psi(\vec{x}) = e^{ikz}\ \Pi(r - z) \tag{8}$$

for the wavefunction, where $z = r\cos(\theta)$, show that the Schrödinger equation takes the form

$$\rho\Pi^{''}(\rho) + (1 - ik\rho)\Pi^{'}(\rho) - k\xi\Pi(\rho) = 0 \ , \tag{9}$$

where $\rho = r - z$ and $\xi = \frac{Z_1 Z_2 e^2 \mu}{\hbar^2 k}$. *(4 Pts)*

b) We want to solve the second order differential equation (9), also known as the confluent hypergeometric equation. For this purpose you can use Frobenius method, which is a method to find power series solutions to second order differential equations.

Show that

$$\psi(\vec{x}) = N e^{ikz}\ {}_1F_1(-i\xi; 1; ik(r - z)) \ . \tag{10}$$

Here

$${}_1F_1(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n\ x^n}{(c)_n\ n!} \ , \tag{11}$$

is the confluent hypergeometric function and $(a)_n$ denotes the Pochhammer symbol,

$$(a)_n = \prod_{m=0}^{n-1} a(a+1)\dots(a+n-1) \quad \text{(with } (a)_0 = 1) \ . \tag{12}$$

(4 Pts)

c) The asymptotic behaviour of the confluent hypergeometric function for large complex argument x is given by

$$ {}_1F_1(a;c;x) \to \frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{-a}\left[1+\mathcal{O}(x^{-1})\right] + \frac{\Gamma(c)}{\Gamma(a)}e^{x}(x)^{a-c}\left[1+\mathcal{O}(x^{-1})\right] \ . \tag{13} $$

Here $\Gamma(z)$ denotes the Gamma function which is defined as

$$ \Gamma(z) = \int_0^\infty dx\ x^{z-1}e^{-x} \tag{14} $$

for $\mathrm{Re}(z) > 0$ and has the property $\Gamma(z+1) = z\,\Gamma(z)$.

Deduce the asymptotic behaviour of the wavefunction $\psi(\vec{x})$ from (13). In particular show that it can be written as

$$ \psi(\vec{x}) \to \frac{Ne^{\xi\pi/2}}{\Gamma(1+i\xi)}\left[e^{ikz+i\xi\ln(kr(1-\cos\theta))} + f_k(\theta)\frac{e^{ikr-i\xi\ln(kr(1-\cos\theta))}}{r}\right] \ , \tag{15} $$

with

$$ f_k(\theta) = -\frac{\Gamma(1+i\xi)}{\Gamma(1-i\xi)}\frac{2Z_1Z_2e^2\mu}{\hbar^2q^2} \ , \tag{16} $$

where $q = 2k\sin(\theta/2)$. *(2 Pts)*

6.3 The Jacobi Identity

Given a commutation relation

$$ [\hat{Q}^a, \hat{Q}^b] = i\sum_c f_c^{ab}\ \hat{Q}^c \tag{17} $$

between operators $\hat{Q}^a$ and $\hat{Q}^b$, prove the Jacobi identity

$$ \sum_c \left(f_a^{bc}f_c^{de} + f_a^{ec}f_c^{bd} + f_a^{dc}f_c^{eb}\right) = 0 \tag{18} $$

of the so-called *structure constants* f_c^{ab}. *(3 Pts)*