Physikalisches Institut
Universität Bonn
Theoretical Physics

Exercise Sheet 7
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Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

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–Exercises–

7.1 The 1d Translation Group

Consider the Hilbert space $\mathcal{H}$ of square-integrable functions $\psi : \mathbb{R} \to \mathbb{C}$ with the inner product

$$\langle \psi | \phi \rangle = \int dx \ \psi^*(x) \ \phi(x) \ . \tag{1}$$

We consider the additive group of real numbers $(\mathbb{R}, +)$, which is the group of 1d translations. Each group element x_0 acts on the Hilbert space $\mathcal{H}$ with the operator

$$\begin{aligned} \mathcal{U}(x_0) \ : \ & \mathcal{H} \to \mathcal{H} \\ & \psi(x) \mapsto \psi(x - x_0) \ . \end{aligned} \tag{2}$$

a) Show that $\mathcal{U}$ is a (non-projective) unitary representation acting on $\mathcal{H}$. *(2 Pts)*

b) Determine the symmetry generator of the action of the translation group on $\mathcal{H}$. *(2 Pts)*

7.2 Rotations in a Plane

a) Consider the Lie group $\mathrm{SO}(2) = \{A \in \mathrm{Mat}(2, \mathbb{R}) | \ A^{\mathrm{T}} A = \mathbb{1}, \det(A) = 1\}$. Determine the manifold of the Lie group SO(2).

   Hint: Parameterise the group elements of SO(2) with suitable matrix elements and deduce the space described by these parameters. *(3 Pts)*

b) Parameterise the elements of the Lie group SO(2) in terms of an angular coordinate θ. *(1 Pt)*

c) Consider a one-dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}$. Classify all one-dimensional non-projective unitary representations, and show that each representation is labelled by an integer.

   Remark: In physics these integers correspond to, for instance, electric charges. The SO(2) symmetry underlying electromagnetism is ultimately the reason for the quantisation of electric charges in nature. *(3 Pts)*

7.3 Projective Representations

Consider a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ with the standard inner product upon which the operators $\mathcal{U}(\theta_1, \theta_2, \theta_3)$ act, where

$$\mathcal{U}(\theta_1,\theta_2,\theta_3) = e^{-i\pi/4}\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\theta_1-\theta_2}{2}\right) + i\cos\left(\frac{\theta_1+\theta_2}{2}\right)\right) & \frac{1}{\sqrt{2}}e^{i\theta_3}\left(\sin\left(\frac{\theta_1-\theta_2}{2}\right) - i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right) \\ \frac{1}{\sqrt{2}}\left(\sin\left(\frac{\theta_1+\theta_2}{2}\right) - i\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right) & \frac{1}{\sqrt{2}}e^{i\theta_3}\left(\cos\left(\frac{\theta_1+\theta_2}{2}\right) + i\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right) \end{pmatrix} .$$

a) Show that $\mathcal{U}(\theta_1, \theta_2, \theta_3)$ is a unitary matrix with $\mathcal{U}(0, 0, 0) = \hat{\mathbb{1}}$. *(2 Pts)*

b) Assuming that $\mathcal{U}(\theta_1, \theta_2, \theta_3)$ paramterises a representation of a Lie group in the vicinity of the identity, compute the commutator relations of the symmetry generators. *(4 Pts)*

c) Argue that the representation of $\mathcal{U}$ is projective. Redefine the symmetry generators and the unitary matrices in such a way that the representation $\tilde{\mathcal{U}}$ and its symmetry generators give rise to a non-projective unitary representation. *(3 Pts)*