Advanced Quantum Theory

Dr. Hans Jockers und Urmi Ninad

http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

Due Date: Dec. 5th, 2019

–Exercises–

Attention: Please submit the solutions to sheet 8 and pick up sheet 9 on Thursday 5.12.2019 by 12:00 from the desk outside office 2.010, BCTP, 2nd floor, PI.

8.1 Symmetry Generators of the Galileo Group in 3d

Consider the Galileo group Gal(3) in 3 dimensions. Let $S \in \mathrm{Gal}(3)$ with the action on the coordinates $(\vec{x}, t)$

$$S(R, \vec{v}, \vec{s}, \tau) : (\vec{x}, t) \longmapsto (\vec{x}', t') = (R\vec{x} - \vec{v}t - \vec{s}, t - \tau) \ , \tag{1}$$

with $R^T = R^{-1}$. In the lecture we derived the commutation relations

$$\begin{aligned} U(S') \cdot \hat{J}_{ij} \cdot U(S')^{-1} &= \sum_{k,l} \left(\hat{J}_{kl} + v'_k \hat{K}_l - v'_l \hat{K}_k + (s'_k + v'_k \tau') \hat{P}_l - (s'_l + v'_l \tau') \hat{P}_k \right) R'_{ki} R'_{lj} \ , \\ U(S') \cdot \hat{K}_i \cdot U(S')^{-1} &= \sum_k \left(\hat{K}_k + \tau' \hat{P}_k \right) R'_{ki} \ , \\ U(S') \cdot \hat{P}_i \cdot U(S')^{-1} &= \sum_k \hat{P}_k R'_{ki} \ , \\ U(S') \cdot \hat{H} \cdot U(S')^{-1} &= \hat{H} + \sum_k v'_k \hat{P}_k \ , \end{aligned} \tag{2}$$

where $\hat{J}_{ij}, \hat{K}_i, \hat{P}_i, \hat{H}$ are the generators for an infinitesimal unitary transformation, i.e.,

$$U(\omega_{ij}, v_i, s_i, \tau) = \mathbb{1} - \frac{i}{2\hbar} \sum_{i,j} \omega_{ij} \hat{J}_{ij} + \frac{i}{\hbar} \sum_i v_i \hat{K}_i + \frac{i}{\hbar} \sum_i s_i \hat{P}_i - \frac{i}{\hbar} \tau \hat{H} \ . \tag{3}$$

a) For $S(R, \vec{v}, \vec{s}, \tau) \in \mathrm{Gal}(3)$ show that

$$S(R, \vec{v}, \vec{s}, \tau)^{-1} = S(R^{-1}, -R^{-1}\vec{v}, -R^{-1}(\vec{s} + \vec{v}t), -\tau) \ . \tag{4}$$

State the infinitesimal inverse transformation. (*2 Pts*)

b) Derive from (2), by inserting for S' an infinitesimal group element, the commutation relations

$$
\begin{aligned}
&[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}_k \ , && [\hat{P}_i, \hat{P}_j] = 0 \ , \\
&[\hat{J}_i, \hat{K}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{K}_k \ , && [\hat{K}_i, \hat{K}_j] = 0 \ , \\
&[\hat{J}_i, \hat{P}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{P}_k \ , && [\hat{K}_i, \hat{P}_j] = 0 \ , \\
&[\hat{J}_i, \hat{H}] = 0 \ , && [\hat{P}_i, \hat{H}] = 0 \ , \\
& && [\hat{K}_i, \hat{H}] = -i\hbar \hat{P}_i \ ,
\end{aligned}
\tag{5}
$$

where $\hat{J}_{ij} = \sum_k \epsilon_{ijk} \hat{J}_k$. *(6 Pts)*

c) We want to extend the commutation relations by central extensions, eg.,

$$
[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k + i\hbar \, c_{ij}^{(JJ)} \cdot \hat{\mathbb{1}} \ . \tag{6}
$$

Show that only the central extensions $c_{[ij]}^{(JJ)}, c_{[ij]}^{(JP)}, c_{[ij]}^{(JK)}, c_i^{(KH)}$ and $c_{ij}^{(KP)} = c^{(KP)} \cdot \delta_{ij}$ are compatible with the Jacobi identity. Here $T_{[ij]}$ denotes the antisymmetric part of the tensor T_{ij}. Furthermore, use the Jacobi identity between $\hat{J}, \hat{K}, \hat{H}$ to derive a relation between $c^{(JP)}$ and $c^{(KH)}$. *(3 + 5(Bonus) Pts)*

d) Find a redefinition of the generators J_i, P_i, K_i such that all the central charges vanish except for $c_{ij}^{(KP)}$. Argue that the central extension $c_{ij}^{(KP)}$ cannot be removed by a redefinition of operators. *(3 Pts)*

8.2 The 1d Galileo Group Gal(1) and the free particle

Consider a free particle in 1d with the Hamiltonian

$$
\hat{H} = \frac{1}{2m} \hat{P}^2 \ . \tag{7}
$$

In the lecture we derived for the boost generators the operator

$$
\hat{K} = t\hat{P} - m\hat{x} \ . \tag{8}
$$

a) Consider the transformed wave function

$$
\psi'(t, x) = \exp\left(\frac{i}{\hbar} v \hat{K}\right) \psi(t, x) \ . \tag{9}
$$

Show that

$$
\psi'(t, x) = \exp\left(-\frac{i}{\hbar} m \left(\frac{1}{2} t v^2 + v x\right)\right) \psi(t, x + vt) \ . \tag{10}
$$

(3 Pts)

b) Let $\psi(t, x)$ be a physical wave function, i.e., $\psi(t, x)$ fulfils the 1d (free) Schrödinger equation, then show that the transformed wave function $\psi'(t, x)$ also fulfils the Schrödinger equation. *(3 Pts)*