Physikalisches Institut
Universität Bonn
Theoretical Physics

Exercise Sheet 9
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Advanced Quantum Theory

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http://www.th.physik.uni-bonn.de/klemm/advancedqm/index.php

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–Exercises–

9.1 The Lorentz Group in 4d

In this exercise we want to derive the commutation relations of the Lorentz group SO(1,3). We use the convention $\eta = (-,+,+,+)$ for the metric tensor. For $S \in \mathrm{SO}(1,3)$,

$$S(\Lambda) : x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \ , \tag{1}$$

and the infinitesimal unitary transformation is given by

$$U(\omega_{\mu\nu}) = \mathbb{1} - \frac{i}{2\hbar}\omega_{\mu\nu}M^{\mu\nu} \ . \tag{2}$$

a) State the composition law, the inverse element and the infinitesimal inverse transformation of the Lorentz group. *(1 Pt)*

b) Compute the operator $U(S')U(\omega_{\mu\nu})U(S')^{-1}$ for infinitesimal S' by considering the infinitesimal transformation corresponding to the element $S' \cdot S(\mathbb{1} - \omega) \cdot S'^{-1} \in \mathrm{SO}(1,3)$. By comparing the coefficients of ω show that *(3 Pts)*

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar \left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\right) \ . \tag{3}$$

c) Define the rotations $J^{ij} := M^{ij}$ and boosts $K^i := M^{0i}$ for $i,j = \{1,2,3\}$ and state their commutation relations. Compare with the commutation relations of the rotations and boosts in the Galileo algebra. *(2 Pts)*

9.2 Spin vs. Special Orthogonal Group

We want to show that the spin group $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$ is a double cover of the special orthogonal group SO(3).

a) Consider the space of traceless hermitian 2×2 matrices $H_{2\times 2}$. Show that any traceless hermitian 2×2 matrix h can be written as

$$h(\vec{x}) = \vec{\sigma} \cdot \vec{x} = \sum_i \sigma_i \, x_i \qquad , \text{with } \ \vec{x} \in \mathbb{R}^3 \ , \tag{4}$$

and compute the determinant of $h(\vec{x})$. *(2 Pts)*

b) Argue that the group SU(2) acts on $H_{2\times 2}$ by conjugation, i.e., $u \in \mathrm{SU}(2)$ yields a map

$$\varphi_u : H_{2\times 2} \to H_{2\times 2} \ , \ \ h \mapsto u \cdot h \cdot u^{-1} \ , \tag{5}$$

which is well defined. In other words, show that the conjugation map φ_u preserves the defining properties of $H_{2\times 2}$. *(1 Pt)*

c) Compute for a general matrix $u \in \mathrm{SU}(2)$ a matrix R_u such that

$$u(\vec{\sigma} \cdot \vec{x})u^{-1} = \vec{\sigma} \cdot (R_u \vec{x}) \ . \tag{6}$$

Show that the computed R_u is a group element of SO(3). *(4 Pts)*

<u>*Hint:*</u> First demonstrate that $R_u \in \mathrm{O}(3)$ and then by observing that $R_{\mathbb{1}_{2\times 2}} = \mathbb{1}_{3\times 3}$, argue that $\det(R_u) = 1$ for any $u \in \mathrm{SO}(3)$. (If you are bored you can also compute the determinant directly, however it is not advised).

d) Finally show that the constructed map

$$\rho : \mathrm{SU}(2) \to \mathrm{SO}(3) \ ; \ \ u \mapsto R_u \tag{7}$$

is 2-to-1. *(2 Pts)*

9.3 The Lagrangian Density of Electrodynamics

In this exercise we want to derive the Maxwell's equation from the Lagrangian density of electrodynamics. We use the convention $\eta = (-, +, +, +)$ for the metric tensor. The action of electrodynamics is given by

$$S = \int d^4x \ \mathcal{L} = \int d^4x \left(-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right) \ , \tag{8}$$

with the Lagrangian density $\mathcal{L}$. Here μ_0 is the vacuum permeability. The electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is defined as

$$\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \ , \tag{9}$$

where c is the speed of light and $\vec{E}$, $\vec{B}$ are the electric and magnetic fields, respectively. The four-current is given by $j^\mu = (c\rho, \vec{J})$ where ρ is the charge density and $\vec{J}$ is current density.

a) Solve the Euler-Lagrange equation for the electromagnetic Lagrangian density to arrive at the Maxwell's equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \ , \tag{10}$$

where ϵ_0 is the vacuum permittivity. Note $\mu_0 \epsilon_0 = 1/c^2$. *(3 Pts)*

b) Prove the Bianchi identity for the electromagnetic tensor

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \ , \tag{11}$$

to arrive at the other two Maxwell's equations

$$\nabla \cdot \vec{B} = 0 \ \ , \ \ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ . \tag{12}$$

(2 Pts)