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Theoretische Physik

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Exercise 1

Exercises General Relativity and Cosmology

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1 Spacetime diagrams and the garage paradoxon

In the following we consider for simplicity $1+1$ dimensional spacetime.

1. Draw a spacetime diagram and draw
   a) an event
   b) a light-ray
   c) the worldline of an object that travels with velocity $v < 1$
   d) the worldline of an object that travels with velocity $v > 1$
   e) the worldline of an accelerated object .

2. Draw a spacetime diagram (x, t) of an observer $\mathcal{O}$ at rest. Into this spacetime diagram draw the worldline of an observer $\mathcal{O}'$ that travels with velocity v measured in the rest-frame of $\mathcal{O}$. What are the coordinate axes of the spacetime diagram of $\mathcal{O}'$?
   Hint: What is his timeaxis? How do you then construct the space-axis?

3. You know from the lecture that an object with length l' in the frame of the observer $\mathcal{O}'$ appears with length l to the observer $\mathcal{O}$ related to l' by

$$l = \sqrt{1 - v^2} l'.$$

   In the following we consider the so-called garage paradoxon. We consider a car and a garage that have both length l at rest. The garage has a front (F) and a back (B) door. The garage is constructed that way that it opens both doors when the front of the car arrives at the front door, closes both doors, if the car is completely inside and opens again both doors, when the car leaves the garage (i.e. the front of the car arrives at the back-door). From the perspective of the garage the car is contracted and nicely fits into the garage. From the point of view of the car, the garage is contracted and the car will not fit into it, but will rather be destroyed. Resolve this paradoxon.
   Hint: Draw a spacetime diagram in which the garage is a rest. Clarify the order the events appear for both observers!

2 The Maxwell equations in a covariant form.

The Maxwell equations read in terms of three-dimensional vector-analysis quantities

$$\nabla \times \vec{B} - \partial_t \vec{E} = \vec{J} \tag{2.1}$$
$$\nabla \cdot \vec{E} = \rho \tag{2.2}$$
$$\nabla \times \vec{E} + \partial_t \vec{B} = 0 \tag{2.3}$$
$$\nabla \cdot \vec{B} = 0. \tag{2.4}$$

In the above form the Maxwell equations are not manifestly covariant. We introduce the following tensors

$$F^{\mu\nu}, \quad F^{0i} = E^i, \quad F^{ij} = \epsilon^{ijk} B_k, \tag{2.5}$$
$$J^\mu, \quad J^0 = \rho, \quad J^i = \vec{J}. \tag{2.6}$$

Using the tensor notation, the Maxwell equations read

$$\partial_\mu F^{\nu\mu} = J^\nu, \tag{2.7}$$
$$\partial_{[\mu} F_{\nu\lambda]} = 0. \tag{2.8}$$

Show that this form reproduces the Maxwell equations in the above form. Introducing the one-form A,

$$A = A_\mu dx^\mu, \quad A_\mu = (\Phi, \vec{A}), \quad F = dA, \tag{2.9}$$

show with the help of your tutor that the Maxwell equations may be written in forms of differential forms as

$$dF = 0, \quad d * F = *J. \tag{2.10}$$

3 Upper and lower indices

This exercise is devoted to explain the relation of upper and lower indices that appear in special relativity. However, we take a more general point of view. Let V be a n-dimensional vector space and $e_1, \ldots, e_n$ be a basis. Let V^* be its dual with basis $e^1, \ldots, e^n$, defined by

$$e^i(e_j) = \delta^i_j. \tag{3.1}$$

Given a bilinear form β: $V \times V \longrightarrow \mathbb{R}$, this induces an isomorphism $\varphi : V \longrightarrow V^*$ by setting

$$\varphi(v) = \beta(v, \cdot), \tag{3.2}$$

as well as a bilinear form β^*: $V^* \times V^* \longrightarrow \mathbb{R}$ given by

$$\beta^*\big(\varphi(v), \varphi(w)\big) = \beta\big(v, w\big). \tag{3.3}$$

We introduce the notation

$$\beta_{ij} = \beta\big(e_i, e_j\big), \quad \beta^*\big(e^i, e^j\big) = \beta^{*ij}. \tag{3.4}$$

Given a vector $v = v^i e_i \in V$, show that its dual $\tilde{v}$ is given by

$$\tilde{v} = \tilde{v}_i e^i, \quad \tilde{v}_i = \beta_{ij} v^j. \tag{3.5}$$

In addition, show that

$$\beta = \beta^{*-1}. \tag{3.6}$$

One also says that an element of V is a covariant vector, whereas an element of V^* is a contravariant vector. Given a linear map $A : V \longrightarrow W$, show that this induces a map $A^* : W^* \longrightarrow V^*$, by setting

$$A^*\alpha(v) = \alpha(Av), \quad v \in V, \quad \alpha \in W^*. \tag{3.7}$$

Work out how the map A^* is expressed in components and show that it is given by the transposed.

Homework

4 Spacetime diagrams for experimental physicists **2 points**

1. Draw a spacetime diagram with two events. Draw for each event the lightcone and mark for both now, future, past and out-of the world, as well as common future and past.

2. You know from the lecture that the four-distance of two events is the same in any coordinate system, i.e.

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (\Delta t')^2, \tag{4.1}$$

   where the (x, y, z, t) and (x', y', z', t') denote the coordinates of the two respective observers. Draw a spacetime diagram with an observer $\mathcal{O}$ at rest and an observer $\mathcal{O}'$ that moves with velocity $v = 0.6$. Draw the worldlines of an object with length 4 at rest in $\mathcal{O}$. Determine its length l' in the frame $\mathcal{O}'$ in two ways.

   a) Algebraic method: Calculate its length using the formula for Lorentz construction.
   b) Geometric method: Construct a hyperbola to gauge the coordinate axes of $\mathcal{O}'$ using the scales of $\mathcal{O}$. Show that the results agree (up to an acceptable tolerance due to your drawing skills.)

5 Lorentz transformations **3 points**

We consider four-dimensional Minkowski spacetime $\mathbb{R}^{3,1}$, which is $\mathbb{R}^4$ equipped with

$$\eta = \mathrm{diag}(-1, 1, 1, 1). \tag{5.1}$$

1. Show that the requirement of an invariant line element leads to the following constraint for a Lorentz transformation $x \mapsto \Lambda x$

$$(x - y)^2 = (\Lambda(x - y))^2. \tag{5.2}$$

   Show that this equation reads in components

$$\Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}. \tag{5.3}$$

2. Show that the set of Lorentz transformations forms a group

$$\mathcal{L} = O(3,1) = \{\Lambda \in \mathbb{R}^{4\times 4} | \Lambda^t \eta \Lambda = \eta\}. \tag{5.4}$$

3. Embed the group of three-dimensional rotations into $\mathcal{L}$.

4. Show that $|\Lambda^0_0| \geq 1$ and that $|\det \Lambda| = 1$. Prove that the Lorentz group consists of four branches (which are not connected among each other).

5. Identify the Lorentz transformations for time and parity reversal and relate them to the respective branches.

6. Construct the Lorentz transformation of a boost with velocity v along the y-axis.

7. Consider the successive transformation of two boosts along the y-axis and of a boost along the y-axis and then along the x-axis. What are the corresponding composite transformations? Derive a formula how to add relativistic velocities. Reproduce for small velocities the Galilein way to add velocities and show that it is impossible to reach velocites larger than 1. Do boosts form a subgroup of the Lorentz-group?

6 The Energy momentum tensor **3 points**

The energy-momentum tensor of the electro-magnetic field is given by

$$T^{\mu\nu} = F^{\mu}{}_{\rho} F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{6.1}$$

Show that

$$\partial_\alpha T^{\alpha\beta} = 0. \tag{6.2}$$

Show that

$$\partial_0 \int T^{0\beta} d^3x = 0. \tag{6.3}$$

Express $T^{0\mu}$ in terms of $\vec{E}$ and $\vec{B}$. Which conserved quantities do you obtain?

7 Some concrete tensor algebra **2 points**

Given the tensor $X^{\mu\nu}$ as well as the vector V^μ specified by their components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^\mu = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix}, \tag{7.1}$$

you are asked to compute

1. $X^{\mu}{}_{\nu}$,

2. $X_{\mu}{}^{\nu}$,

3. $X^{(\mu\nu)}$,
4. $X^{[\mu\nu]}$,
5. X^{μ}_{μ},
6. $V^{\mu}V_{\mu}$,
7. $V_{\mu}X^{\mu\nu}$. Here we have denoted by $(\cdot,\cdot)$ symmetrized indices and by $[\cdot,\cdot]$ anti-symmetrized indices.