Physikalisches Institut
Universität Bonn
Theoretische Physik

ST 2012
15.6.2012
Exercise 10

Exercises General Relativity and Cosmology

Prof. Dr. Albrecht Klemm

1 Maximally symmetric spaces **1 point**

One can argue that the Riemann tensor of a maximally symmetric space has the form

$$R_{\rho\sigma\mu\nu} \propto g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} \,. \tag{1.1}$$

Show that the full expression is given by

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} \left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \,. \tag{1.2}$$

2 Friedmann equations **7 points**

Consider an $(1+N+n)$-dimensional spacetime with coordinates $\{t, x^I, y^i\}$, where $I = 1 \ldots N$ and $i = 1 \ldots N$.
The metric is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{IJ}\mathrm{d}x^I\mathrm{d}x^J + b^2(t)\gamma_{ij}(y)\mathrm{d}y^i\mathrm{d}y^j \,, \tag{2.1}$$

where δ_{IJ} is the Kronecker delta and $\gamma_{ij}(y)$ is the metric in an n-dimensional maximally symmetric spatial manifold. The metric γ is normalized in a way such that the curvature parameter

$$k = \frac{R(\gamma)}{n(n-1)} \tag{2.2}$$

is either $1, 0$ or -1, where $R(\gamma)$ is the Ricci scalar corresponding to the metric γ.

1. Calculate the Ricci tensor for this metric. The nonvanishing components are

$$R_{00} = -\left(\frac{\ddot{a}}{a}\right)^2 N - \left(\frac{\ddot{b}}{b}\right)^2 n \tag{2.3}$$

$$R_{IJ} = \left(\ddot{a}a + (N-1)\dot{a}^2 + na\dot{a}\frac{\dot{b}}{b}\right)\delta_{IJ} \tag{2.4}$$

$$R_{ij} = \left(\ddot{b}b + n\dot{b}^2 + Nb\dot{b}\frac{\dot{a}}{a} + b^2k(n-1)\right)\gamma_{ij} \,. \tag{2.5}$$

2. Define an energy-momentum tensor in terms of an energy density ρ and pressure in the x^I and y^i directions, $p^{(N)}$ and $p^{(n)}$:

$$T_{00} = \rho \tag{2.6}$$

$$T_{IJ} = a^2 p^{(N)} \delta_{IJ} \tag{2.7}$$

$$T_{ij} = b^2 p^{(n)} \gamma_{ij} \,. \tag{2.8}$$

Plug the metric, $T_{\mu\nu}$ and the Riemann Tensor you derived into Einstein's equations to derive Friedmann-like equations for a and b. You will find three independent equations.

3. Derive equations for ρ, $p^{(N)}$ and $p^{(n)}$ at a static solution, where $\dot{a} = \dot{b} = \ddot{a} = \ddot{b} = 0$, in terms of k, n and N. Use these to compute $w^{(N)} = p^{(N)}/\rho$ and $w^{(n)} = p^{(n)}/\rho$, the equation-of-state parameters.

3 Geodesics **2 points**

Consider de Sitter space in coordinates where the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \,. \tag{3.1}$$

- Solve the geodesic equation to find the affine parameter as a function of t. To do this, you should consult exercise 1 on sheet 3.
- Show that the geodesics reach $t = -\infty$ in a finite affine parameter. What does this mean?