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Exercise 11

Exercises General Relativity and Cosmology

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1 Reissner Nordström black hole **4 point**

In this problem you will derive the metric for a electrically charged black hole. It is called the Reissner Nordström black hole. Instead of a vanishing energy momentum tensor, you will now have the energy momentum tensor of the electromagentic field. This means, the Maxwell equations and Einstein's equations are coupled in this case.
Use the coordinates $\{r, t, \theta, \phi\}$ and start with the following general metric for a spherically symmetric problem

$$\mathrm{d}s^2 = -e^{2\alpha(r,t)}\mathrm{d}t^2 + e^{2\beta(r,t)}\mathrm{d}t^2 + r^2\mathrm{d}\Omega^2\,. \tag{1.1}$$

Even though the energy momentum tensor does not vanish, you still can simplify Einstein's equation. Show that

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.2}$$

holds for this problem.

1.1 The electromagnetic field strength tensor

In a spherical symmetric problem the only nonzero components of the electric and the magnetic field are

$$E_r = f(r,t) \tag{1.3}$$
$$B_r = g(r,t) \tag{1.4}$$

Let $\tilde{\epsilon}_{\mu\nu\sigma\rho}$ be the known Levi Civita symbol. This is not a tensor. We define a tensor ϵ by setting

$$\epsilon_{\mu\nu\sigma\rho} = \sqrt{|g|}\tilde{\epsilon}_{\mu\nu\sigma\rho}\,. \tag{1.5}$$

Write down the electrodynamic field strength tensor $F_{\mu\nu}$.

1.2 The energy momentum tensor

Compute the energy momentum tensor and show that its nonvanishing components are given by

$$T_{tt} = \frac{1}{2}e^{-2\beta(r,t)}\left(f^2(r,t) + g^2(r,t)\right) \tag{1.6}$$
$$T_{rr} = -\frac{1}{2}e^{-2\alpha(r,t)}\left(f^2(r,t) + g^2(r,t)\right) \tag{1.7}$$
$$T_{\theta\theta} = \frac{1}{2}e^{-2(\alpha(r,t)+\beta(r,t))}r^2\left(f^2(r,t) + g^2(r,t)\right) \tag{1.8}$$
$$T_{\phi\phi} = \sin^2\theta T_{\theta\theta} \tag{1.9}$$

Use Einstein's equation to show

$$\beta(r,t) = \beta(r)\,. \tag{1.10}$$

Knowing this find a convenient relation between R_{rr} and R_{tt} to show

$$\partial_r \alpha(r,t) + \partial_r \beta(r) = 0 \tag{1.11}$$

Use this knowledge to get rid of α entirely, by redefining the time coordinate like $\mathrm{d}t \to e^{h(t)}\mathrm{d}t$.

1.3 Solving the Maxwell equations

Now take the Maxwell equations, with vanishing current J_μ, and replace the common derivatives by covariant ones.

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0 \tag{1.12}$$

$$\nabla_{[\mu} F_{\mu\rho]} = 0 \tag{1.13}$$

Use the r-component of (1.12) to show that f is time independent, then solve the differential equation you obtain from the t-component to find f. The constant will be the total electric charge of the black hole and will be set to $Q/\sqrt{4\pi}$.
Now do the same using (1.13). This time the constant will be the total magnetic charge and will be set to $P/\sqrt{4\pi}$.

1.4 The Reissner-Nordström metric

Now you are only left with the unknown function $\alpha(r)$. Use Einstein's equation once again to show

$$e^{2\alpha} = 1 + \frac{\tilde{R}}{r} + \frac{G}{r^2}(Q^2 + P^2)\,. \tag{1.14}$$

Compare this to the Schwarzschild solution to fix the constant $\tilde{R}$.

2 Tolmann-Oppenheimer-Volkov equation **3 points**

Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional spacetime.

1. Derive the analogue of the Tolman-Oppenheimer-Volkov (TOV) equation for (2+1)-dimensions.
2. Show that the vacuum solution can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{1-8GM}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\,. \tag{2.1}$$

3. Show that another way to write the same solution is

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\xi^2 + \xi^2\mathrm{d}\phi^2 \tag{2.2}$$

   where $\phi \in [0, 2\pi(1-8GM)^{1/2}]$.

4. Solve the (2+1) TOV equation for a constant density star. Find $p(r)$ and solve for the metric.
5. Solve the $(2+1)$ TOV euqation for a star with equation of state $p = \kappa\rho^{3/2}$. Find $p(r)$ and solve for the metric.
6. Find the mass $M(R) = \int_0^{2\pi}\int_0^R \rho\,\mathrm{d}r\mathrm{d}\theta$ and the proper mass $\bar{M} = \int_0^{2\pi}\int_0^R \rho\sqrt{-g}\,\mathrm{d}r\mathrm{d}\theta$ for the solutions in 4 and 5.

3 Age of the universe **3 points**

Consider a matter dominated universe. Calculate

- how long a positively curved universe would be and how long it would last. Set $\Omega_0 = 1.1$.
- how old a negatively curved universe would be. Set $\Omega_0 = 0.9$.
- how old a flat universe would be. Set $\Omega_0 = 1$.

Use $H_0^{-1} = 14 \cdot 10^9 y$. Useful parameterizations:

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta), \quad \Omega_0 > 1 \tag{3.1}$$

$$a(\eta) = \frac{1}{2}\frac{\Omega_0}{1 - \Omega_0}(\cosh\eta - 1), \quad \Omega_0 < 1 \tag{3.2}$$

A Spherically symmetric metric.

Here the nonvanishing components for the Christoffel symbols, the Riemann tensor and the Ricci tensor for the metric

$$\mathrm{d}s^2 = -e^{2\alpha(r,t)}\mathrm{d}t^2 + e^{2\beta(r,t)}\mathrm{d}t^2 + r^2\mathrm{d}\Omega^2\,. \tag{A.1}$$

are given.

A.1 Christoffel symbols

$$\Gamma^t_{tt} = \partial_t\alpha \qquad \Gamma^t_{tr} = \partial_r\alpha \qquad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta$$

$$\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha \qquad \Gamma^r_{tr} = \partial_t\beta \qquad \Gamma^r_{rr} = \partial_r\beta$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta} \qquad \Gamma^\phi_{r\phi} = \frac{1}{r}$$

$$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

A.2 Riemann tensor

$$
\begin{aligned}
R^t{}_{rtr} &= e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + [\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2] \\
R^t{}_{\theta t\theta} &= -re^{-2\beta}\partial_r\alpha \\
R^t{}_{\phi t\phi} &= -re^{-2\beta}\sin^2\theta\ \partial_r\alpha \\
R^t{}_{\theta r\theta} &= -re^{-2\alpha}\partial_t\beta \\
R^t{}_{\phi r\phi} &= -re^{-2\alpha}\sin^2\theta\ \partial_t\beta \\
R^r{}_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
R^r{}_{\phi r\phi} &= re^{-2\beta}\sin^2\theta\ \partial_r\beta \\
R^\theta{}_{\phi\theta\phi} &= (1 - e^{-2\beta})\sin^2\theta
\end{aligned}
$$

A.3 Ricci tensor

$$
\begin{aligned}
R_{tt} &= [\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] + e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_1\alpha] \\
R_{rr} &= -[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta] + e^{2(\beta-\alpha)}[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta] \\
R_{tr} &= \frac{2}{r}\partial_t\beta \\
R_{\theta\theta} &= e^{-2\beta}[r(\partial_t\beta - \partial_t\alpha) - 1] + 1 \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta
\end{aligned}
$$