Physikalisches Institut
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Theoretische Physik

ST 2012
13.4.2012
Exercise 2

Exercises General Relativity and Cosmology

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1 Derivation of the energy momentum tensor

In this exercise you will derive the energy momentum tensor appearing in the Noether theorem. Consider a scalar field ϕ and the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi, x^\mu)$. The action will be

$$S = \int_R \mathrm{d}^4 x \mathcal{L}(\phi, \partial_\mu \phi, x^\mu) \tag{1.1}$$

with arbitrary volume R.

1. To start, we want to see what variations of x and ϕ will look like.

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \tag{1.2}$$

$$\phi(x^\mu) \to \phi'(x^\mu) = \phi(x^\mu) + \delta\phi(x^\mu) \tag{1.3}$$

   Now we want to compute the total variation in ϕ. Hence we need to think about the variation of

$$\phi \to \phi'(x') = \phi(x) + \Delta\phi(x)\,. \tag{1.4}$$

   What is $\Delta\phi(x)$ to first order in δ?

   Hint: $f(x^\nu + \delta x^\nu) = f(x^\nu) + \delta x^\mu \partial_\mu f(x^\nu) + \cdots$

2. The next step is to compute δS. Which will be

$$\delta S = \int_R \mathrm{d}^4 x' \, \mathcal{L}(\phi', \partial_\mu \phi', x'^\mu) - \int_R \mathrm{d}^4 x \, \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\,. \tag{1.5}$$

   We want rewrite the first integral in terms of x^μ and ϕ. In order to do this we now need to compute the Jacobian of the change of variables $x^\mu \to x'^\mu$.

$$\mathrm{d}^4 x' \to J \mathrm{d}^4 x \quad \text{where} \quad J = \det\left(\frac{\partial x'^\mu}{\partial x^\lambda}\right) = 1 + \partial_\mu(\delta x^\mu)\,. \tag{1.6}$$

   Plug this in, compute the variations and keep only terms up to first order in δ.

3. Write down δS so that $\delta\phi$ and δx^μ are factored out. To do this you have to use partial integration, which will give you a term containing a total derivative.

4. The total derivative can be changed into a surface term by using

$$\int_R \partial_\mu (\cdots)\, \mathrm{d}^4 x = \int_{\partial R} (\cdots) \mathrm{d}\sigma_\mu\,. \tag{1.7}$$

   Having this ready you are able to derive the Euler-Lagrange equation. What are the necessary conditions?

5. From now on we do not use these conditions on $\delta\phi$ and δx^μ anymore and further analyze δS.

   After writing the total derivative as a surface term you should be able to put the result into the following form
$$\int_{\partial R}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - T^\mu_\nu\delta x^\nu\right]\mathrm{d}\sigma_\mu\,. \tag{1.8}$$
   What is the explicit form of T^μ_ν? T^μ_ν is called the energy-momentum tensor.

6. Let us now consider transformations on x^μ and ϕ which let the action S invariant and can be written infinitesimally as
$$\Delta x^\mu = \delta x^\mu = X^\mu_\nu \delta\omega^\nu \tag{1.9}$$
$$\Delta\phi = \Phi_\mu \delta\omega^\mu\,. \tag{1.10}$$
   There may be more general transformations with more parameters.

   Assuming that these obey the Euler-Lagrange equation, we find
$$\delta S = \int_{\partial R}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Phi_\nu - T^\mu_\kappa X^\kappa_\nu\right]\delta\omega^\nu\mathrm{d}\sigma_\mu = 0\,. \tag{1.11}$$
   Check this! This suggests the definition of
$$J^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Phi_\nu - T^\mu_\kappa X^\kappa_\nu\,. \tag{1.12}$$
   The variation $\delta\omega^\nu$ is arbitrary, therefore
$$\int_{\partial R} J^\mu_\nu \mathrm{d}\sigma_\mu = 0 \tag{1.13}$$
   holds. Use (1.7) to rewrite this as a total derivative. Argue now that $\partial_\mu J^\mu_\nu = 0$.

Homework

2 Maxwell equations **5 points**

The quantities used here were defined on the last exercise sheet.

1. What is the definition of the field strength tensor $F_{\mu\nu}$, known from electrodynamics, in terms of A_μ? What is the unit of A_μ in natural units?

2. Now consider the action
$$S = \int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu\right) \tag{2.1}$$
   and derive the Maxwell equations from this, using the Euler-Lagrange equation.

Hints:

- Here, every component A_μ is a field for which you have to solve the Euler Lagrange equation!
- Additionally, you should keep in mind the index structure, when taking the derivative with respect to A_μ. You might have to lower some indices!
- This means also to think about if $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\kappa)}$ has an upper or lower index μ and κ.

3. Compare this to exercise 2 of the last exercise sheet.

3 Energy-momentum tensor of electrodynamics **5 points**

1. Compute the energy-momentum tensor for

$$S = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{3.1}$$

2. The resulting energy momentum tensor $T_{\mu\nu}$ is not symmetric in μ and ν. In order to symmetrize it, add

$$\partial_\lambda K^{\lambda\mu\nu} = \partial_\lambda (F^{\mu\lambda} A^\nu) \tag{3.2}$$

to it. Show that that it is symmetric now. Why are you allowed to add this?

3. Rewrite the new $T_{\mu\nu}$ in a manner so that it corresponds to equation (6.1) on the last exercise sheet.