Exercises General Relativity and Cosmology

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1 Charts for S^2 **2 points**

1. Use the defining equation

$$x^2 + y^2 + z^2 = 1 \tag{1.1}$$

to construct *open* charts for S^2. Give the transition functions and check that they are smooth.

Hint: The idea is clearly to use the square-root function. However, note that you need more than two charts!

2. The projective space. Here we explore another example of a manifold, namely $\mathbb{C}P^1$. This is defined as the space of all lines in $\mathbb{C}^2$ that pass through the origin. Note that we refer here to a complex line, i.e. a copy of $\mathbb{C}$. We denote an element of $\mathbb{C}P^1$ by

$$[z_1 : z_2] = \big\{(z_1, z_2) \neq (0,0) | z_1, z_2 \in \mathbb{C}\big\} / \sim . \tag{1.2}$$

Here we have denoted the equivalence relation $\sim$ by

$$(z_1, z_2) \sim (w_1, w_2) \Leftrightarrow \exists \lambda \in \mathbb{C}^* s.t. (z_1, z_2) = \lambda (w_1, w_2) \tag{1.3}$$

- Show that each element in $\mathbb{C}P^1$ can be represented as either $[1 : a]$ or as $[b : 1]$.
- For the moment we restrict to a real picture. Consider $\mathbb{R}^2$ and draw the lines $x = 1$ as well as $y = 1$ (Note that these lines are no elements of the projective space!). How can these lines be identified with the representatives from the previous task? How many lines are there that do not pass through $x = 1$ respectively $y = 1$?
- Conclude that we can endow $\mathbb{C}P^1$ with two charts both being isomorphic to $\mathbb{C}$. Show that the transition function is given by

$$\varphi : \mathbb{C}^* \to \mathbb{C}^*, \quad z_1 \mapsto z_2 = z_1^{-1}. \tag{1.4}$$

Here z_i denotes the coordinate on the respective copy of $\mathbb{C}$.

3. Show that $\mathbb{C}P^1$ is diffeomorphic to S^2. To do so you may use the charts that are provided by the stereographic projection.

Hint: The charts for the stereographic projection are constructed in Carroll and may be used without deriving them again.

2 Pull-back und push-forward **1 point**

We consider two smooth manifolds M, N of dimension m and n respectively, as well as a smooth map

$$\Phi : M \to N. \tag{2.1}$$

Consider two charts given by maps[1]

$$x, y : M \longrightarrow \mathbb{R}^n. \tag{2.2}$$

Recall that a tangent vector(field) v acts on a function $f : M \longrightarrow \mathbb{R}$ as

$$v(f) = v^i \frac{\partial(f \circ x^{-1})}{\partial x^i}. \tag{2.3}$$

In addition, the map Φ induces a map (push-forward) of the respective tangent spaces

$$\Phi_* : TM \to TN, \quad \Phi_*(v)f = v(f \circ \Phi) \tag{2.4}$$

This can be used to pull-back a contravariant vector $\omega \in T^*N$ by setting

$$\Phi^*\omega(v) = \omega(\Phi_* v) \tag{2.5}$$

Express the change of coordinates, as well as the push-forward and the pull-back in local coordinates. How do the transformations for general tensor fields read? What property does the map Φ in this case have to have?

3 Pseudo-Riemannian metrics **1 point**

Given a manifold M with a Riemannian metric g. Given a submanifold N of M, one obtains again a Riemannian metric $\tilde{g}$ for N by pulling back g via the inclusion $\iota : N \to M$.
Consider the half-circle $S^1_h \subset \mathbb{R}^2$, that is defined by

$$x^2 + y^2 = r^2, \quad y > 1. \tag{3.1}$$

Compute the induced metric in two ways, once using x as a coordinate and once using polar coordinates. You should find that

$$\tilde{g} = r^2 d\varphi^2, \quad \text{resp.} \quad \tilde{g} = \frac{x^2}{r^2 - x^2} dx^2. \tag{3.2}$$

Check that these metrics are the same. In addition, find an example where the pull-back of a metric, that is not positiv definite, is a degenerate one on the submanifold.

[1] We are a bit sloppy with the notation here. In principle one has to be very careful about where this functions are defined.

4 Differential forms **2 points**

4.1 Wedge product and outer derivative **1 point**

Given two differential forms

$$\eta = \frac{1}{p!}\eta_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \quad \omega = \frac{1}{q!}\omega_{\nu_1\dots\nu_q} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_q}, \tag{4.1}$$

the wedge product $\eta \wedge \omega$ is defined by

$$\eta\wedge\omega = \frac{1}{(p+q)!}(\eta\wedge\omega)_{\mu_1\dots\mu_{p+q}} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p+q}} = \frac{1}{p!q!}\eta_{\mu_1\dots\mu_p}\omega_{\nu_1\dots\nu_q} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}. \tag{4.2}$$

Here we have introduced

$$(\eta \wedge \omega)_{\mu_1\dots\mu_{p+q}} = \frac{1}{p!q!}\eta_{[\mu_1\dots\mu_p}\omega_{\mu_{p+1}\dots\mu_{p+q}]}. \tag{4.3}$$

The outer derivative acts on a form as

$$d\omega = \frac{1}{q!}\partial_\mu \omega_{\mu_1\dots\mu_q} dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}. \tag{4.4}$$

Show that

1. $$\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega \tag{4.5}$$

2. $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q \omega \wedge d\eta. \tag{4.6}$$

4.2 The $*$-operator **1 point**

In this exercise we want to explore some properties of the $*$-operator. We consider an n-dimensional manifold with metric g of signature s (i.e. the number of negative eigenvalues of the metric). Given a differential p-form

$$\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \dots dx^{\mu_p}, \tag{4.7}$$

$*\omega$ is defined by

$$* \, \omega = \frac{1}{p!(n-p)!}\sqrt{|g|}\omega^{\mu_1\dots\mu_p}\epsilon_{\mu_1\dots\mu_n} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n}. \tag{4.8}$$

1. Show that
$$** = (-1)^{p(n-p)+s}. \tag{4.9}$$

2. Explicitly compute for polar coordinates of two-dimensional Euclidean space
$$* \, 1, \quad *dr, \quad *d\varphi, \quad *dr \wedge d\varphi. \tag{4.10}$$

5 The deRham sequence for $\mathbb{R}^3$ **2 points**

We consider the de Rham complex for $\mathbb{R}^3$ that is given by

$$0 \xrightarrow{d} \Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3) \xrightarrow{d} 0. \tag{5.1}$$

Here we have denoted by $\Omega^i(\mathbb{R}^3)$ the vector space of differential forms of degree i on $\mathbb{R}^3$. We construct an isomorphism of complexes by

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{d} & \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) & \xrightarrow{d} & 0 \\ & & \downarrow \delta_0 & & \downarrow \delta_1 & & \downarrow \delta_2 & & \downarrow \delta_3 & & \\ 0 & \longrightarrow & \mathcal{C}^0(\mathbb{R}^3) & \xrightarrow{\alpha} & \mathcal{V}^\infty(\mathbb{R}^3) & \xrightarrow{\beta} & \mathcal{V}^\infty(\mathbb{R}^3) & \xrightarrow{\gamma} & \mathcal{C}^0(\mathbb{R}^3) & \xrightarrow{d} & 0. \end{array} \tag{5.2}$$

$$\delta_0 : \Omega^0(\mathbb{R}^3) \xrightarrow{\sim} \mathcal{C}^\infty(\mathbb{R}^3) \qquad \text{(trivial)} \tag{5.3}$$

$$\delta_1 : \Omega^1(\mathbb{R}^3) \xrightarrow{\sim} \mathcal{V}^\infty(\mathbb{R}^3), \qquad \delta_1(a_i dx^i) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{5.4}$$

$$\delta_2 : \Omega^2(\mathbb{R}^3) \xrightarrow{\sim} \mathcal{V}^\infty(\mathbb{R}^3), \qquad \delta_2(b_{ij} dx^i \wedge dx^j) = \begin{pmatrix} b_1 \\ b_2 \\ b_3, \end{pmatrix}, \qquad b_k = \epsilon^{ijk} b_{ij} \tag{5.5}$$

$$\delta_3 : \Omega^3(\mathbb{R}^3) \xrightarrow{\sim} \mathcal{C}^\infty(\mathbb{R}^3), \qquad \delta_3(c_{ijk} dx^i \wedge dx^j \wedge dx^k) = \epsilon^{ijk} c_{ijk} \tag{5.6}$$

Here $\mathcal{C}^\infty(\mathbb{R}^3)$ and $\mathcal{V}^\infty(\mathbb{R}^3)$ denote the spaces of smooth functions respectively smooth vector-fields on $\mathbb{R}^3$. Compute the maps α, β, γ, so that the diagram gets commutative. How can they be identified with the three-dimensional vector-analysis operators $grad, div, rot$?

6 Integration on manifolds **2 points**

6.1 The Stokes theorem **1 point**

The Stokes theorem states that the integral over an exact form can be converted into an integral over a boundary

$$\int_M d\omega = \int_{\partial M} \omega. \tag{6.1}$$

Show for the special case that the domain one integrates over is either a rectangular box or a square this reproduces the Gauss- respectively the (two-dimensional) Stokes theorem

$$\int_M div\vec{V} d^3x = \int_{\partial M} \vec{V}\vec{\nu} dA, \qquad \int_F rot\vec{W}\vec{\nu} dA = \int_{\mathcal{C}} \vec{F}. \tag{6.2}$$

Here $\vec{\nu}$ denotes the unit normal vector and $\mathcal{C}$ is the boundary curve of the area F.

6.2 Integration of forms **1 point**

Given an n-dimensional Riemannian manifold M only the integration of n-forms over M is well-defined. Check that

$$\int_M \omega \wedge *\eta \tag{6.3}$$

defines a scalar product on M. Here the result of the integral is understood to be zero, if the degree of the wedged forms does not fit. Furthermore we define the integral of a function f over M as

$$\int_M f = \int_M f * 1. \tag{6.4}$$

Check for the example of three-dimensional polar coordinates that this reproduces the usual integration with the volume element

$$dV = r^2 \sin\theta dr d\theta d\varphi. \tag{6.5}$$