Exercises General Relativity and Cosmology

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1 Expanding universe **2 points**

Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] . \tag{1.1}$$

The function $a(t)$ is called the scale factor and here it its form is assumed to be

$$a(t) = t^q \quad 0 < q < 1 . \tag{1.2}$$

- What happens if you send t to zero? What are the consequences?
- Analyze the distance of two points at fixed spatial coordinates as a function of time.

Light cones are defined by null paths, i. e.

$$\mathrm{d}s^2 = 0 . \tag{1.3}$$

The task is now to draw a spacetime diagram for the case of null paths for which y and z are held constant. Taking y and z constant gives the equation

$$0 = -\mathrm{d}t^2 + t^{2q}\mathrm{d}x^2 . \tag{1.4}$$

Now prepare for solving this equation. First notice that this equation defines a $(0, 2)$-tensor.

- Write down the tensor defined by this expression.

Define the vector

$$V = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\partial_\mu . \tag{1.5}$$

- What is the meaning of V?

Now analyze

$$\mathrm{d}s^2(V, V) \tag{1.6}$$

by considering the $\mathrm{d}t^2$ and $\mathrm{d}x^2$-part individually. For instance

$$\mathrm{d}t^2(V, V) = (\mathrm{d}t \otimes \mathrm{d}t)(V, V) = \mathrm{d}t(V) \cdot \mathrm{d}t(V) . \tag{1.7}$$

Finally you should find

$$0 = -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + t^{2q}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^2 . \tag{1.8}$$

From which follows

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm t^{-q} \tag{1.9}$$

by using the chain rule.

- Now solve this and write the solutions in terms of $t(x)$.
- Then draw a spacetime-diagram for two distinct points in space. It is sufficient to sketch the general structure, you do not need to make an exact drawing.
- In addition draw such a spacetime diagram of two points in two dimensional Minkowski space. What can you say about the behavior of the light cones when comparing these two cases?

2 Causality **2 points**

Consider a two-dimensional geometry with coordinates t, x such that points with coordinates (t, x) and $(t, x+1)$ are identified.

- What is the topology of such a space?

Consider the metric

$$\mathrm{d}s^2 = -\cos(\lambda)\mathrm{d}t^2 - \sin(\lambda)\left[\mathrm{d}t\mathrm{d}x + \mathrm{d}x\mathrm{d}t\right] + \cos(\lambda)\mathrm{d}x^2 \tag{2.1}$$

where

$$\lambda = \mathrm{arccot}(t)\,. \tag{2.2}$$

Analyze λ in the interval from $t = -\infty$ to $t = \infty$.

- What happens to the sign of the spatial and the time coordinate?

Now think of the future light cone which envelops all positions a particle can reach in the future and remember the classification of intervals with respect to the light cone. Think in the same manner about curves.

- What does the case $t = -\infty$ correspond to?
- What happens to the light-cone on the way to $t = \infty$? What happens, in particular, at $t = \infty$.

Analyze the consequences for timelike curves. When thinking about this, keep in mind the topology of the considered space.

3 Christoffel symbols **4 points**

3.1 Transformation properties

Consider a $(1,0)$-tensor V. How does such a tensor transform? Now take the derivative

$$\partial_\mu V^\nu \tag{3.1}$$

and carry out a coordinate transformation on this. What can you say about the transformation properties of this object?
In the lecture the covariant derivative was defined by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma\,. \tag{3.2}$$

Carry out a coordinate transformation once more. What are the transformation properties of this object?

3.2 More properties of the Christoffel symbols

Show that the covariant derivative of the metric vanishes.

$$\nabla_\sigma g_{\mu\nu} = 0\,. \tag{3.3}$$

Show that this is also true for the inverse metric

$$\nabla_\sigma g^{\mu\nu} = 0\,. \tag{3.4}$$

Using this show that raising and lowering indices commutes with taking the covariant derivative

$$g_{\mu\nu}\nabla_\rho V^\lambda = \nabla_\rho(g_{\mu\lambda}V^\lambda) = \nabla_\rho V_\mu\,. \tag{3.5}$$

Now there is a convenient relation we will need later on, which you should prove now. Given an invertible matrix B we may write it as $B = e^A$. Using this, show that

$$\det(e^A) = e^{\mathrm{Tr}\,A} \tag{3.6}$$

holds.
Hint: The idea is to use a similarity transformation to transform A to Jordan normal form. You need to use the power series of the exponential function. Should you find this troubling you may assume A to be diagonalizable.
Using this show

$$\partial_\mu g = g g^{\alpha\beta}\partial_\mu g_{\beta\alpha}\,. \tag{3.7}$$

Knowing this you are now able to derive the following relations:

1. $\Gamma^\mu_{\mu\nu} = \frac{1}{2}\partial_\nu \ln(|g|)$
2. $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -\partial_\beta\left(g^{\alpha\beta}\sqrt{-g}\right)/\sqrt{-g}$
3. $g^{\alpha\beta}\partial_\nu(g_{\beta\mu}) = -(\partial_\nu g^{\alpha\beta})g_{\beta\mu}$ *Hint:* What is $g^{\alpha\beta}g_{\beta\mu}$
4. $\partial_\alpha g^{\mu\nu} = -\Gamma^\mu_{\beta\alpha}g^{\beta\nu} - \Gamma^\nu_{\beta\alpha}g^{\mu\beta}$ *Hint:* You may want to use 3.

Here repeated indices are summed over.
And now assume the metric to be diagonal and compute the following relations:

5. $\Gamma^\lambda_{\mu\nu} = 0$
6. $\Gamma^\lambda_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}$
7. $\Gamma^\lambda_{\mu\lambda} = \partial_\mu(\ln\sqrt{|g_{\lambda\lambda}|})$
8. $\Gamma^\lambda_{\lambda\lambda} = \partial_\lambda(\ln\sqrt{|g_{\lambda\lambda}|})$

Here $\mu \neq \nu \neq \lambda$ and repeated indices are not summed over.

4 Curvature tensor **2 points**

4.1 Curvature tensor of a two-sphere

In this exercise you will compute the Riemann curvature tensor of a two-dimensional sphere. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi\,. \tag{4.1}$$

1. Compute the inverse metric $g^{\mu\nu}$.
2. How many components does the Riemann curvature tensor have in this case?
3. Before you calculate anything, use the symmetries of the indices of the Riemann curvature tensor to simplify the problem! Which components can you argue to vanish? Find relations among the non-vanishing components. After analyzing this carefully, calculate all components of $R_{\rho\sigma\mu\nu}$.

4.2 Transformation properties

Show that $R_{\rho\sigma\mu\nu}$ transforms like a tensor.