Exercises General Relativity and Cosmology

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1 Geodesics for the expanding universe **3 points**

The metric for the expanding universe is given by

$$ds^2 = -dt^2 + a^2(t)\big(dx^2 + dy^2 + dz^2\big) \tag{1.1}$$

Using the variational method, show that the non-vanishing Christoffel-Symbols are given by

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij}, \quad \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j. \tag{1.2}$$

Write down the geodesic equations. Show that for a null geodesic in x-direction, i.e. a curve of the form

$$x^\mu(\lambda) = \big(t(\lambda), x(\lambda), 0, 0\big) \tag{1.3}$$

one finds

$$\frac{dt}{d\lambda} = \frac{\omega_0}{a}, \tag{1.4}$$

where ω_0 is a constant.

2 The curvature of a sphere **5 points**

We consider the two-sphere given by the equation

$$x^2 + y^2 + z^2 = R^2. \tag{2.1}$$

Introducing spherical coordinates

$$x = r\cos\varphi\sin\theta, \tag{2.2}$$
$$y = r\sin\varphi\sin\theta, \tag{2.3}$$
$$z = r\cos\theta, \tag{2.4}$$

compute the induced metric on the sphere. You should find the well-known result

$$ds^2 = R^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{2.5}$$

Using the formula for the Christoffel symbols (i.e. NOT the variational calculus)

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\big), \tag{2.6}$$

determine all Christoffel symbols. You should find that the only non-vanishing Christoffel symbols are

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta. \tag{2.7}$$

In addition, compute the Riemann tensor

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.8}$$

as well as the Ricci tensor

$$R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu} \tag{2.9}$$

and the scalar curvature

$$R = g^{\sigma\nu} R_{\sigma\nu}. \tag{2.10}$$

Be explicit in your calculations.

3 Some identities for Christoffel symbols **2 points**

Given a diagonal metric $g_{\mu\nu}$. Show that the Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = 0, \tag{3.1}$$

$$\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda g_{\mu\mu}, \tag{3.2}$$

$$\Gamma^{\lambda}_{\mu\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right), \tag{3.3}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right). \tag{3.4}$$

In this exercise there is no summing over repeated indices and we have that $\mu \neq \nu \neq \lambda$.