Physikalisches Institut
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Theoretische Physik

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Exercise 6

Exercises General Relativity and Cosmology

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1 Bianchi identity **2 points**

Derive the Bianchi identity

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0\,.$$

In order to do this, consider the covariant derivative of the Riemann tensor evaluated in locally inertial coordinates

$$\nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}\,.$$

2 Variation of the Riemann tensor **3 points**

Show that the variation of the Riemann tensor can be written as a total derivative:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu})$$

3 Killing equation **4 points**

The so called *Killing equation* is given by

$$\nabla_{(\mu} K_{\nu)} = \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0\,.$$

Where a K_μ satisfying this equation is called *Killing vector*. Let K_μ a Killing vector, show

$$\nabla_\mu \nabla_\sigma K^\rho = R^{\rho}{}_{\sigma\mu\nu} K^\nu\,.$$

Contracting indices you can find a formula for the Ricci tensor, which you should use now to show

$$K^\lambda \nabla_\lambda R = 0\,.$$

4 Killing vectors of the three sphere **1 points**

Now consider the three Killing vectors of the three sphere, which are given by

$$\begin{aligned} R &= \partial_\phi \\ S &= \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \\ T &= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi \end{aligned}$$

and show that the relations

$$[R,S] = T, \quad [S,T] = R, \quad [T,R] = S$$

hold.